

ON THE MOMENT GENERATING FUNCTION OF PILLAI'S $V^{(s)}$ CRITERION¹

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1. Introduction and summary. The moment generating function (mgf) of Pillai's $V^{(s)}$ criterion in the central case has been considered by Pillai [8], [9] and James [3]. In the present paper, the mgf's of $V^{(s)}$ are considered in the following non-central cases: (i) MANOVA and (ii) canonical correlation. The lower order moments of $V^{(s)}$ for (i) and (ii) in the non-central case were obtained earlier by Pillai [10] for $s = 2$ in the linear case i.e. when there is only one non-zero population root. These results for (i) were extended by Khatri and Pillai [5], [6], [7] to general s in the linear case and in the planar case i.e. when there are two non-zero population roots. Only the first two moments were considered in the planar case while the first four moments were obtained in the linear case. The results of this paper further facilitates the derivation of the general moments.

2. Preliminaries. The following lemmas will be used in the sequel for the derivation of the mgf's:

LEMMA 1. Let $\mathbf{S}(p \times p)$ be a positive definite (pd) symmetric matrix, $\mathbf{T}(p \times p)$, a complex matrix whose real part is pd symmetric, and $\mathbf{U}(p \times p)$, a symmetric matrix, then

$$(2.1) \quad \int_{R(\mathbf{T}) > 0} e^{\text{tr} \mathbf{T}} |\mathbf{T}|^{-b} {}_qF_r(a_1, \dots, a_q; b_1, \dots, b_r; \mathbf{T}^{-1}\mathbf{S}, \mathbf{U}) d\mathbf{T} \\ = {}_qF_{r+1}(a_1, \dots, a_q; b_1, \dots, b_r, b; \mathbf{S}, \mathbf{U}), \quad (\text{James [3], Constantine [1]}),$$

where the hypergeometric function of matrix argument is defined by James [3] as

$$(2.2) \quad {}_qF_r(a_1, \dots, a_q; b_1, \dots, b_r; \mathbf{S}, \mathbf{T}) \\ = \sum_{k=0}^{\infty} \sum_{\kappa} (a_1)_{\kappa} \cdots (a_q)_{\kappa} C_{\kappa}(\mathbf{S}) C_{\kappa}(\mathbf{T}) [(b_1)_{\kappa} \cdots (b_r)_{\kappa} C_{\kappa}(I_p) k!]^{-1},$$

where $a_1, \dots, a_q, b_1, \dots, b_r$ are real or complex constants and the multivariate coefficient (a_{κ}) is given by

$$(a)_{\kappa} = \prod_{i=1}^p (a - \frac{1}{2}(i-1))_{\kappa_i},$$

where $(a)_k = a(a+1) \cdots (a+k-1)$. The partition κ of k is such that $\kappa = (k_1, k_2, \dots, k_p)$ ($k_1 \geq k_2 \geq \dots \geq k_p \geq 0$), $k_1 + \dots + k_p = k$, and the zonal polynomials, $C_{\kappa}(\mathbf{S})$, are expressible in terms of elementary symmetric functions of the latent roots of \mathbf{S} [3].

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LEMMA 2. If \mathbf{S} , \mathbf{T} and \mathbf{U} are as in Lemma 1, then

$$(2.3) \quad (\Gamma_p(a))^{-1} \int_{\mathbf{S}>0} e^{-\text{tr}\mathbf{S}} |\mathbf{S}|^{a-\frac{1}{2}(p+1)} {}_qF_r(a_1, \dots, a_q; b_1, \dots, b_r; \mathbf{ST}, \mathbf{U}) d\mathbf{S} \\ = {}_{q+1}F_r(a_1, \dots, a_q, a; b_1, \dots, b_r; \mathbf{T}, \mathbf{U}),$$

(James [3], Constantine [1]).

LEMMA 3. If $\mathbf{S}(p \times p)$ is pd, then

$$(2.4) \quad {}_1F_1(a, b; \mathbf{S}) = \Gamma_p(b)[\Gamma_p(a)\Gamma_p(b-a)]^{-1} \\ \cdot \int_0^1 e^{\text{tr}\mathbf{ST}} |\mathbf{T}|^{a-\frac{1}{2}(p+1)} |\mathbf{I} - \mathbf{T}|^{b-a-\frac{1}{2}(p+1)} d\mathbf{T}, \quad (\text{James [3]}).$$

LEMMA 4. Let \mathbf{Z} be a complex symmetric matrix whose real part is pd, and let \mathbf{T} be an arbitrary complex symmetric matrix. Then

$$(2.5) \quad \int_{\mathbf{S}>0} e^{-\text{tr}\mathbf{ZS}} |\mathbf{S}|^{t-\frac{1}{2}(p+1)} C_\kappa(\mathbf{ST}) d\mathbf{S} \\ = (t)\kappa\Gamma_p(t)|\mathbf{Z}|^{-t} C_\kappa(\mathbf{TZ}^{-1}), \quad (\text{Constantine [1]})$$

where $R(t) > \frac{1}{2}(p-1)$.

It is easy to see that Lemma 2 follows from Lemma 4. A lemma similar to Lemma 4 has been used to prove Lemma 1 which can be referred to [1]. Also note that ${}_0F_0(\mathbf{S}) = e^{\text{tr}\mathbf{S}}$ and

$$(2.6) \quad C_\kappa(\mathbf{I} + \mathbf{A})/C_\kappa(\mathbf{I}) = \sum_{n=0}^k \sum_\eta a_{\kappa,\eta} C_\eta(\mathbf{A})/C_\eta(\mathbf{I}),$$

where $a_{\kappa,\eta}$ are constants (Constantine [2]).

3. Mgf of $V^{(s)}$ for MANOVA. Let \mathbf{X} be a $p \times f_2$ matrix variate ($p \leq f_2$) and \mathbf{Y} a $p \times f_1$ matrix variate ($p \leq f_1$) and let the columns be all independently normally distributed with covariance matrix $\mathbf{\Sigma}$, $E(\mathbf{X}) = \mathbf{M}$ and $E(\mathbf{Y}) = \mathbf{0}$. Let l_1, \dots, l_p be the characteristic roots of

$$|\mathbf{XX}' - l(\mathbf{YY}' + \mathbf{XX}')| = 0,$$

and $\omega_1, \dots, \omega_p$ those of $|\mathbf{MM}' - \omega\mathbf{\Sigma}| = 0$, then the joint density function of l_1, \dots, l_p is given by Constantine [1] and by James [3] in the form

$$(3.1) \quad e^{-\frac{1}{2}\text{tr}\mathbf{\Omega}} C(p, f_1, f_2) {}_1F_1(\frac{1}{2}\nu; \frac{1}{2}f_2; \frac{1}{2}\mathbf{\Omega}, \mathbf{L}) |\mathbf{L}|^{\frac{1}{2}(f_2-p-1)} |\mathbf{I} - \mathbf{L}|^{\frac{1}{2}(f_1-p-1)} \\ \cdot \prod_{i>j} (l_i - l_j), \quad 0 < l_1 \leq \dots \leq l_p < 1,$$

where $\mathbf{L} = \mathbf{X}'(\mathbf{YY}' + \mathbf{XX}')^{-1}\mathbf{X}$, $\mathbf{\Omega} = \mathbf{M}'\mathbf{\Sigma}^{-1}\mathbf{M}$, $\nu = f_1 + f_2$,

$$(3.2) \quad C(p, f_1, f_2) = \{\pi^{\frac{1}{2}p^2} \Gamma_p(\frac{1}{2}\nu)\} / \{\Gamma_p(\frac{1}{2}f_1)\Gamma_p(\frac{1}{2}f_2)\Gamma_p(\frac{1}{2}p)\},$$

and the determinants in (3.1) are expressed as products of the characteristic roots of their respective matrices. In the context of (i), $f_2 = l - 1$ and $f_1 = N - l$, N being the pooled sample size of the samples from l populations. Here $V^{(s)} = V^{(p)} = \sum_{i=1}^p l_i$. Now, by an application of Lemmas 1 and 2, we get

$$(3.3) \quad E(e^{\text{tr}\mathbf{L}}) = e^{-\frac{1}{2}\text{tr}\mathbf{\Omega}} 2^{\frac{1}{2}p(p-1)} [\Gamma_p(\frac{1}{2}f_1)(2\pi_i)^{\frac{1}{2}p(p+1)}]^{-1} \int_{\mathbf{S}>0} e^{-\text{tr}\mathbf{S}} |\mathbf{S}|^{\frac{1}{2}(p-1)} \\ \cdot \int_{\mathbf{R}(\mathbf{T})>0} e^{\text{tr}\mathbf{T}} |\mathbf{T}|^{-\frac{1}{2}f_2} \int_{\mathbf{L}>0} e^{\text{tr}(\mathbf{I}t + \mathbf{ST}^{-1}\frac{1}{2}\mathbf{\Omega})\mathbf{L}} |\mathbf{L}|^{\frac{1}{2}(f_2-p-1)} \\ \cdot |\mathbf{I} - \mathbf{L}|^{\frac{1}{2}(f_1-p-1)} d\mathbf{L} d\mathbf{T} d\mathbf{S}.$$

Further, using Lemma 3 and integrating with respect to \mathbf{L} ,

$$(3.4) \quad E(e^{t\mathbf{r}\mathbf{L}}) = e^{-\frac{1}{2}t\mathbf{r}\mathbf{\Omega}} \Gamma_p(\frac{1}{2}f_2) 2^{\frac{1}{2}p(p-1)} [\Gamma_p(\frac{1}{2}\nu) (2\pi i)^{\frac{1}{2}p(p+1)}]^{-1} \int_{\mathbf{R}(\mathbf{T}) > 0} e^{t\mathbf{r}\mathbf{T}} |\mathbf{T}|^{-\frac{1}{2}f_2} \cdot \int_{\mathbf{S} > 0} e^{-t\mathbf{r}\mathbf{S}} |\mathbf{S}|^{\frac{1}{2}(\nu-p-1)} {}_1F_1(\frac{1}{2}f_2; \frac{1}{2}\nu; \mathbf{I}t + \mathbf{S}\mathbf{T}^{-1}\frac{1}{2}\mathbf{\Omega}) d\mathbf{S} d\mathbf{T}.$$

Again, with the help of (2.6), Lemma 4 and a similar lemma, we get

$$(3.5) \quad E(e^{t\mathbf{r}\mathbf{L}}) = e^{-\frac{1}{2}t\mathbf{r}\mathbf{\Omega}} \sum_{k=0}^{\infty} \sum_{\kappa} \sum_{n=0}^k \sum_{\eta} (\frac{1}{2}f_2)_{\kappa} (\frac{1}{2}\nu)_{\eta} a_{\kappa, \eta} t^{k-n} C_{\kappa}(\mathbf{I}) C_{\eta}(\frac{1}{2}\mathbf{\Omega}) \cdot [(\frac{1}{2}\nu)_{\kappa} (\frac{1}{2}f_2)_{\eta} k! C_{\eta}(\mathbf{I})]^{-1}.$$

4. Mgf of $V^{(s)}$ for canonical correlation. Let the columns of $(\mathbf{X}_1 \mathbf{X}_2)'$ be ν independent normal $(p + q)$ -variates, $(p \leq q, p + q \leq \nu, \nu + 1 = n$, the sample size) with zero means and covariance matrix

$$\mathbf{\Sigma} = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma'_{12} & \Sigma_{22} \end{pmatrix}.$$

Let $\mathbf{R} = \text{diag}(r_i)$, where r_1^2, \dots, r_p^2 , are the characteristic roots of the equation

$$|\mathbf{X}_1 \mathbf{X}_2' (\mathbf{X}_2 \mathbf{X}_2')^{-1} \mathbf{X}_2 \mathbf{X}_1' - r^2 \mathbf{X}_1 \mathbf{X}_1'| = 0$$

and $\mathbf{P}^2 = \text{diag}(\rho_1^2, \dots, \rho_p^2)$ are characteristic roots of the equation

$$|\mathbf{\Sigma}_{12} \mathbf{\Sigma}_{22}^{-1} \mathbf{\Sigma}'_{12} - \rho^2 \mathbf{\Sigma}_{11}| = 0.$$

Then, the distribution of r_1^2, \dots, r_p^2 is given by Constantine [1], in the following form [3]

$$(4.1) \quad |\mathbf{I} - \mathbf{P}^2|^{\frac{1}{2}\nu} C(p, f_1, f_2) {}_2F_1(\frac{1}{2}\nu, \frac{1}{2}\nu; \frac{1}{2}f_2; \mathbf{P}^2, \mathbf{R}^2) |\mathbf{R}^2|^{\frac{1}{2}(f_2-p-1)} |I - R^2|^{\frac{1}{2}(f_1-p-1)} \cdot \prod_{i>j} (r_i^2 - r_j^2) \prod_{i=1}^p dr_i^2, \quad 0 < r_1^2 \leq \dots \leq r_p^2 < 1,$$

where $f_2 = q$ and $f_1 = \nu - q$. Again, in this case $V^{(s)} = V^{(p)} = \sum_{i=1}^p r_i^2$.

Now using Lemma 1 once and Lemma 2 twice and integrating with respect to \mathbf{R}^2 with the help of Lemma 3 we get the mgf of $V^{(p)} = \text{tr} \mathbf{R}^2$ in the form

$$(4.2) \quad E(e^{t\mathbf{r}\mathbf{R}^2}) = 2^{\frac{1}{2}p(p-1)} \Gamma_p(\frac{1}{2}f_2) |\mathbf{I} - \mathbf{P}^2|^{\frac{1}{2}\nu} [\Gamma_p(\frac{1}{2}\nu) (2\pi i)^{\frac{1}{2}p(p+1)}]^{-1} \cdot \int_{\mathbf{R}(\mathbf{T}) > 0} e^{t\mathbf{r}\mathbf{T}} |\mathbf{T}|^{-\frac{1}{2}f_2} \int_{\mathbf{S}_2 > 0} e^{-t\mathbf{r}\mathbf{S}_2} |\mathbf{S}_2|^{\frac{1}{2}(\nu-p-1)} \cdot \int_{\mathbf{S}_1 > 0} e^{-t\mathbf{r}\mathbf{S}_1} |\mathbf{S}_1|^{\frac{1}{2}(\nu-p-1)} {}_1F_1(\frac{1}{2}f_2; \frac{1}{2}\nu; \mathbf{I}t + \mathbf{S}_1 \mathbf{S}_2 \mathbf{T}^{-1} \mathbf{P}^2) d\mathbf{S}_1 d\mathbf{S}_2 d\mathbf{T}.$$

Again, using (2.6), Lemma 4 and a similar lemma we have

$$(4.3) \quad E(e^{t\mathbf{r}\mathbf{R}^2}) = |I - \mathbf{P}^2|^{\nu/2} \sum_{k=0}^{\infty} \sum_{\kappa} \sum_{n=0}^k \sum_{\eta} (\frac{1}{2}f_2)_{\kappa} ((\frac{1}{2}\nu)_{\eta})^2 \cdot a_{\kappa, \eta} t^{k-n} C_{\kappa}(\mathbf{I}) C_{\eta}(\mathbf{P}^2) [(\frac{1}{2}\nu)_{\kappa} (\frac{1}{2}f_2)_{\eta} k! C_{\eta}(\mathbf{I})]^{-1}.$$

5. Remarks. Khatri [4] has obtained the non-central mgf of $V^{(p)}$ associated with the test $\lambda \mathbf{\Sigma}_1 = \mathbf{\Sigma}_2$, $\lambda > 0$, where $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$ are the covariance matrices of two p -variate normal populations. However, a factor $(\frac{1}{2}\nu)_{\eta}$ has been omitted in the expression for the mgf and hence the correct expression is given by

$$E(e^{tV^{(p)}}) = |\lambda \mathbf{\Lambda}|^{-\frac{1}{2}f_1} \sum_{k=0}^{\infty} \sum_{\kappa} \sum_{n=0}^k \sum_{\eta} (\frac{1}{2}f_1)_{\kappa} (\frac{1}{2}\nu)_{\eta} a_{\kappa, \eta} t^{k-n} C_{\kappa}(\mathbf{I}) C_{\eta}(\mathbf{I} - (\lambda \mathbf{\Lambda})^{-1}) \cdot [(\frac{1}{2}\nu)_{\kappa} k! C_{\eta}(\mathbf{I})]^{-1},$$

where $\Lambda = \text{diag ch } \Sigma_1 \Sigma_2^{-1}$, $f_1 = n_1 - 1$ and $f_2 = n_2 - 1$, where n_1 and n_2 are the respective sizes of the samples from each of the two populations.

Further it should be pointed out that the moments of $V^{(p)}$ obtained by Pillai [10] for (i) and (ii) for $p = 2$ in the linear case and those by Khatri and Pillai [5], [6], [7] for (i) for general linear and planar cases were verified to follow from (3.5) and (4.3) to the extent $a_{\kappa, \eta}$ coefficients are available in Constantine [2] and further tabulations carried out by us.

REFERENCES

- [1] CONSTANTINE, A. G. (1963). Some non-central distribution problems in multivariate analysis. *Ann. Math. Statist.* **34** 1270-1285.
- [2] CONSTANTINE, A. G. (1966). The distribution of Hotelling's generalized T_v^2 . *Ann. Math. Statist.* **37** 215-225.
- [3] JAMES, A. T. (1964). Distributions of matrix variates and latent roots derived from normal samples. *Ann. Math. Statist.* **35** 475-501.
- [4] KHATRI, C. G. (1967). Some distribution problems connected with the characteristic roots of $S_1 S_2^{-1}$. *Ann. Math. Statist.* **38** 944-948.
- [5] KHATRI, C. G. and PILLAI, K. C. S. (1965). Some results on the non-central multivariate beta distribution and moments of traces of two matrices. *Ann. Math. Statist.* **36** 1511-1520.
- [6] KHATRI, C. G. and PILLAI, K. C. S. (1967). On the moments of traces of two matrices in multivariate analysis. *Ann. Inst. Statistical Math.* **19** 143-156.
- [7] KHATRI, C. G. and PILLAI, K. C. S. (1968). On the non-central distributions of two test criteria in multivariate analysis of variance. *Ann. Math. Statist.* **39** 215-226.
- [8] PILLAI, K. C. S. (1954). On some distribution problems in multivariate analysis. Mimeograph Series No. 88, Institute of Statistics, Univ. North Carolina, Chapel Hill.
- [9] PILLAI, K. C. S. (1956). Some results useful in multivariate analysis. *Ann. Math. Statist.* **27** 1106-1114.
- [10] PILLAI, K. C. S. (1966). Non-central multivariate beta distribution and the moments of traces of some matrices. *Multivariate Analysis*. Academic Press, New York.