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ON THE PROBABILITY DISTRIBUTION OF A FILTERED RANDOM TELEGRAPH SIGNAL

BY DAVID S. NEWMAN

The Boeing Company, Aerospace Group

1. Introduction. The two-state Markov process known as the random telegraph signal has been applied to a number of problems in communication engineering. Most investigations have been concerned with the spectral density function of the process. Wonham and Fuller [6] treat a special case of the problem treated in this paper; namely, the probability distribution of a filtered random telegraph signal. The analogous problem for a secondary or filtered Poisson process (in the terminology of L. Takacs [5] or E. Parzen [3], respectively) has a closed form solution which apparently was first derived by S. O. Rice [4]. His result can be obtained rigorously from the Lévy-Khintchine theory of infinitely divisible distributions (see Feller [1]). The method used here is more elementary.

2. Statement of the problem. The random telegraph signal is a Markov process $X(t)$ with two values which we shall take to be 0 and 1. The representation of $X(t)$ which is most convenient for our purposes is

$$X(t) = \sum_{k=0}^{\infty} (-1)^k H(t - S_k), \tag{1}$$

where $H(u)$ is the unit step function

$$H(u) = 1 \quad \text{for} \quad u \geqq 0$$
$$= 0 \quad \text{for} \quad u < 0,$$

and $S_k = \sum_{j=1}^{k} T_j$, $S_0 = 0$, and $\{T_j\}$ is a sequence of independent, exponentially distributed random variables, with mean $1/\lambda$ for j odd and $1/\mu$ for j even. The assignment $X(0) = 1$ in equation (1) is purely a matter of convenience.

If the random signal $X(t)$ is passed through a linear filter, the output may be represented in the form

$$Y(t) = \sum_{k=1}^{\infty} (-1)^k A(t - S_k) \tag{2}$$

analogous to equation (1). $A(u)$ is the response of the filter at time u to a unit impulse applied at time zero; various assumptions may be made about its behavior.

The results of this paper are threefold: (a) The moment generating function (mgf)

$$\Phi(\theta, t) = E(\exp \theta Y(t))$$

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is shown to be an entire function of θ for each value of t, and a bound is obtained on $\Phi(\theta, t)$. (b) $\Phi(\theta, t)$ satisfies a linear, second order differential equation in t. (c) $\Phi(\theta) = \lim_{t\to\infty} \Phi(\theta, t)$ exists and is analytic for θ in a neighborhood of zero, and therefore in a strip $|\text{Re}\,\theta| < \theta_0$ for some θ_0. The technique used in the proof of (c) provides a workable though cumbersome method for the explicit calculation of the limit $\Phi(\theta)$ of the generating function. The result (c), Lévy's continuity theorem and analytic continuation together imply that the limit of the distribution of $Y(t)$ exists, and has mgf $\Phi(\theta)$.

3. Analyticity of the moment generating function. Let N_t be the alternating renewal process associated with the sequence of sums $\{S_k\}$—that is, the total number of jumps in the random telegraph signal up to time t. Formally $N_t = \max\{k : S_k \leqq t\}$, which exists with probability 1. Let A_{nt} be the event $\{N_t = n\}$, and let I_{nt} be the indicator function of this event. Define, if it exists,

$$\begin{aligned}(3)\qquad \Phi_k(r, n) &= E[(Y(t))^k I_{nt}] \\ &= E[(Y(t))^k \mid N_t = n] P[N_t = n].\end{aligned}$$

If $K_n(t)$ is the distribution function of the random variable S_n, then

$$P[N_t = n] = K_n(t) - K_{n+1}(t).$$

From (2) it follows that

$$|Y(t)|^k \leqq (\textstyle\sum_{j=1}^n |A(t - S_j)|)^k \leqq (n\bar{A})^k$$

whenever $N_t = n$, if we assume $\bar{A} = \sup_{0 \leqq u < \infty} |A(u)| < \infty$. These results imply that $\Phi_k(t, n)$ exists and

$$|\Phi_k(t, n)| \leqq (n\bar{A})^k (K_n(t) - K_{n+1}(t)).$$

We may now show that the kth moment

$$\Phi_k(t) = E((Y(t))^k)$$

exists, and obtain a bound on it (and on the kth absolute moment, for that matter). Consider the partial sums of the series $\sum_{n=0}^\infty n^k[K_n(t) - K_{n+1}(t)]$. By Abel's summation formula,

$$(4)\quad \textstyle\sum_{n=1}^N n^k(K_n - K_{n+1}) + (N + 1)^k K_{N+1} = \sum_{n=0}^N [(n + 1)^k - n^k] K_{n+1}$$

for all t.

Let $\lambda^* = \max(\lambda, \mu)$. Then the sum of n exponential random variables with mean $1/\lambda^*$ is stochastically smaller than S_n, or in other words,

$$(5)\qquad K_{n+1}(t) \leqq \int_0^{\lambda^* t} (x^n/n!) e^{-x}\, dx.$$

It follows that $\lim_{N\to\infty} (N + 1)^k K_{N+1}(t) = 0$ for each t, so that the two series whose partial sums occur in (4) diverge or converge together, to the same limit.

Consider now the double series

$$\textstyle\sum_{k=1}^\infty \sum_{n=0}^\infty (1/k!)(\bar{A}\theta)^k [(n + 1)^k - n^k] K_{n+1}(t) + 1,$$

with $\theta > 0$. Since the terms are positive, the order of summation is immaterial. But

$$\begin{aligned}\sum_{n=0}^{\infty} K_{n+1}(t) \sum_{k=0}^{\infty} ((\bar{A}\theta)^k/k!)[(n+1)^k - n^k] + 1 &= \sum_{n=0}^{\infty} K_{n+1}(t)[e^{(n+1)\bar{A}\theta} - e^{nA\theta}] + 1 \\ &= (e^{\bar{A}\theta} - 1)\sum_{n=0}^{\infty} (e^{\bar{A}\theta})^n K_{n+1}(t) + 1.\end{aligned}$$

But by the inequality (5) the last series converges, and furthermore

$$\sum_{n=0}^{\infty} (e^{\bar{A}\theta})^n K_{n+1}(t) \leqq \int_0^{\lambda^* t} \sum_0^{\infty} ((xe^{\bar{A}\theta})^n/n!)e^{-x}\, dx$$

and the right side is simply

$$(e^{\bar{A}\theta} - 1)^{-1}\{\exp[\lambda^* t(e^{\bar{A}\theta} - 1)] - 1\}.$$

Consequently the double series sums to $\exp[\lambda^* t(e^{\bar{A}\theta} - 1)]$ for all values of t and θ. Now it follows that $\sum_{n=0}^{\infty} n^k[K_n(t) - K_{n+1}(t)]$ converges uniformly in t on every finite subinterval of $[0, \infty)$. From this fact and the convergence of the double series one has

THEOREM 1. *$Y(t)$ has moments of every order. The moment generating function has a Taylor series expansion*

$$\Phi(\theta, t) = \sum_{k=0}^{\infty} \Phi_k(t)\theta^k/k!$$

converging for every finite $t \in [0, \infty)$ and every complex θ; furthermore

$$|\Phi(\theta, t)| \leqq \sum_{k=0}^{\infty} |\Phi_k(t)|(|\theta|^k/k!) \leqq \exp[\lambda^* t(e^{\bar{A}|\theta|} - 1)]$$

where $\lambda^* = \max(\lambda, \mu)$. *Consequently $\Phi(\theta, t)$ is an entire function of θ for each t.*

4. The differential equation for $\Phi(\theta, t)$. In this section we shall prove

THEOREM 2. *The* mgf *$\Phi(\theta, t)$ is the unique solution to the differential equation*

$$e^{g(t)}(d/dt)(e^{-g(t)}(d/dt)(e^{-\lambda t}\Phi(\theta, t))) - \lambda\mu e^{-\lambda t}\Phi(\theta, t) = 0 \tag{6}$$

with the initial conditions

$$\Phi(\theta, 0) = 1; \qquad \Phi'(\theta, 0) = \lambda(e^{\theta A(0)} - 1), \tag{7}$$

where $g(t) = (\lambda - \mu)t + \theta A(t)$. (The prime in (7) denotes a derivative with respect to t.)

PROOF. Let A_{nt} be the event $N_t = n$ defined in the preceding section and I_{nt} its indicator function. Then by the complete additivity of the integral

$$\Phi(\theta, t) = \sum_{n=0}^{\infty} \Phi(\theta, t, n)$$

where

$$\Phi(\theta, t, n) = E(e^{\theta Y(t)} I_{nt}).$$

Now $\Phi(\theta, t, n)$ may be expressed as an integral over Euclidean n-space R^n, since $Y(t)$ restricted to A_{nt} is a Borel function of the random variables $(S_1, \cdots, S_n)$, or equivalently $(T_1, \cdots, T_n)$.

If we define the differential probability element

$$dP_n = P[s_i < S_i \leqq s_i + ds_i, \quad 1 \leqq i \leqq n, \text{ and } N_t = n],$$

then in terms of the density function $f_j(t)$ of T_j and $F_j(t) = \int_0^t f_j(u)\,du$, we have $dP_n = [\prod_{j=1}^n f_j(s_j - s_{j-1})\,ds_j][1 - F_{n+1}(t - s_n)]$.

Note that $dP_n = 0$ unless $0 \leqq s_j \leqq \cdots \leqq s_n \leqq t$. Then

$$\Phi(\theta, t, n) = \int_{R^n} \prod_{j=1}^n \exp[(-1)^{j+1}\theta A(t - s_j)]\,dP_n .$$

These integrals suggest that a recurrence relation exists among the $\Phi(\theta, t, n)$. The relation becomes clear when the change of variables $y_j = t - s_{n-j+1}$ is made, and the resulting integral has the form

$$\text{(8)} \quad \Phi(\theta, t, n) = e^{-\lambda t} \prod_{i=1}^n \lambda_i \int_{\Omega_{nt}} \exp[(-1)^n \textstyle\sum_{k=1}^n (-1)^k g(y_k)]\,dy_1 \cdots dy_n$$

where $\lambda_i = \lambda$ for i even and μ for i odd, and $g(y) = (\lambda - \mu)y + \theta A(y)$. Ω_{nt} is the point set $0 \leqq y_1 \leqq \cdots \leqq y_n \leqq t$. The recurrence relation is as follows: Define $G_0(t) = H_0(t) = 1$, and

$$\text{(9)} \quad \begin{aligned} G_k(t) &= \int_0^t e^{(-1)^{k-1}g(y)} G_{k-1}(y)\,dy, \\ H_k(t) &= \int_0^t e^{(-1)^k g(y)} H_{k-1}(y)\,dy. \end{aligned}$$

Then $\Phi(\theta, t, n) = \lambda(\lambda\mu)^{(n-1)/2} G_n(t) e^{-\lambda t}$ for n odd, and

$$\Phi(\theta, t, n) = (\lambda\mu)^{n/2} H_n(t) e^{-\lambda t}$$

for n even. The series

$$\text{(10a)} \quad \chi_e(t) = \textstyle\sum_{n=0}^\infty (\lambda\mu)^n H_{2n}(t), \qquad \chi_0(t) = \sum_{n=0}^\infty \lambda(\lambda\mu)^n G_{2n+1}(t);$$

$$\text{(10b)} \quad \chi_h(t) = \textstyle\sum_{n=1}^\infty (\lambda\mu)^n H_{2n-1}(t), \qquad \chi_g(t) = \sum_{n=0}^\infty \lambda(\lambda\mu)^n G_{2n}(t)$$

converge uniformly in t on every finite subinterval of $[0, \infty]$ by their relation to the series $\sum_{n=0}^\infty \Phi(\theta, t, n)$ and $\sum_0^\infty \Phi(-\theta, t, n)$. For the same reason the differential equations

$$\text{(11)} \quad \begin{aligned} \chi_e'(t) &= e^{g(t)} \chi_h(t), \\ \chi_h'(t) &= (\lambda\mu) e^{-g(t)} \chi_e(t), \\ \chi_0'(t) &= e^{g(t)} \chi_g(t), \\ \chi_g'(t) &= (\lambda\mu) e^{-g(t)} \chi_0(t), \end{aligned}$$

with initial conditions $\chi_h(0) = \chi_0(0) = 0$, $\chi_e(0) = 1$, $\chi_g(0) = \lambda$, are an immediate consequence of (9). Since

$$\Phi(\theta, t) = e^{-\lambda t}[\chi_e(t) + \chi_0(t)]$$

it is convenient to reduce (11) to the single second-order differential equation

$$e^{g(t)}(d/dt)[e^{-g(t)}\chi'(t)] - \lambda\mu\chi(t) = 0,$$

with the initial conditions

$$\chi(0) = 1, \qquad \chi'(0) = \lambda e^{g(0)} = \lambda e^{\theta A(0)}.$$

This result is equivalent to (6) and (7), and the theorem is proved.

It is worth noting that there is no need to assume the continuity or differentiability of $g(t)$ [that is, $A(t)$]. The differentiability of $\chi(t)$ and $e^{-g(t)}\chi'(t)$ is a consequence of the uniform convergence of the differentiated series.

5. Existence and analyticity of the limiting characteristic function. Existing results on the asymptotic behavior of differential equations (for example, see Levinson [2]) imply only the existence of the limits $\Phi(\theta) = \lim_{t\to\infty}\Phi(\theta, t)$ when restrictions on $A(t)$ similar to those given below are made. Here it will be shown that the limit is analytic in a strip containing the imaginary θ-axis.

Let $A(t)$ have second derivative on $[0, \infty)$. Then the transformation

$$f(\theta, t) = \exp(\tfrac{1}{2}\theta A(t))\Phi(\theta, t) \tag{12}$$

brings (6) and (7) into the form

$$f''(\theta, t) + (\lambda + \mu)f'(\theta, t) + [\theta q_1(t) + \theta^2 q_2(t)]f(\theta, t) = 0 \tag{13}$$

and

$$\begin{aligned} f(\theta, 0) &= \exp(\tfrac{1}{2}\theta A(0)), \\ f'(\theta, 0) &= \tfrac{1}{2}\theta A'(0)\exp(\tfrac{1}{2}\theta A(0)) - 2\lambda\sinh(\tfrac{1}{2}\theta A(0)). \end{aligned} \tag{14}$$

The primes denote derivatives with respect to t, and

$$\begin{aligned} q_1(t) &= \tfrac{1}{2}A''(t) - \tfrac{1}{2}(\lambda - \mu)A'(t), \\ q_2(t) &= -\tfrac{1}{4}(A'(t))^2. \end{aligned} \tag{15}$$

THEOREM. *If $A(t)$ has a second derivative on $[0, \infty)$, and $q_1(t)$, $q_2(t)$ defined by (15) are absolutely integrable on $[0, \infty)$, and $\lim_{t\to\infty} A(t)$ exists and is finite, then $\Phi(\theta) = \lim_{t\to\infty}\Phi(\theta, t)$ is analytic at least for*

$$|\mathrm{Re}\,\theta| < \theta_0 = [(R_1^2 + 2)/2R_2]^{\frac{1}{2}} - R_1/2R_2 \tag{16}$$

where

$$R_i = (\lambda + \mu)^{-1}\int_0^\infty |q_i(t)|\,dt \qquad (i = 1, 2).$$

PROOF. Let

$$\begin{aligned} M(x) &= (\lambda + \mu)^{-1}[1 - e^{-(\lambda+\mu)x}], & x > 0, \\ &= 0, & x < 0, \end{aligned}$$

and let $M * h(t) = \int_0^t M(t - s)h(s)\,ds$ be the convolution of M with any integrable function h. Then (13) and (14) together are, by elementary methods, equivalent to the integral equation

$$f(\theta, t) = \alpha(\theta)e^{-(\lambda+\mu)t} + \beta(\theta) - M * [\theta q_1 + \theta^2 q_2 f(\theta, \cdot)](t) \tag{17}$$

$\alpha(\theta)$ and $\beta(\theta)$ carry the initial conditions (14) as

$$\alpha(\theta) + \beta(\theta) = f(\theta, 0), \qquad -(\lambda + \mu)\alpha(\theta) = f'(\theta, 0). \tag{18}$$

Since $f(\theta, t)$ is an entire function of θ for each t, it may be expanded in a power series

$$f(\theta, t) = \sum_{k=0}^{\infty} f_k(t)\theta^k$$

which converges for all θ. The coefficients $\alpha(\theta)$ and $\beta(\theta)$ are also analytic and have expansions $\sum \alpha_k\theta^k$, $\sum \beta_k\theta^k$. By repeated differentiation of (17) with respect to θ and setting $\theta = 0$ one obtains

$$f_k(t) = \alpha_k e^{-(\lambda+\mu)t} + \beta_k - M * [q_1 f_{k-1} + q_2 f_{k-2}](t) \tag{19}$$

for all $k \geqq 0$ if we define $f_{-1}(t) = f_{-2}(t) = 0$.

It is easy to show that if $h(t)$ is integrable on $[0, \infty)$,

$$\lim_{t\to\infty} M * h(t) = (\lambda + \mu)^{-1} \int_0^\infty h(t)\, dt.$$

By mathematical induction and (19) it follows that

$$f_k(\infty) = \beta_k - (\lambda + \mu)^{-1} \int_0^\infty [q_1(t) f_{k-1}(t) + q_2(t) f_{k-2}(t)]\, dt. \tag{20}$$

The existence of the integral on the right is assured by the fact that $f_k(t)$ is bounded for $0 \leqq t \leqq \infty$ which we shall now prove, and which also completes the proof that $f(\theta, t)$ is analytic. By induction and (19) we have $|f_k(t)| \leqq g_k$ for all k and $0 \leqq t \leqq \infty$, where g_k satisfies the difference equation

$$g_k = |\alpha_k + \beta_k| + R_1 g_{k-1} + R_2 g_{k-2}, \tag{21}$$

with R_1 and R_2 as in (16). From the boundary conditions (14) and (18) one has

$$|\alpha_k + \beta_k| = |f_k(0)| = (1/k!)|\tfrac{1}{2}A(0)|^k = c^k/k!.$$

The difference equation (14) may be solved formally by introducing the generating function

$$G(\theta) = \sum_{k=0}^{\infty} g_k\theta^k.$$

If

$$G(\theta) = e^{c\theta} + R_1\theta G(\theta) + R_2\theta^2 G(\theta) \tag{22}$$

so that

$$G(\theta) = e^{c\theta}/(1 - R_1\theta - R_2\theta^2), \tag{23}$$

then the Taylor series expansion of $G(\theta)$ converges for $|\theta| < \theta_0$ defined in (16). The coefficients g_k must satisfy (21) by the argument which leads from (17) to (19)—successive differentiation of G with respect to θ at $\theta = 0$.

The conclusion that

$$f(\theta) = \sum_{k=0}^{\infty} f_k(\infty) n^k = \lim_{t\to\infty} f(\theta, t)$$

follows by applying what is, in effect, the dominated convergence theorem. Let $\{t_n\}$ be any sequence of positive real numbers with $t_n \to \infty$. Then $f_k(t_n)\theta^k$ is a summable (i.e., Lebesgue integrable) function of k for each n, since $|f_k(t_n)\theta^k| \leqq$

$g_k |\theta|^k$ and $g_k |\theta|^k$ is summable provided $|\theta| < \theta_0$. Since $f_k(t_n)\theta^k \to f_k(\infty)\theta^k$ for all k (that is, "almost everywhere" in k), the dominated convergence theorem yields

$$\sum_{k=0}^{\infty} |f_k(t_n)\theta^k - f_k(\infty)\theta^k| \to 0$$

as $n \to \infty$, or in other words,

$$f(\theta, t_n) \to \sum_{k=0}^{\infty} f_k(\infty)\theta^k = f(\theta).$$

Furthermore the convergence is uniform in θ on any closed subset of $|\theta| < \theta_0$, so by a classical theorem from analytic function theory $f(\theta)$ is analytic for $|\theta| < \theta_0$. The same is also true of $\Phi(\theta)$. But since $\Phi(i\theta)$ is a characteristic function it follows that $\Phi(\theta)$ is analytic at least for $|\text{Re}\cdot\theta| < \theta_0$.

6. Remark. The sequence of equations (19) and the limiting form (20) provide a technique for the explicit calculation of the limiting distribution—at least in theory. This technique may be usable for some forms of $A(t)$ which are of interest in engineering applications.

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