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A SYSTEM OF LINEAR DIFFERENTIAL EQUATIONS FOR THE DISTRIBUTION OF HOTELLING'S GENERALIZED T_0^2

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1. Introduction and summary. Let $\mathcal{S}_1$, $\mathcal{S}_2$ be independent $m \times m$ matrices on n_1, n_2 degrees of freedom respectively, $\mathcal{S}_2$ having a Wishart distribution and $\mathcal{S}_1$ having a possibly non-central Wishart distribution with the same covariance matrix. Hotelling's generalized T_0^2 statistic is then defined [7] by

$$T = n_2^{-1} T_0^2 = \operatorname{tr} \mathcal{S}_1 \mathcal{S}_2^{-1}. \tag{1.1}$$

The complete distribution of this statistic is known only in particular cases. If $m = 1$, then $(n_2/n_1)T$ is simply non-central F. In the case $n_1 = 1$, T reduces to Hotelling's generalization of "Student's" t, which also has a non-central F distribution. When $m = 2$, Hotelling [7] has shown that in the null case the density function of T is

$$f(T) = [\Gamma(n_1 + n_2 - 1)/\Gamma(n_1)\Gamma(n_2 - 1)](\tfrac{1}{2}T)^{n_1-1}(1 + \tfrac{1}{2}T)^{-(n_1+n_2)} \cdot {}_2F_1(1, \tfrac{1}{2}(n_1 + n_2); \tfrac{1}{2}(n_2 + 1); v), \tag{1.2}$$

where $v = T^2/(T + 2)^2$, and ${}_2F_1$ is the Gaussian hypergeometric function.

When n_2 becomes large, the distribution of T_0^2 approaches that of χ^2 based on mn_1 degrees of freedom. Ito [9] has derived asymptotic expansions both for the cumulative distribution function (cdf) of T_0^2, and for the percentiles of T_0^2 in terms of the corresponding $\chi^2_{mn_1}$ percentiles.

Other approximations to the distribution requiring large n_2 for validity have been obtained by Pillai and Samson [12]. These authors have used the method of fitting a Pearson curve by means of moment quotients to tabulate upper 5% and 1% points for $m = 2, 3, 4$.

The exact distribution of T over the range $0 \leqq T < 1$ has been obtained in the general non-central case by Constantine [3], using the methods of zonal polynomials and hypergeometric functions of matrix argument developed by James and Constantine ([2] and [10], for example). Constantine's solution has the form

$$f(T) = [\Gamma_m(\tfrac{1}{2}(n_1 + n_2))/\Gamma(\tfrac{1}{2}mn_1)\Gamma_m(\tfrac{1}{2}n_2)]T^{\frac{1}{2}mn_1-1}\mathcal{P}(T), \tag{1.3}$$

where $\mathcal{P}(T)$ is a power series in T convergent in the unit circle, and

$$\Gamma_m(z) = \pi^{\frac{1}{4}m(m-1)} \prod_{i=0}^{m-1} \Gamma(z - \tfrac{1}{2}i). \tag{1.4}$$

In Section 2 of the present paper, it is shown that in the null case the density function $f(T)$ (or rather, its analytic continuation into the complex T-plane) satisfies an ordinary linear differential equation of degree m of Fuchsian type, having regular singularities at $T = 0, -1, \cdots, -m$ and infinity. More specifi-

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cally, an equivalent first-order system is obtained, and the problem is most conveniently treated in this form. Constantine's series (1.3) in the null case is shown in Section 3 to be the relevant solution for $f(T)$ in the neighbourhood of the regular singularity at $T = 0$. The differential equations lead to convenient recurrence relations for the coefficients in $\mathcal{P}(T)$. In Section 4 an alternative derivation of Ito's asymptotic formula is presented. Preliminary results are then given (Section 5) for the regular singularity at $T = \infty$, and a heuristic treatment of the limiting distribution as $n_1 \to \infty$ is presented in Section 6. Finally, it is shown in Section 7 that the moments of T may be obtained from the differential equations for the Laplace transform of $f(T)$ given in Section 1.

One objective in deriving the differential equations for $f(T)$ has been to obtain a convenient exact method for computing the distribution and its percentiles. This work is in progress, and it is hoped that results will be available shortly.

2. The system of linear differential equations. Let $w_1, \cdots, w_m$ denote the latent roots of $\mathcal{S}_1\mathcal{S}_2^{-1}$; then from (1.1)

$$T = \textstyle\sum_{i=1}^{m} w_i. \tag{2.1}$$

Assuming that $\mathcal{S}_1$ has the central Wishart distribution, the joint density function of the w_i when $n_1, n_2 \geqq m$ is

$$\begin{aligned} \phi_{m;n_1,n_2}(\mathbf{w}) &= [\pi^{\frac{1}{2}m^2}\Gamma_m(\tfrac{1}{2}(n_1+n_2))/\Gamma_m(\tfrac{1}{2}n_1)\Gamma_m(\tfrac{1}{2}n_2)\Gamma_m(\tfrac{1}{2}m)] \\ &\quad \cdot(\textstyle\prod_{i=1}^{m} w_i)^{\frac{1}{2}(n_1-m-1)}\prod_{i=1}^{m}(1+w_i)^{-\frac{1}{2}(n_1+n_2)}\prod_{i<j}^{m}(w_i-w_j), \\ &\quad (0 < w_m < \cdots < w_1 < \infty). \end{aligned} \tag{2.2}$$

(See [5], [8], [13]). The case of singular $\mathcal{S}_1$, $n_1 < m$, does not require separate treatment, since the distribution of T then has a simple relation to the case $n_1 \geqq m$ ([3] Section 4). The following proof holds strictly for $m \geqq 2$.

Throughout this section, the suffixes on ϕ will be omitted for convenience. The Laplace transform (Lt) $L_0(s)$ of $f(T)$ may then be written in the form

$$L_0(s) = \textstyle\int_{\mathfrak{D}_m} e^{-s\Sigma w_i}\,\phi(\mathbf{w})\,d\mathbf{w}, \qquad (s \geqq 0), \tag{2.3}$$

where $\mathfrak{D}_m$ denotes the region $\{0 < \omega_m < \cdots < w_1 < \infty\}$. In general, it may be seen that the functional

$$\mathcal{L}(\psi) = \textstyle\int_{\mathfrak{D}_m} e^{-s\Sigma w_i}\,\psi(\mathbf{w})\,d\mathbf{w} \tag{2.4}$$

is the ordinary Lt of the following function of a single variable:

$$\Psi(T) = \textstyle\int_{\mathfrak{D}_{m-1}(T)}\psi(T - w_2 - \cdots - w_m, w_2, \cdots, w_m)\,dw_2\cdots dw_m, \tag{2.5}$$

where

$$\begin{aligned} \mathfrak{D}_{m-1}(T) &= \mathfrak{D}_{m-1} \cap \{2w_2 + w_3 + \cdots + w_m < T\} \\ &= \{0 < w_m < m^{-1}T;\, w_m < w_{m-1} < (m-1)^{-1}(T - w_m);\, \cdots; \\ &\quad w_3 < w_2 < \tfrac{1}{2}(T - w_3 - \cdots - w_m)\}. \end{aligned} \tag{2.6}$$

Taking $\psi = \phi$ in (2.5), we obtain an integral form of $f(T)$.

Thus $L_0(s) = \mathfrak{L}(\phi)$, and the following Lt's will also be introduced:

$$(2.7)\quad L_r(s) = \mathfrak{L}\{\phi(\mathbf{w}) \textstyle\sum_{k_1<k_2<\cdots<k_r}^m [(1 + w_{k_1}) \cdots (1 + w_{k_r})]^{-1}\}, \quad (r = 1, 2, \cdots, m).$$

The summation in curly brackets is extended over the $\binom{m}{r}$ selections of r distinct roots, and is a symmetric function of the w_i. Clearly, the $L_r(s)$ exist for all $s \geqq 0$. Differentiating under the sign of integration:

$$(2.8)\quad -L_r'(s) = \mathfrak{L}\{\phi(\mathbf{w}) \textstyle\sum_{k_1<\cdots<k_r}^m [(1 + w_{k_1}) \cdots (1 + \omega_{k_r})]^{-1} \cdot[(1 + w_{k_1}) + \cdots + (1 + w_{k_1})^{\cdot} - r + (w_{l_1} + \cdots + w_{l_{m-r}})]\},$$

where $(l_1, \cdots, l_{m-r})$ is the set of suffixes complementary to $(k_1, \cdots, k_r)$. Hence, writing

$$(2.9)\quad \Phi_r(s) = \mathfrak{L}\{\phi(\mathbf{w}) \textstyle\sum_{k_1<\cdots<k_r}^m (w_{l_1} + \cdots + w_{l_{m-r}}) \cdot[(1 + w_{k_1}) \cdots (1 + w_{k_r})]^{-1}\},$$

it is seen that

$$(2.10)\quad -L_r'(s) = (m - r + 1)L_{r-1}(s) - rL_r(s) + \Phi_r(s), \quad (r = 1, 2, \cdots, m).$$

Now let l denote any suffix distinct from each of $k_1, \cdots, k_r$. It follows by integration by parts that

$$\begin{aligned} & s\mathfrak{L}\{\phi(\mathbf{w})w_l[(1 + w_{k_1}) \cdots (1 + w_{k_r})]^{-1}\} \\ &= -\textstyle\int_{\mathfrak{D}_m} ((\partial/\partial w_l)e^{-s\Sigma w_i})w_l\phi(\mathbf{w})/(1 + w_{k_1}) \cdots (1 + w_{k_r})\, d\mathbf{w} \\ (2.11)\quad &= \mathfrak{L}\{\phi(\mathbf{w})[(1 + w_{k_1}) \cdots (1 + w_{k_r})]^{-1}[-\tfrac{1}{2}(n_2 + m - 1) \\ &\quad + \tfrac{1}{2}(n_1 + n_2)(1 + w_l)^{-1} + (\textstyle\prod_{i<j}^m (w_i - w_j))^{-1}w_l(\partial/\partial w_l) \\ &\quad \cdot\textstyle\prod_{i<j}^m (w_i - w_j)]\}. \end{aligned}$$

Since

$$(2.12)\quad (\textstyle\prod_{i<j}^m (w_i - w_j))^{-1}(\partial/\partial w_l) \prod_{i<j}^m (w_i - w_j) = \sum_{i=1, i\neq l}^m (w_l - w_i)^{-1},$$

it is necessary to consider the following summation in connection with (2.9):

$$(2.13)\quad \textstyle\sum_{k_1<\cdots<k_r}^m [(1 + w_{k_1}) \cdots (1 + w_{k_r}')]^{-1} \cdot\{w_{l_1} \sum_{i=1, i\neq l_1}^m (w_{l_1} - w_i)^{-1} + \cdots + w_{l_{m-r}} \sum_{i=1, i\neq l_{m-r}}^m (w_{l_{m-r}} - w_i)^{-1}\}.$$

The coefficient of $(w_i - w_j)^{-1}$, $(i < j)$, is seen to be

$$w_i \textstyle\sum_{k_1<\cdots<k_r (\text{all } k_n\neq i)}^m [(1 + w_{k_1}) \cdots (1 + w_{k_r})]^{-1} - w_j \sum_{k_1<\cdots<k_r (\text{all } k_n\neq j)}^m [(1 + w_{k_1}) \cdots (1 + w_{k_r})]^{-1}$$

$$
\begin{aligned}
&= w_i\{\textstyle\sum_{k_1<\cdots<k_r(\text{all } k_n\neq i,j)}^m [(1+w_{k_1})\cdots(1+w_{k_r})]^{-1} \\
(2.14)\quad &\quad + (1+w_j)^{-1}\textstyle\sum_{k_1<\cdots<k_{r-1}(\text{all } k_n\neq i,j)}^m ([1+w_{k_1})\cdots(1+w_{k_{r-1}})]^{-1}\} \\
&\quad - (\text{similar term with } i, j \text{ interchanged}) \\
&= (w_i - w_j)\textstyle\sum_{k_1<\cdots<k_r(\text{all } k_n\neq i,j)}^m [(1+w_{k_1})\cdots(1+w_{k_r})]^{-1} \\
&\quad + (w_i - w_j)(1+w_i+w_j)[(1+w_i)(1+w_j)]^{-1} \\
&\qquad \cdot\textstyle\sum_{k_1<\cdots<k_{r-1}(\text{all } k_n\neq i,j)}^m [(1+w_{k_1})\cdots(1+w_{k_{r-1}})]^{-1}.
\end{aligned}
$$

The summation (2.13) may therefore be written as

$$
\begin{aligned}
&\textstyle\sum_{i<j}^m\sum_{k_1<\cdots<k_n(\text{all } k_n\neq i,j)}^m [(1+w_{k_1})\cdots(1+w_{k_r})]^{-1} \\
&\quad + \textstyle\sum_{i<j}^m\{(1+w_i)^{-1}+(1+w_j)^{-1}\} \\
(2.15)\quad &\qquad \cdot\textstyle\sum_{k_1<\cdots<k_{r-1}(\text{all } k_n\neq i,j)}^m [(1+w_{k_1})\cdots(1+w_{k_{r-1}})]^{-1} \\
&\quad - \textstyle\sum_{i<j}^m [(1+w_i)(1+w_j)]^{-1} \\
&\qquad \cdot\textstyle\sum_{k_1<\cdots<k_{r-1}(\text{all } k_n\neq i,j)}^m [(1+w_{k_1})\cdots(1+w_{k_{r-1}})]^{-1}.
\end{aligned}
$$

Since the second term in (2.15) is

$$
\begin{aligned}
(2.16)\quad &\textstyle\sum_{i\neq j}^m (1+w_i)^{-1}\sum_{k_1<\cdots<k_{r-1}(\text{all } k_n\neq i,j)}^m [(1+w_{k_1})\cdots(1+w_{k_{r-1}})]^{-1} \\
&= r(m-r)\textstyle\sum_{k_1<\cdots<k_r}^m [(1+w_{k_1})\cdots(1+w_{k_r})]^{-1},
\end{aligned}
$$

it follows that (2.15) reduces to

$$
\begin{aligned}
(2.17)\quad &[\tbinom{m-r}{2} + r(m-r)]\textstyle\sum_{k_1<\cdots<k_r}^m [(1+w_{k_1})\cdots(1+w_{k_r})]^{-1} \\
&\quad - \tbinom{r+1}{2}\textstyle\sum_{k_1<\cdots<k_{r+1}}^m [(1+w_{k_1})\cdots(1+w_{k_{r+1}})]^{-1}.
\end{aligned}
$$

Hence, from (2.11) and (2.17):

$$
(2.18)\quad s\Phi_r(s) = -\tfrac{1}{2}(m-r)(n_2-r)L_r(s) + \tfrac{1}{2}(r+1)(n_1+n_2-r)L_{r+1}(s).
$$

Substituting in (2.10) we obtain:

$$
\begin{aligned}
(2.19)\quad &(m-r+1)sL_{r-1} + [s((d/ds)-r) - \tfrac{1}{2}(m-r)(n_2-r)]L_r \\
&\quad + \tfrac{1}{2}(r+1)(n_1+n_2-r)L_{r+1} = 0, \quad (r = 1, 2, \cdots, m-1).
\end{aligned}
$$

The same result holds for $r = 0$ and m if L_{-1} and L_{m+1} are defined to be identically zero. If $r = m$, a factor s may be cancelled, yielding

$$
(2.20)\quad ((d/ds) - m)L_m + L_{m-1} = 0.
$$

It now remains to invert the Laplace transforms. As seen earlier, $L_r(s)$ is the Lt of a certain function of a single variable which will be denoted by $H_r(T)$, $(H_{-1} \equiv 0, H_0 = f)$. In virtue of (2.5) and (2.7), it is seen that $H_r(T)$ is dominated for all $T \geqq 0$ by a constant multiple of $f(T)$. Constantine's result (1.3) shows that $f(T) = O(T^{\frac{1}{2}mn_1-1})$ as $T \to 0+$. (This may also be obtained by taking $\psi = \phi$ in (2.5)). Hence $sL_r(s)$ is the Lt of $H_r'(T)$, and since also $L_r'(s)$ is the Lt of

$-TH_r(T)$, the required system of first order differential equations is obtained:

$$(2.21)\quad -(m - r + 1)\, dH_{r-1}/dT + \{(T + r)\, d/dT + a_r\}H_r - b_rH_{r+1} = 0, \qquad (r = 0, 1, \cdots, m - 1),$$

$$(2.22)\quad -H_{m-1} + (T + m)H_m = 0,$$

where

$$(2.23)\quad a_r = \tfrac{1}{2}(m - r)(n_2 - r) + 1, \qquad b_r = \tfrac{1}{2}(r + 1)(n_1 + n_2 - r), \qquad (r = 0, 1, \cdots, m).$$

Although these equations have been derived for $m \geqq 2$, they also hold for $m = 1$.

Elimination of $H_1, \cdots, H_m$ from equations (2.21–22) will clearly yield a linear homogeneous differential equation of order m for $f = H_0$. The coefficient of $f^{(r)}$ is a polynomial in T of order $r + 1$, that of the highest derivative $f^{(m)}$ being $T(T + 1) \cdots (T + m)$. The differential equation is therefore of Fuchsian type with regular singularities at $0, -1, \cdots, -m$, and infinity. In particular, when $m = 2$:

$$(2.24)\quad T(T+1)(T+2)f'' + [\tfrac{1}{2}(3n_2 + 5)T^2 + 2(n_2 - n_1 + 4)T - 2(n_1 - 2)]f' + \tfrac{1}{2}(n_2 + 1)[(n_2 + 1)T - 2(n_1 - 2)]f = 0.$$

If the transformations

$$(2.25)\quad f(T) = T^{n_1-1}(1 + \tfrac{1}{2}T)^{-(n_1+n_2)}g(T), \qquad v = T^2/(T + 2)^2$$

are made in (2.24), g may be shown to satisfy a hypergeometric equation in conformity with Hotelling's result (1.2).

In the general case, however, it is preferable to work with the linear system (2.21–22) itself. An extensive literature exists dealing with such systems (see [1]). To express the result in matrix form, the following notation will be introduced for $(m + 1) \times (m + 1)$ matrices, all of whose elements are zero except those on their leading, upper and lower diagonals:

$$(2.26)\quad \{(\lambda_0, \cdots, \lambda_{m-1}), (\mu_0, \cdots, \mu_m), (\nu_1, \cdots, \nu_m)\} = \begin{bmatrix} \mu_0, & \lambda_0, & 0, & \cdots, & 0 \\ \nu_1, & \mu_1, & \lambda_1, & \cdots, & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & & \nu_{m-1}, & \mu_{m-1}, & \lambda_{m-1} \\ 0, & \cdots, & 0, & \nu_m, & \mu_m \end{bmatrix}.$$

Columns and rows will be numbered $0, 1, \cdots, m$.

Differentiating (2.22) with respect to T to achieve symmetry, and introducing the column vector

$$\mathbf{H} = (H_0, H_1, \cdots, H_m)', \tag{2.27}$$

the system may be written in the form

$$(T\mathbf{E}_{m+1} + \mathbf{A})\, d\mathbf{H}/dT = \mathbf{B}\mathbf{H}, \tag{2.28}$$

where $\mathbf{E}_{m+1}$ is the $(m+1) \times (m+1)$ unit matrix, and

$$\begin{aligned} \mathbf{A} &= \{(0, 0, \cdots, 0), (0, 1, 2, \cdots, m), (-m, -(m-1), \cdots, -1)\}, \\ \mathbf{B} &= \{(b_0, \cdots, b_{m-1}), (-a_0, -a_1, \cdots, -a_m), (0, \cdots, 0)\}. \end{aligned} \tag{2.29}$$

3. The regular singularity at the origin. Equation (2.28) may also be written as

$$d\mathbf{H}/dT = (T^{-1}\mathbf{R} + \textstyle\sum_{r=0}^{\infty} \mathbf{S}_r T^r)\mathbf{H}, \tag{3.1}$$

where $\mathbf{R}$, $\mathbf{S}_r$ are constant $(m+1) \times (m+1)$ matrices. The standard procedure for discussing the solution of (3.1) in the vicinity of the origin is to reduce $\mathbf{R}$ to its canonical Jordan form by means of a suitable linear transformation of $\mathbf{H}$ ([1], Chapter 4). For present purposes it is sufficient to find a matrix $\mathbf{P}$ reducing $\mathbf{A}$ to its canonical form:

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{\Lambda} = \text{diag}\,\{0, 1, \cdots, m\}. \tag{3.2}$$

The right-hand side denotes an $(m+1) \times (m+1)$ diagonal matrix. A suitable $\mathbf{P}$, together with its inverse, is given by

$$\begin{aligned} \mathbf{P} &= \{p_{ij}\}, \qquad p_{ij} = \tbinom{m-j}{m-i}, \\ \mathbf{P}^{-1} &= \{p_{ij}^*\}, \qquad p_{ij}^* = (-1)^{i+j}\tbinom{m-j}{m-i}, \qquad (i, j = 0, 1, \cdots, m). \end{aligned} \tag{3.3}$$

Clearly, both $\mathbf{P}$ and $\mathbf{P}^{-1}$ are lower triangular. It may be shown without difficulty that

$$\mathbf{P}^{-1}\mathbf{B}\mathbf{P} = \mathbf{C} = \{(\beta_0, \cdots, \beta_{m-1}), (\alpha_0, \cdots, \alpha_m), (\gamma_1, \cdots, \gamma_m)\}, \tag{3.4}$$

where

$$\begin{aligned} \alpha_i &= \tfrac{1}{2}[(m - 2i)n_1 - in_2 + (2i^2 - mi - i - 2)], \\ \beta_i &= \tfrac{1}{2}(i + 1)(n_1 + n_2 - i), \\ \gamma_i &= -\tfrac{1}{2}(m - i + 1)(n_1 - i + 1). \end{aligned} \tag{3.5}$$

Thus, if we write

$$\mathbf{H} = \mathbf{P}\mathbf{M}, \tag{3.6}$$

equation (2.28) becomes

$$(T\mathbf{E}_{m+1} + \mathbf{\Lambda})\, d\mathbf{M}/dT = \mathbf{C}\mathbf{M}, \tag{3.7}$$

or, alternatively,

$$d\mathbf{M}/dT = \{T^{-1}\mathbf{V}_0 + \textstyle\sum_{r=1}^{m} (T + r)^{-1}\mathbf{V}_r\}\mathbf{M}, \tag{3.8}$$

where

$$
\begin{aligned}
\mathbf{V}_0 &= \{(\tfrac{1}{2}(n_1 + n_2), 0, \cdots, 0), (\tfrac{1}{2}mn_1 - 1, 0, \cdots, 0), (0, 0, \cdots, 0)\}, \\
\mathbf{V}_r &= \{(0, \cdots, 0, \beta_r, 0, \cdots, 0), (0, \cdots, 0, \alpha_r, 0, \cdots, 0), \\
&\qquad (0, \cdots, 0, \gamma_r, 0, \cdots, 0)\} \quad (r = 1, \cdots, m).
\end{aligned} \tag{3.9}
$$

The characteristic roots of $\mathbf{V}_0$ are zero (with multiplicity m) and $\frac{1}{2}mn_1 - 1$. The presence of equal roots makes discussion of the complete solution difficult. However, in view of Constantine's result (1.3), the relevant solution in the vicinity of $T = 0$ is of the form

$$\mathbf{M} = k(m; n_1, n_2)T^{\frac{1}{2}mn_1-1} \textstyle\sum_{r=0}^{\infty} \mathbf{W}_r T^r, \qquad (|T| < 1), \tag{3.10}$$

where $k(m; n_1, n_2)$ is the constant in square brackets in (1.3). The vectors $\mathbf{W}_r = (W_{0r}, \cdots, W_{mr})'$ may be determined without difficulty since $\frac{1}{2}mn_1 - 1$ is the largest root ([1] Chapter 4, Problem 13) except when $m = n_1 = 1$. Substitution in (3.7) yields the following recurrence relations:

$$
\begin{aligned}
&\mathbf{W}_0 = (1, 0, \cdots, 0)'; \\
&i(r + \tfrac{1}{2}mn_1 - 1)W_{ir} \\
&\qquad = \gamma_i W_{i-1,r-1} + [\alpha_i - (r + \tfrac{1}{2}mn_1 - 2)]W_{i,r-1} + \beta_i W_{i+1,r-1} \\
&\qquad\qquad (i = 1, \cdots, m;\ r = 1, 2, \cdots); \\
&rW_{0r} = \tfrac{1}{2}(n_1 + n_2)W_{1r}, \qquad (r = 1, 2, \cdots).
\end{aligned} \tag{3.11}
$$

When $m = n_1 = 1$, the roots are $-\frac{1}{2}$, 0, and do not differ by an integer.

4. Ito's asymptotic expansion for large n_2. Let us write

$$t = n_2 T = T_0^2. \tag{4.1}$$

Then the cdf $F(t)$ of t has a power series representation, convergent for $|t| < n_2$, which may be obtained by term-by-term integration of Constantine's series. Essentially, Ito's expansion of $F(t)$ is obtained by rearranging this series as a convergent power series in n_2^{-1}, and multiplying by the Stirling-type asymptotic expansion of $k(m; n_1, n_2)$.

It is readily seen by induction from (3.11) that each W_{ir} is a polynomial in n_2 of order $(r - i)$ at most. Hence, making the substitution (4.1) in (3.10),

$$\mathbf{M} = k(m; n_1, n_2)(t/n_2)^{\frac{1}{2}mn_1-1} \textstyle\sum_{r=0}^{\infty} \mathbf{W}_r^* t^r, \qquad (\mathbf{W}_r^* = n_2^{-r}\mathbf{W}_r), \tag{4.2}$$

where the components W_{ir}^* of W_r^* are polynomials in n_2^{-1}, lower powers $\geqq i$. To verify that rearrangement of (4.2) as a power series in n_2^{-1} is valid, we note that this series is dominated by the corresponding solution of

$$d\mathbf{M}/dt = \{t^{-1}\mathbf{V}_0 + \textstyle\sum_{r=1}^{m} (r - n_2^{-1}t)^{-1} n_2^{-1} \mathbf{V}_r^+\}\mathbf{M}, \tag{4.3}$$

where the $\mathbf{V}_r^+$ are obtained from the $\mathbf{V}_r$ $(r = 1, \cdots, m)$ by replacing all negative signs by positive signs in the expressions for α_i, β_i, γ_i. This solution is a double power-series in t and n_2^{-1} with positive coefficients, convergent for $|t| < n_2$.

In order to obtain the rearranged series for the component $M_0 = f$, it is convenient to first remove the factor n_2^{-i} from each W_{ir}^*, i.e. we make the shearing transformation

$$\mathbf{M} = \text{diag}\,\{1, n_2^{-1}, \cdots, n_2^{-m}\}\mathbf{N}. \tag{4.4}$$

Then

$$\mathbf{N} = n_2^{-(\frac{1}{2}mn_1-1)}k(m; n_1, n_2)\sum_{r=0}^{\infty} n_2^{-r}\mathbf{Y}_r(t), \tag{4.5}$$

where

$$\begin{aligned} \mathbf{Y}_0(t) &= t^{\frac{1}{2}mn_1-1}\mathbf{y}_0(t), & y_{00}(0) &= 1, \\ \mathbf{Y}_r(t) &= t^{\frac{1}{2}mn_1}\mathbf{y}_r(t), & (r &= 1, 2, \cdots), \end{aligned} \tag{4.6}$$

and the components $y_{ir}(t)$ of the $\mathbf{y}_r(t)$ are power series in t. The second requirement in (4.6) arises because $\mathbf{W}_0^*$ and its transform under (4.4) are both $(1, 0, \cdots, 0)'$, which is independent of n_2^{-1}.

From (3.7), $\mathbf{N}$ satisfies the equation:

$$\begin{aligned} &(n_2^{-1}t\mathbf{E}_{m+1} + \mathbf{\Lambda})\, d\mathbf{N}/dt \\ &= \{(n_2^{-2}\beta_0, \cdots, n_2^{-2}\beta_{m-1}), (n_2^{-1}\alpha_0, \cdots, n_2^{-1}\alpha_m), (\gamma_1, \cdots, \gamma_m)\}\mathbf{N} \\ &= \{\mathbf{\Delta}_0 + n_2^{-1}\mathbf{\Delta}_1 + n_2^{-2}\mathbf{\Delta}_2\}\mathbf{N}. \end{aligned} \tag{4.7}$$

The matrices $\mathbf{\Delta}_i$ are given by:

$$\begin{aligned} \mathbf{\Delta}_0 &= \{(0, \cdots, 0)(0, -\tfrac{1}{2}, -1, \cdots, -\tfrac{1}{2}m), (\gamma_1, \cdots, \gamma_m)\}, \\ \mathbf{\Delta}_1 &= \{(\tfrac{1}{2}, 1, \cdots, \tfrac{1}{2}m), (\bar{\alpha}_0, \bar{\alpha}_1, \cdots, \bar{\alpha}_m), (0, \cdots, 0)\}, \\ \mathbf{\Delta}_2 &= \{(\bar{\beta}_0, \cdots, \bar{\beta}_{m-1}), (0, \cdots, 0), (0, \cdots, 0)\}, \end{aligned} \tag{4.8}$$

where

$$\begin{aligned} \bar{\alpha}_i &= \tfrac{1}{2}[(m - 2i)n_1 + 2i^2 - mi - i - 2], \\ \bar{\beta}_i &= \tfrac{1}{2}(i + 1)(n_1 - i). \end{aligned} \tag{4.9}$$

Substituting (4.5) in (4.7), it is found that:

$$[t(d/dt) - (\tfrac{1}{2}mn_1 - 1)]Y_{0,r} - \tfrac{1}{2}Y_{1,r} = \tfrac{1}{2}n_1\, Y_{1,r-1}, \quad (r = 0, 1, \cdots), \tag{4.10}$$

while for $i = 1, \cdots, m$:

$$\begin{aligned} &i((d/dt) + \tfrac{1}{2})Y_{i,r} - \gamma_i Y_{i-1,r} \\ &\qquad = (-t(d/dt) + \bar{\alpha}_i)Y_{i,r-1} + \tfrac{1}{2}(i + 1)Y_{i+1,r-1} + \bar{\beta}_i Y_{i+1,r-2}. \end{aligned} \tag{4.11}$$

Y_{-1} and Y_{-2} are taken to be identically zero.

These equations determine the $Y_{i,r}$ uniquely in virtue of (4.6). It is seen that for $r = 0, 1, \cdots$, $Y_{0,r}$ and $Y_{1,r}$ are given by a pair of simultaneous linear differential equations, inhomogeneous except when $r = 0$. The corresponding homogeneous equations are of the form:

$$\begin{bmatrix} t, & 0 \\ 0, & 1 \end{bmatrix} d\mathbf{Z}/dt = \begin{bmatrix} \frac{1}{2}\, mn_1 - 1, & \frac{1}{2} \\ -\frac{1}{2}\, mn_1, & -\frac{1}{2} \end{bmatrix} \mathbf{Z}, \qquad (\mathbf{Z} = (Z_0, Z_1)'), \tag{4.12}$$

or, equivalently:

$$d\mathbf{Z}/dt = \left\{ t^{-1} \begin{bmatrix} \frac{1}{2}mn_1 - 1, & \frac{1}{2} \\ 0, & 0 \end{bmatrix} + \begin{bmatrix} 0, & 0 \\ -\frac{1}{2}mn_1, & -\frac{1}{2} \end{bmatrix} \right\} \mathbf{Z}. \tag{4.13}$$

The solution corresponding to the root $\frac{1}{2}mn_1 - 1$ of the leading matrix is easily verified to be

$$\mathbf{Z} = \text{const}\ \gamma(t)(1, -t)', \tag{4.14}$$

where

$$\gamma(t) = e^{-\frac{1}{2}t} t^{\frac{1}{2}mn_1 - 1}. \tag{4.15}$$

For $(Y_{0,0}, Y_{1,0})$ it follows from (4.6) that the zero root does not apply, and that the constant in (4.14) is unity in this case. Hence it follows from (4.11) that for $r = 0$:

$$\begin{aligned} Y_{i,0} = {} & (-t)^i \gamma(t)(m-1)(m-2) \cdots (m-i+1)(n_1-1)(n_1-2) \\ & \cdots (n_1 - i + 1)[i!\,(mn_1+2)(mn_1+4) \cdots (mn_1 + 2(i-1))]^{-1}, \\ & \qquad (i = 2, \cdots, m). \end{aligned} \tag{4.16}$$

Similarly, in solving for $Y_{0,r}$, $Y_{1,r}$, $(r \geqq 1)$, it is found that only the solution (4.14) of the homogeneous part applies. The constant is determined by the requirement that the lowest power of t occurring is at least $\frac{1}{2}mn_1$. The same requirement eliminates the general term const $e^{-\frac{1}{2}t}$ in the solution for $Y_{i,r}$ $(i = 2, \cdots, m)$.

In order to derive the expansion ([9], equation (4.3)) of Ito's paper, it is sufficient to calculate the following $Y_{i,r}$:

$$\begin{aligned} Y_{0,1} &= t\gamma(t)\{-\tfrac{1}{2}n_1 + \tfrac{1}{4}t(m + n_1 + 1)(mn_1 + 1)^1\}, \\ Y_{0,1} &= t^2\gamma(t)(mn_1+2)^{-1}\{\tfrac{1}{2}[m(n_1^2+2) + (4n_1+2)] - \tfrac{1}{4}t(m+n_1+1)\}, \\ Y_{2,1} &= (m-1)(n_1-1)t^3\gamma(t)[(mn_1+2)(mn_1+4)]^{-1} \\ &\quad \cdot\{-\tfrac{1}{4}[m(n_1^2+2) + (4n_1+2)] \\ &\quad + \tfrac{1}{8}t(mn_1+6)^{-1}[mn_1^2 + m(n_1^2+n_1+8) + (8n_1+4)]\}, \\ Y_{0,2} &= t^2\gamma(t)(mn_1+2)^{-1}\{\tfrac{1}{8}n_1\,[m(n_1^2+2) + (4n_1+2)] \\ &\quad - \tfrac{1}{24}t(mn_1+4)^{-1}[m^2(3n_1^2+4) + 3m(n_1^3+n_1^2+8n_1+4) \\ &\quad + (16n_1^2 + 24n_1 + 16)] \\ &\quad + \tfrac{1}{32}t^2[(mn_1+4)(mn_1+6)]^{-1}[m^3n_1 + m^2(2n_1^2+2n_1+8) \\ &\quad + m(n_1^3 + 2n_1^2 + 21n_1 + 20) + (8n_1^2 + 20n_1 + 20)]\}. \end{aligned} \tag{4.17}$$

The density function of t is then

$$f(t) = n_2^{-\frac{1}{2}mn_1}k(m; n_1, n_2)\{\gamma(t) + n_2^{-1}Y_{0,1} + n_2^{-2}Y_{0,2} + \cdots\}, \quad (|t| < n_2). \tag{4.18}$$

Integrating with respect to t to obtain $F(t)$, and using the expansion

$$\begin{aligned} n_2^{-\frac{1}{2}mn_1}k(m; n_1, n_2) \sim [2^{\frac{1}{2}mn_1}\Gamma(\tfrac{1}{2}mn_1)]^{-1}\{1 &+ \tfrac{1}{4}mn_1n_2^{-1}(n_1 - m - 1) \\ &+ mn_1(96n_2^2)^{-1}[3m^3n_1 - 2m^2(3n_1^2 - 3n_1 + 4) \\ &+ 3m(n_1^3 - 2n_1^2 + 5n_1 - 4) \\ &+ (-8n_1^2 + 12n_1 + 4)] + \cdots\}, \end{aligned} \tag{4.19}$$

Ito's result is obtained as an asymptotic expansion uniformly valid for t in any bounded interval. The corresponding expansion of the percentiles of t for large n_2 ([9] equation (3.33)) may be derived formally from that of the cdf by means of an algorithm found by G. W. Hill and the present author ([6]).

5. The regular singularity at infinity. Letting

$$z = T^{-1}, \tag{5.1}$$

equation (3.8) takes the form

$$d\mathbf{M}/dz = \{-z^{-1}\mathbf{C} + \textstyle\sum_{r=1}^{m} (z + r^{-1})^{-1}\mathbf{V}_r\}\mathbf{M}. \tag{5.2}$$

Thus (3.8) has a regular singularity at $T = \infty$, with linearly independent solutions corresponding to the $(m + 1)$ latent roots of $-\mathbf{C}$, convergent for $|T| > m$. Since $\mathbf{C}$ is similar to $\mathbf{B}$, these roots are

$$a_r = \tfrac{1}{2}(m - r)(n_2 - r) + 1, \qquad (r = 0, 1, \cdots, m). \tag{5.3}$$

The a_r form a decreasing sequence for increasing r. We now seek to relate these solutions to $f(T)$. Let $l(m; n_1, n_2)$ denote the constant in (2.2). From (2.5) we have, as $T \to \infty$,

$$\begin{aligned} &f(T)T^{\frac{1}{2}(n_2-m+3)} \\ &= l(m; n_1, n_2) \int_{\mathfrak{D}_{m-1}(T)} (1 - T^{-1} \textstyle\sum_{i=2}^{m} w_i)^{\frac{1}{2}(n_1-m-1)} \\ &\quad \cdot (\textstyle\prod_{i=2}^{m} w_i)^{\frac{1}{2}(n_1-m-1)}[1 + T^{-1}(1 - \textstyle\sum_{i=2}^{m} w_i)]^{-\frac{1}{2}(n_1+n_2)} \\ &\quad \cdot \textstyle\prod_{i=2}^{m} (1 + w_i)^{-\frac{1}{2}(n_1+n_2)} \textstyle\prod_{j=2}^{m} [1 - T^{-1}(\textstyle\sum_{i=2}^{m} w_i + w_j)] \\ &\quad \cdot \textstyle\prod_{2 \leq i < j \leq m} (w_i - w_j)\, dw_2 \cdots dw_m \\ &\to [l(m; n_1, n_2)/l(m - 1; n_1 - 1, n_2 + 1)] \int_{\mathfrak{D}_{m-1}} \phi_{m-1;n_1-1,n_2+1}(\mathbf{w})\, d\mathbf{w} \\ &= \pi^{\frac{1}{2}}\Gamma(\tfrac{1}{2}(n_1 + n_2 - m + 1))\Gamma(\tfrac{1}{2}(n_2 + 1))/[\Gamma(\tfrac{1}{2}m) \\ &\quad \cdot\Gamma(\tfrac{1}{2}n_1)\Gamma(\tfrac{1}{2}(n_2 - m + 1))\, \Gamma(\tfrac{1}{2}(n_2 - m + 2))] \\ &= \kappa(m; n_1, n_2), \quad \text{say}. \end{aligned} \tag{5.4}$$

In order to justify the limit, we note that since

$$w_2 + \cdots + w_m < T - w_2 < \tfrac{1}{2}T, \tag{5.5}$$

the integrand is dominated on $\mathfrak{D}_{m-1}$ by a multiple of $\phi_{m-1;n_1-1,n_2+1}$. Since

$$a_{m-1} = \tfrac{1}{2}(n_2 - m + 3), \tag{5.6}$$

equation (5.4) shows that in the neighbourhood of $T = \infty$, the relevant **M** may be a linear combination of the solutions corresponding to all roots of $-\mathbf{C}$ except $a_m = 1$. Although (5.4) yields the coefficient of the a_{m-1} solution, the problem of determining the other coefficients in this linear combination remains unsolved. A further complication consists in the fact that the a_r differ by integers if $n_2 - m + 1$ is an even integer, while if $n_2 - m + 1$ is odd then $a_r - a_{r+2}$ is integral. The solutions therefore involve logarithmic terms. This situation is unfortunate because the tabulation of higher percentile points of $f(T)$ is of considerable interest.

If $n_2 > m + 1$, the first four terms in the expansion of $F(T)$ for large T are:

$$\begin{aligned} F(T) &= 1 - \kappa(m; n_1, n_2)T^{-\frac{1}{2}(n_2-m+1)}\{2(n_2 - m + 1)^{-1} \\ &\quad + [T(n_2 - m + 3)]^{-1}[m(n_1 - 1) - (2n_1 + n_2 - 1)] \\ &\quad + [4T^2(n_2 - m + 5)]^{-1}[m^2(n_1 - 1)^2 - 2m(n_1 - 1)(3n_1 + n_2 - 2) \\ &\quad + n_2^2 - 2n_2(n_1^2 - 5n_1 + 2) + 3(2n_1^2 - 2n_1 + 1) \\ &\quad + 2(n_1 - 1)(n_1 - 2)n_2(n_2 + 1)(n_2 - m + 2)^{-1}] + \cdots\}. \end{aligned} \tag{5.7}$$

This result may be derived by applying to **H** the transformation given later in Section 7. However, the details will be omitted.

6. The limiting distribution for increasing n_1. As n_1 becomes large, it is clear that the random variable

$$\tau = n_1^{-1}T = \operatorname{tr}\{(n_1^{-1}\mathbb{S}_1)\mathbb{S}_2^{-1}\} \tag{6.1}$$

will converge in distribution. The limiting distribution is perhaps of mainly mathematical interest, but it has the merit of giving completeness to tables of the Hotelling statistic. Our discussion in this section will be rather heuristic.

We first obtain an integral form of the limiting density function, which will be denoted by $\theta(\tau)$. Take $\psi = \phi_{m;n_1,n_2}$ in (2.5). Substituting

$$w_i = n_1 u_i, \qquad (i = 2, \cdots, m), \tag{6.2}$$

and letting $n_1 \to \infty$ it is found that

$$\begin{aligned} \theta(\tau) &= \lim_{n_1\to\infty} n_1 f(n_1\tau) \\ &= \alpha(m; n_2) \int_{\mathfrak{D}_{m-1}(\tau)} \exp\{-\tfrac{1}{2}[(\tau - \textstyle\sum_{i=2}^m u_i)^{-1} + \sum_{i=2}^m u_i^{-1}]\} \\ &\quad \cdot [(\tau - \textstyle\sum_{i=2}^m u_i) \prod_{i=2}^m u_i]^{-\frac{1}{2}(n_2+m-1)} \prod_{j=2}^m (\tau - \sum_{i=2}^m u_i - u_j) \\ &\quad \cdot \textstyle\prod_{2\leq j<i\leq m} (u_i - u_j)\, du_2 \cdots du_m, \end{aligned} \tag{6.3}$$

where

$$\alpha(m; n_2) = \lim_{n_1 \to \infty} n_1^{-\frac{1}{2}mn_2} l(m; n_1, n_2) = \pi^{\frac{1}{2}m^2}/2^{\frac{1}{2}mn_2}\Gamma_m(\tfrac{1}{2}n_2)\Gamma_m(\tfrac{1}{2}m). \tag{6.4}$$

For large τ, one easily shows that

$$\theta(\tau) = O(\tau^{-\frac{1}{2}(n_2-m+3)}). \tag{6.5}$$

The behaviour for small τ is more complicated. Setting

$$v_i = \tau u_i, \qquad (i = 2, \cdots, m), \tag{6.6}$$

in equation (6.3), we obtain:

$$\begin{aligned} \theta(\tau) &= \alpha(m; n_2)\tau^{-\frac{1}{2}mn_2-1} \\ &\cdot \int_{\mathcal{R}_{m-1}} \exp\{-(2\tau)^{-1}[(1 - \textstyle\sum_{i=2}^m v_i)^{-1} + \sum_{i=2}^m v_i^{-1}]\} \\ &\cdot [(1 - \textstyle\sum_{i=2}^m v_i)\prod_{i=2}^m v_i]^{-\frac{1}{2}(n_2+m+1)} \prod_{j=2}^m (1 - \sum_{i=2}^m v_i - v_j) \\ &\cdot \textstyle\prod_{2 \leq i < j \leq m} (v_i - v_j)\, dv_2 \cdots dv_m \end{aligned} \tag{6.7}$$

where

$$\mathcal{R}_{m-1} = \{0 < v_m < m^{-1}; v_m < v_{m-1} < (m-1)^{-1}(1 - v_m); \cdots; v_3 < v_2 < \tfrac{1}{2}(1 - v_2 - \cdots - v_m)\}. \tag{6.8}$$

An asymptotic estimate of $\theta(\tau)$ as $\tau \to 0+$ may be obtained by means of a method due to Laplace ([4] Chapter I, Section 3). Consider an integral of the form

$$\mathfrak{F}(\tau) = \int_\eta^\xi e^{\tau^{-1}\omega(v)}\rho(v)\, dv. \tag{6.9}$$

Suppose that $\omega(v)$ is real on (η, ξ) and has its greatest value in the interval at $v = \xi$, with $\omega'(\xi) = 0$, $\omega''(\xi) < 0$. Under wide conditions on ρ, ω, we may expect that, as $\tau \to 0+$,

$$\mathfrak{F}(\tau) \sim \tfrac{1}{2}e^{\tau^{-1}\omega(\xi)} \textstyle\sum_{r=0}^\infty \Gamma(\frac{1}{2}(r+1))c_r\tau^{\frac{1}{2}(r+1)}, \tag{6.10}$$

where

$$\textstyle\sum_{r=0}^\infty c_r u^r = \rho(\zeta(u))\zeta'(u) \tag{6.11}$$

and $v = \zeta(u)$ is the inverse of

$$u = +[\omega(\xi) - \omega(v)]^{\frac{1}{2}}. \tag{6.12}$$

The integral involving v_2 in (6.7) is:

$$\begin{aligned} \mathcal{J}_2 &= \int_{v_2=v_3}^{\frac{1}{2}(1-\Sigma_3)} \exp\{-(2\tau)^{-1}[(1 - \textstyle\sum_3 - v_2)^{-1} + v_2^{-1}]\} \\ &\cdot [(1 - \textstyle\sum_3 - v_2)v_2]^{-\frac{1}{2}(n_2+m+1)} \\ &\cdot (1 - \textstyle\sum_3 - 2v_2)\prod_{j \geq 3} [(1 - \sum_3 - v_j - v_2)(v_2 - v_j)]\, dv_2, \end{aligned} \tag{6.13}$$

where $\sum_3$ denotes $\sum_3^m v_i$.

Clearly

$$\omega(v_2) = -\tfrac{1}{2}[(1 - \textstyle\sum_3 - v_2)^{-1} + v_2^{-1}] \tag{6.14}$$

has its greatest value at $\xi = \frac{1}{2}(1 - \sum_3)$:

$$\omega(\xi) = -2(1 - \textstyle\sum_3)^{1}. \tag{6.15}$$

Also,

$$\begin{aligned} u &= +[\omega(\xi) - \omega(v_2)]^{\frac{1}{2}} \\ &= [\tfrac{1}{2}(1 - \textstyle\sum_3)]^{-\frac{3}{2}}(\xi - v_2) + \cdots , \end{aligned} \tag{6.16}$$

whence

$$v_2 = \zeta(u) = \tfrac{1}{2}(1 - \textstyle\sum_3) - [\tfrac{1}{2}(1 - \textstyle\sum_3)]^{\frac{3}{2}}u + \cdots . \tag{6.17}$$

Hence, obtaining ρ from (6.13):

$$\begin{aligned} &\rho(\zeta(u))\zeta'(u) \\ &\quad = u \cdot 2^{n_2 - m + 3}(1 - \textstyle\sum_3)^{-(n_2+m-2)} \prod_{j \geqq 3} [(1 - \textstyle\sum_3) - 2v_j]^2 + \cdots . \end{aligned} \tag{6.18}$$

It follows from (6.10) that

$$\begin{aligned} \mathcal{G}_2 \sim d_2\tau \exp \{-2\tau^{-1}(1 - \textstyle\sum_3)^{-1}\} \\ \cdot (1 - \textstyle\sum_3)^{-(n_2+m-2)} \prod_{j \geqq 3} [(1 - \textstyle\sum_3) - 2v_j]^2, \end{aligned} \tag{6.19}$$

where

$$d_2 = 2^{n_2 - m + 2}. \tag{6.20}$$

The same method may be applied successively to the $v_3, \cdots, v_m$ integrals. The integral with respect to v_r is found by induction to be:

$$\begin{aligned} \mathcal{G}_r \sim d_r\tau^{\frac{1}{2}r} \exp \{-r^2(2\tau)^{-1}(1 - \textstyle\sum_{r+1})^{-1}\}(1 - \textstyle\sum_{r+1})^{-\mu_r} \\ \cdot \prod_{j \geqq r+1} [(1 - \textstyle\sum_{r+1}) - rv_j]^r, \end{aligned} \tag{6.21}$$

where

$$\begin{aligned} d_r &= 2^{\frac{1}{2}r-1}\Gamma(\tfrac{1}{2}r)r^{\frac{1}{2}r(n_2-m)+\frac{1}{4}(r^2+r+2)}(r-1)^{-\frac{1}{2}(r-1)(n_2-m)-\frac{1}{4}(r^2-r+4)} \\ \mu_r &= \tfrac{1}{2}r(n_2 + m - 1) - \tfrac{3}{4}(r - 1)(r + 2), \ \textstyle\sum_{r+1} \equiv \sum_{r+1}^m v_i . \end{aligned} \tag{6.22}$$

Hence, as $\tau \to 0+$,

$$\begin{aligned} \theta(\tau) &\sim \alpha(m; n_2)(\textstyle\prod_{r=2}^m d_r)e^{-m^2/2\tau}\tau^{-\frac{1}{2}mn_2 - 1 + \frac{1}{2}\sum_2^m r} \\ &= [\pi^{\frac{1}{4}(m-1)(m+2)}m^{\frac{1}{4}m(2n_2-m+1)+1}/2^{\frac{1}{2}mn_2 - \frac{1}{4}(m-1)(m-2)}(m!)^{\frac{1}{2}}\Gamma_m(\tfrac{1}{2}n_2)] \\ &\quad \cdot e^{-m^2/2\tau}\tau^{-\frac{1}{2}mn_2 + \frac{1}{4}(m-2)(m+3)}. \end{aligned} \tag{6.23}$$

Although the above derivation strictly requires that $m \geqq 2$, it is seen that when $m = 1$ (6.23) reduces to

$$\theta(\tau) \sim [2^{\frac{1}{2}n_2}\Gamma(\tfrac{1}{2}n_2)]^{-1}e^{-1/2\tau}\tau^{-(\frac{1}{2}n_2+1)}, \tag{6.24}$$

which is the density function of $\tau = 1/\chi^2$, where χ^2 is based on n_2 degrees of freedom.

A system of linear differential equations for $\theta(\tau)$ will now be derived. Let the following substitutions be made in (2.28):

$$\tau = n_1^{-1}T, \qquad \mathbf{H} = \operatorname{diag}\{1, n_1^{-1}, \cdots, n_1^{-m}\}\,\mathbf{J}. \tag{6.25}$$

Then

$$\begin{aligned}[\tau\mathbf{E}_{m+1} + \{(0, \cdots, 0), (0, n_1^{-1}, 2n_1^{-1}, \cdots, mn_1^{-1}),\\ (-m, -(m-1), \cdots, -1)\}]\, d\mathbf{J}/d\tau\\ = \{(n_1^{-1}b_0, \cdots, n_1^{-1}b_{m-1}), (-a_0, \cdots, -a_m), (0, \cdots, 0)\}\mathbf{J}.\end{aligned} \tag{6.26}$$

Formally letting $n_1 \to \infty$, we obtain

$$(\tau\mathbf{E}_{m+1} + \mathbf{\Omega})\, d\mathbf{J}/d\tau = -\mathbf{\Gamma}\mathbf{J}, \tag{6.27}$$

where

$$\begin{aligned}\mathbf{\Omega} &= \{(0, \cdots, 0), (0, \cdots, 0), (-m, -(m-1), \cdots, -1)\},\\ \mathbf{\Gamma} &= \{(\tfrac{1}{2}, 1, \cdots, \tfrac{1}{2}m), (a_0, a_1, \cdots, a_m), (0, 0, \cdots, 0)\}.\end{aligned} \tag{6.28}$$

In order to show that $\mathbf{J}$ has a regular singularity at $\tau = \infty$, set $x = \tau^{-1}$. Then

$$\begin{aligned}d\mathbf{J}/dx &= x^{-1}(\mathbf{E}_{m+1} + x\mathbf{\Omega})^{-1}\mathbf{\Gamma}\mathbf{J}\\ &= \{x^{-1}\mathbf{\Gamma} + \textstyle\sum_{r=0}^{m} (-\mathbf{\Omega})^{r+1}\mathbf{\Gamma}x^r\}\mathbf{J}.\end{aligned} \tag{6.29}$$

The $(m + 1)$ linearly independent solutions in the neighbourhood of ∞ correspond to the latent roots of $\mathbf{\Gamma}$, viz a_r $(r = 0, 1, \cdots, m)$. In view of (6.5), the required $\mathbf{J}$ may be a linear combination of the solutions obtained from $a_0, \cdots, a_{m-1}$, and the problem of determining the coefficients arises as in Section 5.

Turning next to consider the solution near $\tau = 0$, let

$$\mathbf{J} = \operatorname{diag}\{1, \tau^{-1}, \cdots, \tau^{-m}\}\mathbf{K}. \tag{6.30}$$

Then

$$d\mathbf{K}/d\tau = \tau^{-2}(\mathbf{D}_0 + \tau\mathbf{D}_1)\mathbf{K}, \tag{6.31}$$

where

$$\begin{aligned}\mathbf{D}_0 &= \tfrac{1}{2}(\mathbf{E}_{m+1} + \mathbf{\Omega})^{-1}\mathbf{\Xi},\\ \mathbf{\Xi} &= \{(1, 2, \cdots, m), (0, \cdots, 0), (0, \cdots, 0)\},\\ \mathbf{D}_1 &= -(\mathbf{E}_{m+1} + \mathbf{\Omega})^{-1}\{(0, \cdots, 0, \cdots, 0), (a_0, \cdots, a_{j-j}, \cdots, a_m - m),\\ &\qquad (0, \cdots, (j-1)(m-j+1), \cdots, 0)\}.\end{aligned} \tag{6.32}$$

The system (6.31) is seen to have a singular point of the second kind (in fact an irregular singularity) at $\tau = 0$ ([1], Chapter 5). In general, such systems have formal solutions which provide asymptotic expansions of the actual solutions for small τ. As in the case of the regular singular points, we first seek the latent roots of the leading matrix $\mathbf{D}_0$. These may be obtained from the determinantal equation

$$\det \{\lambda(\mathbf{E}_{m+1} + \mathbf{\Omega}) - \mathbf{\Xi}\} = 0. \tag{6.33}$$

The left-hand side is a continuant ([11], Chapter XIII), and may be written in centrosymmetric form (loc. cit. Section 549):

$$\det \{(-\lambda^{\frac{1}{2}}, -2\lambda^{\frac{1}{2}}, \cdots, -m\lambda^{\frac{1}{2}}), (\lambda, \lambda, \cdots, \lambda), (-m\lambda^{\frac{1}{2}}, \cdots, -2\lambda^{\frac{1}{2}}, -\lambda^{\frac{1}{2}})\} = 0. \tag{6.34}$$

Hence, a theorem on continuants (loc. cit. Section 576) may be used to evaluate the determinant:

$$\prod_{i=0}^{m} [\lambda - (m - 2i)\lambda^{\frac{1}{2}}] = \lambda^{[\frac{1}{2}m]+1} \prod_{i=0}^{[\frac{1}{2}(m-1)]} [\lambda - (m - 2i)^2] = 0, \tag{6.35}$$

where [ ] denotes the greatest integer part. The matrix $\mathbf{D}_0$ therefore has $[\frac{1}{2}(m + 1)]$ positive latent roots $\frac{1}{2}m^2, \frac{1}{2}(m - 2)^2, \cdots$, and $[\frac{1}{2}m] + 1$ zero roots.

Again, the presence of equal roots makes any discussion of the complete solution of (6.31) extremely difficult (see [14]). However, (6.23) implies that the relevant solution is that corresponding to the largest latent root $\frac{1}{2}m^2$, since this solution approaches zero more rapidly as $\tau \to 0+$ than any other.

Noting that $\mathbf{D}_0$ has rank m, there exists a non-singular matrix $\mathbf{Q} = \{q_{ij}\}$ reducing $\mathbf{D}_0$ to its canonical Jordon form:

$$\begin{aligned} \mathbf{Q}^{-1}\mathbf{D}_0\mathbf{Q} = \{&(0, 0, \cdots, 0, \cdots, 0), \\ &(\tfrac{1}{2}m^2, \tfrac{1}{2}(m - 2)^2, \cdots, \tfrac{1}{2}(m - 2\nu)^2, 0, \cdots, 0), \\ &(0, 0, \cdots, 0, 1, \cdots, 1)\} = \mathbf{\Upsilon}_0' \end{aligned} \tag{6.36}$$

where $\nu = [\frac{1}{2}(m - 1)]$, and there are $[\frac{1}{2}m]$ ones in the lower diagonal.

A suitable set of q_{ij} may be obtained from the following recurrence relations:

$$\begin{aligned} &q_{0j} = 1, \quad (j = 0, \cdots, m); \quad q_{im} = 0, \ (i = 1, \cdots, m); \\ &q_{ij} = i^{-1}(m - 2j)^2[q_{i-1,j} - (m - i + 2)q_{i-2,j}], \\ &\qquad\qquad (i = 1, \cdots, m; j = 0, \cdots, \nu); \\ &q_{ij} = i^{-1}[q_{i-1,j+1} - (m - i + 2)q_{i-2,j+1}], \\ &\qquad\qquad (i = 1, \cdots, m; j = \nu + 1, \cdots, m - 1). \end{aligned} \tag{6.37}$$

Writing

$$\mathbf{K} = \mathbf{Q}\mathbf{G}, \tag{6.38}$$

equation (6.31) takes the form

$$d\mathbf{G}/d\tau = \tau^{-2}(\mathbf{\Upsilon}_0 + \tau\mathbf{\Upsilon}_1)\mathbf{G}, \tag{6.39}$$

where

$$\Upsilon_1 = \mathbf{Q}^{-1}\mathbf{D}_1\mathbf{Q} = (v_{ij}). \tag{6.40}$$

We now seek a formal solution

$$\mathbf{G} = \beta(m; n_2)e^{-m^2/2\tau}\tau^{\delta} \sum_{r=0}^{\infty} \mathbf{X}_r\tau^r, \qquad (X_{00} = 1), \tag{6.41}$$

where the constant $\beta(m; n_2)$ is given by (6.23). The $\mathbf{X}_r = (X_{0r}, \cdots, X_{mr})'$ and δ are to be determined. Substitution in (6.39) yields:

$$\mathbf{X}_0 = (1, 0, \cdots, 0)'; \qquad (v_{00} + \delta)\, X_{00} = 0; \tag{6.42}$$

while for $r = 1, 2, \cdots$:

$$\begin{aligned} X_{ir} &= A_{i,r-1}/i(2m - i), && (i = 1, \cdots, \nu); \\ X_{\nu+1,r} &= m^{-2}A_{\nu+1,r-1}; && \\ X_{ir} &= m^{-2}(A_{i,r-1} + X_{i-1,r}), && (i = \nu + 2, \cdots, m); \\ (v_{00} + r + \lambda)X_{0r} &= -\textstyle\sum_{j=1}^{m} v_{0j}X_{jr}, && \end{aligned} \tag{6.43}$$

where

$$A_{i,r} = -2[\textstyle\sum_{j=0}^{m} v_{ij}X_{jr} + (r + \delta)X_{ir}], \quad (i = 1, \cdots, m; r = 0, 1, \cdots). \tag{6.44}$$

From (6.42),

$$\delta = -v_{00} \tag{6.45}$$

so that the last relation in (6.43) becomes

$$X_{0r} = -r^{-1} \textstyle\sum_{j=1}^{m} v_{0j}X_{jr}. \tag{6.46}$$

The author has not succeeded in proving in general that

$$v_{00} = \tfrac{1}{2}mn_2 - \tfrac{1}{4}(m - 2)(m + 3) \tag{6.47}$$

as required by (6.23) and (6.45). However, this has been verified for some early values of m.

It has thus been shown that a formal solution (6.41) of (6.39) exists, and we conjecture that this provides an asymptotic expansion of the required solution as $\tau \to 0+$. The cdf, $\Theta(\tau)$ say, is given by an expansion

$$\Theta(\tau) \sim \beta(m; n_2)e^{-m^2/2\tau}\tau^{\delta+2} \textstyle\sum_{r=0}^{\infty} \mathbf{U}_r\tau^r, \tag{6.48}$$

where the $\mathbf{U}_r$ may be obtained from the relation

$$d\Theta/d\tau = \theta(\tau) = \textstyle\sum_{j=0}^{m} G_j, \quad (\mathbf{G} = (G_0, \cdots, G_m)'). \tag{6.49}$$

An asymptotic development of $f(T)$ for large n_1 based on $\theta(\tau)$ would clearly present a difficult problem, and will not be attempted here.

7. The moments of T. Constantine has shown ([3], Section 5) that the moments of T exist up to the jth, where j is the largest integer such that $j < \frac{1}{2}(n_2 - m + 1)$, and he has obtained expressions for these moments in terms of zonal polynomials. In the present section it is shown that recurrence relations for the moments may be derived from the differential equations (2.19–20) for the Laplace transforms $L_r(s)$. These equations may be written:

$$d\mathbf{L}/ds = [s^{-1}\mathbf{B}^* + \mathbf{A}]\mathbf{L}, \tag{7.1}$$

where $\mathbf{A}$ was defined in (2.29) and

$$\mathbf{B}^* = \{(-b_0, \cdots, -b_{m-1}), (a_0^*, \cdots, a_m^*), (0, \cdots, 0)\}, \tag{7.2}$$
$$a_r^* = a_r - 1 = \tfrac{1}{2}(m - r)(n_2 - r).$$

Since $\mathbf{L}(0) = (1, \cdots)'$ and $a_m^* = 0$ is the smallest root of $\mathbf{B}^*$, it follows that $\mathbf{L}$ may be a linear combination of the independent solutions corresponding to the latent roots a_r^* ($r = 0, \cdots, m$) (which differ by integers in the same manner as the a_r (Section 5)). In any case, however,

$$\mathbf{L}(s) = \textstyle\sum_{r=0}^{j^*} \mathbf{l}_r s^r + o(s^{j^*}), \ (\mathbf{l}_r = (l_{0r}, \cdots, l_{mr})') \tag{7.3}$$

where j^* is the largest integer such that

$$j^* < a_{m-1}^* = \tfrac{1}{2}(n_2 - m + 1). \tag{7.4}$$

Thus $j^* = j$, and all moments of T up to the jth exist by a standard result on Laplace transforms, in agreement with Constantine's result.

The matrix $\mathbf{B}^*$ may be reduced to diagonal form by the transformation

$$\mathbf{L} = \mathbf{\Pi}\mathbf{L}^*, \tag{7.5}$$

where

$$\mathbf{\Pi} = \{\pi_{ik}\}, \qquad \pi_{ik} = \binom{k}{i}(n_1 + n_2 - i)![(m + n_2 - i - k)!]^{-1} \tag{7.6}$$

It may also be shown that

$$\mathbf{\Pi}^{-1} = \{\pi_{ik}^*\}, \qquad \pi_{ik}^* = (-1)^{i+k}\binom{k}{i} \tag{7.7}$$
$$\cdot(m + n_2 - i - k - 1)![(n_1 + n_2 - k)!]^{-1}(m + n_2 - 2i),$$

and that

$$\mathbf{\Pi}^{-1}\mathbf{A}\mathbf{\Pi} = \mathbf{\Sigma} = \{(\lambda_0^*, \cdots, \lambda_{m-1}^*), (\mu_0^*, \cdots, \mu_m^*), (\nu_1^*, \cdots, \nu_m^*)\}, \tag{7.8}$$

where

$$\lambda_r^* = (r + 1)(n_2 - r - 1)(m + n_2 - r)(n_1 - m + 1 + r)$$
$$\cdot[(m + n_2 - 2r - 2)(m + n_2 - 2r - 1)]^{-1},$$
$$\mu_r^* = [-r(m + n_2 - r)(m + 2n_1 + n_2 + 1) + m(n_1 + n_2)(m + n_2 + 1)] \tag{7.9}$$
$$\cdot[(m + n_2 - 2r - 1)(m + n_2 - 2r + 1)]^{-1},$$
$$\nu_r^* = -(m - r + 1)(n_1 + n_2 - r + 1)[(m + n_2 - 2r + 1)$$
$$\cdot(m + n_2 - 2r + 2)]^{-1}.$$

The differential equation (7.1) reduces to

$$d\mathbf{L}^*/ds = [s^{-1} \operatorname{diag}\{a_0^*, \cdots, a_m^*\} + \mathbf{\Sigma}]\mathbf{L}^*. \tag{7.10}$$

Taking

$$\mathbf{L}^* = \sum_{r=0}^{j} \mathbf{l}_r{}^* s^r + o(s^j), \tag{7.11}$$

the following recurrence relations are obtained for the $\mathbf{l}_r{}^* = (l_{0r}^*, \cdots, l_{mr}^*)'$

$$(7.12)\qquad \begin{aligned} l_0{}^* &= n_2![(n_1+n_2)!]^{-1}(0, \cdots, 0, 1)', \\ l_r{}^* &= \operatorname{diag}\{(r-a_0{}^*)^{-1}, \cdots, (r-a_m{}^*)^{-1}\}\boldsymbol{\Sigma} l_{r-1}^*, \qquad (r = 1, \cdots, j). \end{aligned}$$

The moments of T are then given by

$$\mathcal{E}(T^r) = (-1)^r r!\, l_{0r} = (-1)^r r!(n_1+n_2)! \sum_{k=0}^{m} l_{kr}^*/(m+n_2-k)!. \tag{7.13}$$

In particular, taking $r = 1$,

$$\mathbf{l}_1{}^* = n_2![(n_1+n_2)!]^{-1}(0, \cdots, 0, -2\lambda_{m-1}^*/(n_2-m-1), \mu_m{}^*)', \tag{7.14}$$

whence it is easily found that

$$\mathcal{E}(T) = mn_1/(n_2-m-1). \tag{7.15}$$

(Constantine, loc. cit).

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