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ON CONTINUITY PROPERTIES OF INFINITELY DIVISIBLE DISTRIBUTION FUNCTIONS

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Let $\varphi(\xi)$ be the characteristic function of an infinitely divisible distribution function F on R_1. Suppose F has no Gaussian component. Then one has the representation

$$(1) \qquad -\log|\varphi(\xi)| = \int_{-\infty}^{\infty}(1 - \cos \xi x)\nu(dx) = 2\int_{-\infty}^{\infty}\sin^2 \tfrac{1}{2}\xi x\nu(dx)$$

where ν is the Levy measure and satisfies

$$(2) \qquad \int_{-\infty}^{\infty} x^2(1 + x^2)^{-1}\nu(dx) < \infty.$$

Continuity properties of F depend on the behavior of ν near the origin. For interesting results and references to previous work see [5]. Related information of a very refined kind is developed in [3].

1. For $0 \leqq \lambda \leqq 2$ introduce the condition

$$(C_\lambda) \qquad \int_{-1}^{1}|x|^\lambda\nu(dx) = \infty.$$

It is easily seen from (2) above that (C_2) never holds. In [2] it is shown that (C_0) is equivalent to the continuity of F. An example is given in [1] in which (C_0) holds but the corresponding F is not absolutely continuous. This example will be modified to show that for any $\lambda < 2$ (C_λ) may hold yet F not be absolutely continuous. It follows that assertion (I) [2], p. 286, and the result given as Corollaries 1–3 of Theorem 2 in [4] are erroneous.

It follows from (1) and the Riemann-Lebesgue lemma that if F is absolutely continuous

$$(3) \qquad \int_{-\infty}^{\infty}\sin^2 \xi x\nu(dx) \to \infty \quad \text{as} \quad |\xi| \to \infty.$$

Let $0 \leqq \lambda < 2$, and let c be an integer exceeding $2/(2 - \lambda)$. Let $a_j = 2^{-c^j}$, $j = 1, 2, \cdots$, and let ν be atomic with atoms of weight $a_j^{-\lambda}$ at $x = a_j$, $j = 1, 2, \cdots$. Note that ν satisfies (2) and (C_λ). Let $\xi_n = \pi a_n^{-1}$ and observe

$$\int_{-\infty}^{\infty}\sin^2 \xi_n x\nu(dx) = \sum_{j=1}^{\infty}(\sin^2 \pi\, 2^{(c^n - c^j)})2^{\lambda c^j}$$
$$= \sum_{j=n+1}^{\infty}(\sin^2 \pi\, 2^{(c^n - c^j)})2^{\lambda c^j} \leqq \pi^2 \sum_{j=n+1}^{\infty} 2^{-((2-\lambda)c^j - 2c^n)}$$

and the last term tends to zero with n, so that (3) is violated.

2. Let $H(r) = \int_{|x|<r}|x|^2\nu(dx)$. Consider the condition

(D_β) There exist $c > 0$ and $r_0 > 0$ such that $H(r) > cr^\beta$ for $0 < r < r_0$.

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Equation (1) together with

$$\int_{-\infty}^{\infty} (1 - \cos \xi x)\nu(dx) \geqq \int_{|x|<\xi^{-1}} (1 - \cos \xi x)\nu(dx)$$
$$\geqq \tfrac{1}{4}\xi^2 \int_{|x|<\xi^{-1}} x^2\nu(dx) = \tfrac{1}{4}\xi^2 \cdot H(\xi^{-1})$$

shows immediately that *if* (D_β) *holds for some* $\beta < 2$, $|\varphi(\xi)| \leqq \exp(-c/4|\xi|^{2-\beta})$ *for some* $c > 0$ *and for* $|\xi|$ *sufficiently big and so* F *has derivatives of all orders.*

Let $G(r) = \int_{|x|>r} \nu(dx)$. It was pointed out in [1] that

$$\inf \{\alpha > 0: \int_{-1}^{1} |x|^\alpha \nu(dx) < \infty\} = \inf \{\alpha > 0: r^\alpha G(r) \to 0 \text{ as } r \to 0\}.$$

Thus the conditions (C_α) are intimately related to the behavior of lim sup $G(r)r^\beta$. Since such conditions can not imply the absolute continuity of F it is natural to consider also lim inf $G(r)r^\beta$.

Using (2) it is easily shown (see the proof of Theorem 2.1 in [1]) that

$$r^2G(r) \to 0 \quad \text{as} \quad r \downarrow 0. \tag{4}$$

It will now be shown that *if for some* $r_0 > 0$, $r^{-\alpha} < G(r) < r^{-\beta}$ *then* (D_γ) *holds with* $\gamma = \beta(2 - \alpha)/\alpha$. *Thus continuous derivatives of all orders exist if* $\alpha > 2\beta/(2 + \beta)$, for this makes $\gamma < 2$; *in particular* $\alpha > 1$ *suffices*, for in view of (4) one may always take $\beta = 2$ and then $(2\beta)/(2 + \beta) = 1$.

For proof, let $0 < r < r_0$ and let $G(x) = y^{-\alpha}$. Then

$$H(x) = -\int_0^x r^2\, dG(r) = -x^2G(x) + 2\int_0^y rG(r)\, dr + 2\int_y^x rG(r)\, dr$$
$$\geqq -x^2y^{-\alpha} + 2\int_0^y r^{1-\alpha}\, dr + 2y^{-\alpha}\int_y^x r\, dr = (\alpha)/(2 - \alpha)y^{2-\alpha}.$$

Since $G(x) < x^{-\beta}$, $y = (G(x))^{-1/\alpha} \geqq x^{\beta/\alpha}$ and one obtains

$$H(x) \geqq \alpha(2 - \alpha)^{-1}x^{\beta(2-\alpha)/\alpha}.$$

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