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ABSTRACTS OF PAPERS

(Abstracts of papers to be presented at the Annual meeting, Madison, Wisconsin, August 26–30, 1968. Additional abstracts appeared in the June and April issues.)

4. Trucated bivariate Poisson distributions.
MUNIR AHMAD and MASASHI OKAMOTO, Iowa State University.

The truncated univariate Poisson distribution $f_X(x) = p(x)/P(T)$, $x \varepsilon T$, where T is a subset of nonnegative integers, $p(x) = e^{-\theta}\theta^x/x!$ and $P(T) = \sum_{x \varepsilon T} [p(x)]$, has been studied by several authors by imposing different kinds of restrictions on the range of X. In this paper some of the special cases of the bivariate correlated Poisson (BCP) distribution have been discussed and their parameters estimated by the method of maximum likelihood and the method of moments. It has also been shown that in a bivariate distribution the marginal truncated (MT) distribution is equivalent to the truncated marginal (TM) distribution if and only if the variates are independent. The estimation of parameters of the truncated BCP distribution has been investigated under various types of truncation. Various recurrence relations of the truncated BCP distribution have been obtained. (Received 29 April 1968.)

5. The central limit theorem in R^k, $k \geqq 1$, and normal approximation to the probabilities of Borel sets.
R. N. BHATTACHARYA, University of California, Berkeley.

Let $X^{(n)}$, $n = 1, 2, \cdots$, be a sequence of independent random vectors in R^k, $k \geqq 1$, each with zero mean vector, and covariance matrix I, the $k \times k$ identity matrix. Let $\beta_s^{(r)}$ denote the sum of the absolute sth moments of the co-ordinates of $X^{(r)}$, and $\beta_s(n) = (\sum_{r=1}^n \beta_s^{(r)})/n$. If P_n is the distribution of $(X^{(1)} + \cdots + X^{(n)})/n^{\frac{1}{2}}$, Φ is the standard k-dimensional normal measure, and B_n, $n = 1, 2, \cdots$, is an arbitrary sequence of Borel sets in R^k, then $|P_n(B_n) - \Phi(B_n)| \leqq [a_1(k, \delta)\beta_{3+\delta}^{3(1+\delta)/(3+\delta)}(n) + (a_2(k)/\epsilon_0)\beta_{3+\delta}^{3/(3+\delta)}(n)]/n^{\frac{1}{2}} + a_3(k)g_n \cdot (a_4(k, \delta)\beta_{3+\delta}^{3/(3+\delta)}(n)n^{-\frac{1}{2}})$, where a_1, a_2, a_3, a_4, as well as b's appearing below, are constants depending only on the indicated arguments, $\delta > 0$, ϵ_0 is a positive constant, and $g_n(\epsilon) = \sup_x (\Phi(\partial(B_n + x))^\epsilon)$, $0 < \epsilon < \epsilon_0$; here $B_n + x$ is the translate of B_n by x, $\partial(B_n + x)$ is the boundary of $B_n + x$, and $(\partial(B_n + x))^\epsilon$ is the ϵ-neighborhood of $\partial(B_n + x)$. Also,

$$|P_n(B_n) - \Phi(B_n)| \leqq [b_1(k, \delta)\beta_{3+\delta}^{6/(3+\delta)}(n) + (b_2(k)/\epsilon_0)\beta_{3+\delta}^{3/(3+\delta)}(n)]/n^{\frac{1}{2}} + h_n([b_3(k)\beta_{3+\delta}^{3/(3+\delta)}(n) \log (n + 1)]/n^{\frac{1}{2}}),$$

where $h_n(\epsilon) = \Phi(\partial(B_n))^\epsilon)$, $0 < \epsilon < \epsilon_0$. These extend some previous results of the author (Ph.D. thesis (1967), Univ. of Chicago, and *Bull. Am. Math. Soc.* **74,** (1968) 285–287). Of particular interest are the cases, (a) $B_n = B$ for all n, where B is a Φ-continuity set, and (b) $\Phi(\partial B_n)$ converges to zero. These results are proved by the same methods as used in the first reference cited above. (Received 10 April 1968.)

6. The central limit theorem in R^k, $k \geqq 1$, and asymptotic expansion for the probability of a Borel set.
R. N. BHATTACHARYA, University of California, Berkeley.

Let $X^{(n)}$, $n = 1, 2, \cdots$, be a sequence of independent random vectors in R^k, $k \geqq 1$, each with mean vector zero, and covariance matrix I, the $k \times k$ identity matrix. Also,

$f^{(r)}$ is the characteristic function of $X^{(r)}$, and $\beta_s^{(r)}$ is the sum of the absolute sth moments of the co-ordinates of $X^{(r)}$. Suppose (i) $\sup_r \beta_s^{(r)} < \infty$ for some integer $s \geqq 3$, and (ii) $\sup_r \lim \sup_{|t|\to\infty} |f^{(r)}(t)| < 1$. If P_n is the distribution of $(X^{(1)} + \cdots + X^{(n)})/n^{\frac{1}{2}}$, Φ is the k-dimensional standard normal distribution, and μ_n is the usual Edgeworth expansion of P_n in terms of Φ and its derivatives, then $(P_n - \mu_n)(A) = o(\log n)^{k/2} n^{-(s-2)/2}$, uniformly over all Borel sets A such that $\Phi((\partial A)^\epsilon) \leqq d\epsilon$, $0 < \epsilon < \epsilon_0$, d and ϵ_0 being two given (arbitrary) positive constants; for any set B, ∂B is the boundary of B, and B^ϵ is the ϵ-neighborhood of B. More generally, if B_n, $n = 1, 2, \cdots$, is an arbitrary sequence of Borel sets in R^k, then $|(P_n - \mu_n)(B_n)| \leqq c_1(n)\Lambda(B_n^{\epsilon_n})n^{-(s-2)/2} + |\mu_n|((\partial B_n)^{2\epsilon_n})$, where $c_1(n) \to 0$ as $n \to \infty$, Λ is Lebesgue measure in R^k, $\epsilon_n = 2a^n$ for an appropriate constant a, $0 < a < 1$, $|\mu_n|$ denotes the total variation of μ_n. One may replace $\Lambda(B_n^{\epsilon_n})$ above by $\min\{\Lambda(B_n^{\epsilon_n}), [\log(n+1)]^{k/2}\}$. These results extend, and are proved much the same way as some previous results of the author (Ph.D. thesis (1967), Univ. of Chicago, and *Bull. Am. Math. Soc.* **74**, (1968) 285–287). (Received 10 April 1968.)

7. Estimation of a certain functional of a probability density function. G. K. BHATTACHARYYA and G. G. ROUSSAS, University of Wisconsin.

A key functional which occurs in the expressions for the asymptotic efficiency of many nonparametric tests and rank estimates is $\Delta(F) = \int_{-\infty}^{\infty} f^2(x)\,dx$, where F is the unknown cumulative distribution function of the population with density $f(x)$. In the present work, Parzen's method of estimation of density function is used to construct a class of estimates for $\Delta(F)$ based on a single sample from $F(x - \theta)$, where θ is an unknown location parameter. The procedure is extended to the case of several samples. Some large sample properties, like asymptotic unbiasedness, consistency in the mean square, etc. are investigated for the proposed class of estimates. (Received 31 May 1968.)

8. Characterization of a class of distributions. K. C. CHANDA, University of Florida.

Let $X_1, \cdots, X_n$ be a sequence of n iid random variables and let $g(x)$ be a function of x such that $\sum_{i=1}^n \{g(X_i) - g(\bar{X})\}$ is distributed independently of $\bar{X} = \sum_{i=1}^n X_i/n$. If $g(x) = x^2$ it is well known that X_1 has a normal distribution. It has been demonstrated by the author that if $g(x)$ be a rational function of the form $p(x)/q(x)$ where $p(x)$ is a polynomial of degree $\leqq 2$ and $q(x)$ is a polynomial of degree $\leqq 1$, then depending on the nature of the coefficients of $p(x)$ and $q(x)$ the distribution of X_1 will be either normal, or inverse normal. The author is presently investigating the situation where $g(x)$ is a rational function, being a ratio of two polynomials of arbitrary degrees and will present his report at a subsequent stage. (Received 9 May 1968).

9. On the exact distributions of Votaw's criteria for testing compound symmetry of a covariance. P. C. CONSUL, University of Calgary.

Consider a sample $(x_{1j}, x_{2j}, \cdots, x_{nj}, j = 1, 2, \cdots, p+q)$ of size $n \geqq p + q$ from the $(p+q)$ stochastic variables which are distributed normally. Defining the covariance $|S|$, where S is the sum of products matrix which may be partitioned into S_{aa}, $S_{aa'}$, S_{bb}, $S_{bb'}$ and S_{ab}, Votaw (1948) defined the likelihood ratio statistic for testing the compound symmetry of the matrix by $L = |S|\,[\{S_{aa} + (p-1)S_{aa'})(S_{bb} + (q-1)S_{bb'}) - pqS_{ab}^2\} \cdot (S_{aa} - S_{aa'})^{p-1}(S_{bb} - S_{bb'})^{q-1}]$ and derived the expected value $E(L^t/H)$. Roy (1951) proved that the expected value could be expressed as the product of a number of gamma quotients and that the distribution could be obtained in the form of an infinite series. Consul (1966), (1967) has obtained the exact distributions of some likelihood criteria by applying the inversion theorem and operational calculus. The same method has been used in this paper to obtain the exact distributions of Votaw's likelihood criteria for (i) $p = q = 2$, (ii) $p = 3$, $q = 2$, (iii) $p = q = 3$, (iv) $p = 5$, $q = 2$ and (v) $p = 5$, $q = 3$. (Received 16 May 1968).

10. On the asymptotic theory of least squares estimation in segmented regression: identified case (preliminary report). PAUL I. FEDER and DAVID L. SYLWESTER, Yale University and University of Vermont.

For fixed n consider the regression model $X_{ni} = \mu(t_{ni}\ ; \theta) + e_{ni}\ i = 1, \cdots, n, 0 \leqq t_{ni} \leqq \cdots \leqq t_{nn} \leqq 1$. The e_{ni} are iid rv's with mean zero, variance σ^2, and finite $2 + \delta$ moment. The function $\mu(t, \theta)$ is continuous, but has different parametric forms on intervals $[\tau_{j-1}, \tau_j)$, $0 < \tau_1 < \cdots < \tau_{r-1} \leqq 1$. It is assumed that $\mu(t; \theta)$ conforms to a linear model within each segment $[\tau_{j-1}, \tau_j)$ and that the τ_j's are unknown. Since the residual sum of squares function may not possess any derivatives with respect to the τ's, one cannot directly apply classical techniques to derive asymptopic distribution theory (as $n \rightarrow \infty$) for the least squares estimates and Wilks-Chernoff type test statistics. In this paper it is shown that by deleting relatively few observations, the problem can be transformed into a new problem in which classical methods are applicable. Asymptotic distribution theory is discussed for the new problem by classical techniques and it is shown that the results are also valid in the original problem. Results analagous to the usual normal theory and χ^2-type distributions are derived. (Received 27 May 1968.)

11. Structural analysis for the first order autoregressive stochastic process (preliminary report). M. SAFIUL HAQ, University of Western Ontario.

In a first order autoregressive stochastic process, the autocorrelation parameter ρ can be treated as a parameter of the error variable of a structural model and the model can be treated as a conditional structural model (Fraser, *Amer. Math. Soc.* (1967) 1456–1465). Or alternately the error variable can be made parameter-free by a suitable transformation depending on ρ and the resultant transformed model can be treated as a structural model. In both the cases the orbit of the responses or the transformed responses depend on the auto-correlation parameter ρ. For known value of ρ the inference concerning the structural parameter θ is based on the conditional structural distribution of the parameter θ. For unknown ρ, a mode of inference concerning the parameter ρ has been studied based on the residual likelihood for the orbital statistic. It has been found that the inference depend on the statistic $r = \sum (x_{i-1} - \bar{x})(x_i - \bar{x})/\sum (x_i - \bar{x})^2$ where $x_1, \cdots, x_n$, are the responses and $\bar{x}$ the sample mean, and the error variable follows normal distribution and the model is a simple location model. The extension of the results for the generalised location model has also been studied. (Received 31 May 1968.)

12. Test for several regression equations. E. H. INSELMANN, Frankford Arsenal, Philadelphia.

This paper deals with a generalization of C. G. Chow's test for comparison of two regression equations. The extension contained herein involves the comparison of k regression equations. The derivation of the test is simplified by using lemmas due to F. M. Fisher. The test statistic for the comparison is a chi square distributed variable similar to that of Chow. A test is also determined to compare subsets of coefficients. (Received 16 May 1968.)

13. Maximum likelihood estimation of multivariate covariance components.

JEROME KLOTZ and JOSEPH PUTTER, University of Wisconsin, Madison.

Maximum-likelihood estimators of multivariate components of variance and covariance are derived for the balanced one-way layout. The model, in terms of p-variate observation row vectors $\mathbf{x}_{jk}$, is $\mathbf{x}_{jk} = \boldsymbol{\mu} + \mathbf{b}_j + \mathbf{w}_{jk}$ $(j = 1, \cdots, J;\ k = 1, \cdots, K)$, where $\boldsymbol{\mu}$ is a fixed mean vector, and $\mathbf{b}_j:N(\mathbf{0}, \mathbf{\Sigma}_b)$ and $\mathbf{w}_{jk}:N(\mathbf{0}, \mathbf{\Sigma}_w)$ are independent. Denoting $\mathbf{x}.. = \Sigma_j\Sigma_k\mathbf{x}_{jk}/JK$, $\mathbf{x}_{j\cdot} = \Sigma_k\mathbf{x}_{jk}/K$, $\mathbf{S}_b = K\Sigma_j(\mathbf{x}_{j\cdot} - \mathbf{x}..)'(\mathbf{x}_{j\cdot} - \mathbf{x}..)$, $\mathbf{S}_w = \Sigma_j\Sigma_k(\mathbf{x}_{jk} - \mathbf{x}_{j\cdot})'(\mathbf{x}_{jk} - \mathbf{x}_{j\cdot})$ and $\mathbf{S}_t = \mathbf{S}_b + \mathbf{S}_w$, let $\mathbf{C}$ be a matrix such that $\mathbf{S}_t/JK = \mathbf{CC}'$ and $\mathbf{S}_b/JK = \mathbf{C}$ diag $(d_1, \cdots, d_p)\mathbf{C}'$. The maximum-likelihood estimators of the parameters are $\hat{\boldsymbol{\mu}} = \mathbf{x}..$, $\hat{\mathbf{\Sigma}}_b = \mathbf{C}\,\text{diag}(e_1, \cdots, e_p)\mathbf{C}'$ and $\hat{\mathbf{\Sigma}}_w = \mathbf{S}_t/JK - \hat{\mathbf{\Sigma}}_b$, where $e_m = \max(0, Kd_m - 1)/(K - 1)$. (Received 12 May 1968.)

14. Distribution of products and quotients of independent Bessel function random variables.

SAMUEL KOTZ and R. SRINIVASAN, Temple University.

A number of papers have been devoted in recent years to the distribution of products and quotients of independent random variables. Comprehensive bibliography till 1963 is given in a monograph by J. D. Donahue (1964), (*Clearinghouse for Federal Scientific and Technical Information, Department of Commerce*, AD-60-3667). The most recent papers by M. D. Springer and W. E. Thompson (1966), *SIAM J. Appl. Math.* **14** 511–526, Z. A. Lomnicki (1967) *J. Roy. Statist. Soc. Ser. B.* **29** 513–523, and A. M. Mathai and R. K. Saxena (1966), *Metrika* **11** 127–132, solve the problem for classes of distributions including normal and gamma distributions. In the present paper, using the technique of the Mellin transforms, explicit expressions are derived for the distributions of the product (and the quotient) of independent Bessel function random variables. As particular cases, distributions of the product (and the quotient) of exponential, folded Gaussian, Rayleigh, chi-squared, gamma, randomized gamma, non-central chi-squared, and several other positive independent random variables are obtained. (Received 1 May 1968.)

15. W statistic in covariate discriminant analysis.

AHMED ZOGO MEMON and MASASHI OKAMOTO, Iowa State University.

Assume that the samples $(x_{11}, \cdots, x_{1N_1})$ and $(x_{21}, \cdots, x_{2N_2})$ drawn from the p-variate normal populations $\pi_1:N(\mu_1, \Sigma)$ and $\pi_2:N(\mu_2, \Sigma)$ are independent and that the multivariate observation x comes from one of these populations. Let $\bar{x}_1$, $\bar{x}_2$ and S be the minimum variance unbiased estimates of the unknown parameters μ_1, μ_2 and Σ. Cochran and Bliss (1948) and Cochran (1964) consider the situation in discriminant analysis where the discriminators x are correlated with other normally distributed characters y having unspecified parameters. They use this additional information in modifying the Anderson classification statistic $W = [x - \frac{1}{2}(\bar{x}_1 + \bar{x}_2)]'S^{-1}(\bar{x}_1 - \bar{x}_2)$ by changing x to $x - \hat{\beta}y$ in W, where $\hat{\beta}$ is the estimate of regression matrix of x on y. The procedure of classification according to the resulting discriminant function is to classify x into π_1 if its value $\geqq 0$ and to π_2 if its value < 0. This paper investigates the effect of covariates on discrimination of the observation by deriving the probabilities of misclassification which arise due to the use of the modified W criterion. (Received 29 April 1968.)

16. On the distribution of the Z statistic in discriminant analysis.

AHMED ZOGO MEMON and MASASHI OKAMOTO, Iowa State University.

The statistic $Z = (N_1/(N_1 + 1))(x - \bar{x}_1)'S^{-1}(x - \bar{x}_1) - (N_2/(N_2 + 1))(x - \bar{x}_2)'S^{-1}\cdot(x - \bar{x}_2)$ is a criterion proposed by Kudô (1959) and John (1960) in discriminant analysis for classification of a multivariate observation x into its correct population when the observation is known to have come from one of the two p-variate normal populations $\pi_1:N(\mu_1, \Sigma)$ and $\pi_2:N(\mu_2, \Sigma)$. The samples $(x_{11}, \cdots, x_{1N_1})$ and $(x_{21}, \cdots, x_{2N_2})$ drawn from π_1 and π_2 are assumed to be independent of each other. $\bar{x}_1$, $\bar{x}_2$ and S are minimum variance unbiased estimates of the unknown parameters μ_1, μ_2 and Σ. The sampling distribution of Z for small samples appears to be extremely complicated for the purpose of numerical use. A large sample approach has therefore been employed in this paper in determining an asymptotic expansion with respect to N_1^{-1}, N_2^{-1} and n^{-1} where n is the total number of degrees of freedom used in estimating the covariance matrix Σ. Further, the two kinds of probabilities of misclassification which arise due to the use of this criterion under the procedure of assigning the observation x to π_1 if $Z \leqq 0$ and to π_2 if $Z > 0$ are also derived. (Received 29 April 1968.)

17. The distribution of weighted linear combinations of cell frequency counts when the cell probabilties are not equal. C. J. PARK, University of Nebraska.

Assume that n objects have been randomly distributed into N cells. Let p_i denote the probability that an object falls in the ith cell. Let s_i denote the number of cells with i entries. It is shown that the asymptotic distribution of weighted linear combinations of s_i's, i.e. $W = \sum_{i=0}^{n} w_i s_i$ is asymptotically normal when n and N tend to infinity with $n/N \rightarrow \alpha$, $0 < \alpha < \infty$, and w_i's and p_i's satisfy condition that $\sum_{\nu=0}^{\infty} (np_k)^{\nu}(\nu!)^{-1}w^s\nu^t$ converges uniformly and absolutely for every fixed $k = 1, 2, \cdots$ and $s, t = 0, 1, 2, \cdots$ in a neighborhood of α_k, $0 < \alpha_k < \infty$ where $\lim_{n\rightarrow\infty} np_k = \alpha_k$. In the proof a generating function is used. (Received 3 May 1968.)

18. The distribution of weighted linear combinations of cell theory frequency counts: two samples from continuous distributions. C. J. PARK, University of Nebraska. (By title.)

Let $X_{(1)}, X_{(2)}, \cdots, X_{(n)}$ be the order statistics of a random sample from a continuous distribution $F(x)$. Let $I_j = (X_{(j-1)} - X_{(j)}]$, $j = 1, 2, \cdots, n+1$, with $X_{(0)} = -\infty$, $X_{(n+1)} = +\infty$. Let $y_1, y_2, \cdots, y_m$ be a random sample from a continuous distribution $G(y)$, and s_k denote the number of I_j's with k y_i's. Under the assumption $F(x) \equiv G(x)$, it is shown that the weighted linear combination of s_i's, i.e., $W = \sum_{i=0}^{m} w_i s_i$, is asymptotically normal when m and n tend to infinity with $m/n \rightarrow \alpha$, $0 < \alpha < \infty$, and w_i's satisfy condition that $\sum_{\nu=0}^{\infty} (x/(1+x))^{\nu} w^s \nu^t$ converges uniformly and absolutely for every fixed s and t in a neighborhood of α. (Received 3 May 1968.)

19. Some striking properties of binomial and beta moments. MORRIS SKININSKY, Brookhaven National Laboratory.

Let M_n $(n = 1, 2, \cdots)$ denote the convex body of n-tuples $(c_1, \cdots, c_n)$ with $c_i = \int_{[0,1]} x^i \, d\sigma(x)$, $i = 1, 2, \cdots, n$, where σ varies over the class of all probability measures on the Borel subsets of the unit interval. For the moment sequence $(c_1, c_2, \cdots)$ corresponding to a σ in this class, write $\nu_n(c_1, c_2, \cdots) = c_n$,

$$\nu_n^{\pm}(c_1, c_2, \cdots) = {}^{\max}_{\min} \{d: (c_1, \cdots c_{n-1}, d) \in M_n\}.$$

Take $R_n = \nu_n^+ - \nu_n^-$, $p_n = (\nu_n - \nu_n^-)/R_n$. THEOREM. *Let N be a positive integer,* $0 \leqq \theta \leqq 1$. $c_i^*(N, \theta) = \sum_{j=0}^{N} (j/N)^i \theta^j (1-\theta)^{N-j}$, $i = 1, 2, \cdots$. $\underset{\sim}{c}_n^*(N, \theta) = (c_1^*(N, \theta), \cdots, c_n^*(N, \theta))$. *Then* $p_{2i-1}(\underset{\sim}{c}_{2i-1}^*(N, \theta)) = \theta$, $p_{2i}(\underset{\sim}{c}_{2i}^*(N, \theta)) = i/N$, $i = 1, 2, \cdots, N$. $R_{2m}(\underset{\sim}{c}_{2m-1}^*(N, \theta)) = N^{-2m+1}\binom{N}{m} m!(m-1)![\theta(1-\theta)]^m$, $R_{2m+1}(\underset{\sim}{c}_{2m+1}^*(N, \theta)) = N^{-2m+1}\binom{N}{m+1} m!(m+1)![\theta(1-\theta)]^m$,

$m = 1, 2, \cdots, N$. The process of proof of this theorem and the result itself combined with previously obtained results ("Extreme nth Moments for Distributions on [0, 1] and the Inverse of a Moment Space Map" to appear December 1968, *J. Appl. Prob.*) upon which it is based yield several families of identities involving Stirling numbers of the second kind. THEOREM. *Let* $\hat{c}_i(a, b)$ *denote the ith moment of a beta distribution with parameters* $a, b > 0$. $\hat{c}_n(a, b) = (\hat{c}_1(a, b), \cdots, \hat{c}_n(a, b))$. *Then* $p_{2i}(\hat{c}_{2i}(a, b)) = i/(a + b + 2i - 1)$, $p_{2i+1}(\hat{c}_{2i+1}(a, b)) = (a + i)/(a + b + 2i)$, $i = 0, 1, \cdots$. Two general theorems are given. The first exhibits the connection between two $\{p_n\}$ sequences whose corresponding distributions on [0, 1] are symmetrically related. The second exhibits the invariance of the $\{p_n\}$ sequence under the standard 1–1 mapping that takes the class of all distributions on [0, 1] onto the class of all distributions on some finite interval. (Received 8 April 1968.)

20. Asymptotic expansions of the distributions of the likelihood ratio criteria for covariance matrix. NARIAKI SUGIURA, University of North Carolina.

Asymptotic expansions of the non-null distributions of the likelihood ratio tests based on a random sample from multivariate normal population for (1) equality of covariance matrix to a given matrix, (2) equality of mean vector and covariance matrix to a given vector and a given matrix, (3) sphericity, are derived with those of null distributions for problems (1) and (2). Non-null distribution of the likelihood ratio test for (4) equality of several covariance matrices is also obtained. Unbiasedness of these test criteria were discussed in the previous paper (Abstract, *Ann. Math. Statist.* **38**, (1967) 1937), which will appear in this Journal. Monotonicity property of the power function of the modified likelihood ratio test for the problem (1) was established by H. Nagao (*J. Sci. Hiroshima Univ.* **31** (1967) 147–150). (Received 16 May 1968.)

21. When do minimum variance estimators coincide? DONALD H. THOMAS, General Motors Research Laboratories, Warren. (Introduced by Andre G. Laurent.)

Considerable interest has recently been shown in the question of when ordinary least squares estimators (OLSE's) coincide with best linear unbiased estimators (BLUE's) in the linear model $Y = X\beta + e$, where Y is an $n \times 1$ vector of observables, X a known $n \times p$ matrix of rank $r \leqq p$, β a $p \times 1$ vector of parameters, and e an $n \times 1$ vector of "errors" with $E(e) = 0$ and $E(ee^1) = 6^2V$. This paper presents necessary and sufficient conditions for BLUE's corresponding to two different assumptions on V to coincide. For example, if the two assumptions are $V = V_1$ and $V = V_2$, respectively, (where V_1, V_2 may be any two positive definite symmetric matrices) then a necessary and sufficient condition for all BLUE's under the first assumption to coincide with the corresponding BLUE's under the second assumption is that the column space of X be spanned by r linearly independent eigenvectors of $V_2V_1^{-1}$ (or $V_1V_2^{-1}$), alternatively, that the column space of $V_2V^{-1}X$ is identical with the column space of X. When $V_1 = I$ (or $V_2 = I$) these conditions reduce to known ones concerning the coincidence of OLSE's with BLUE's. Hence, design matrices can be constructed which lead to minimum variance estimators independent of which hypothesis on the covariance matrix is assumed. (Received 8 April 1968.)

22. Stochastic absolute stability of stochastic automatic systems. CHRIS P. TSOKOS, University of Rhode Island. (Introduced by Edward J. Carney.)

Consider the stochastic automatic system

(1) $\dot{x}(t; \omega) = A(\omega)x(t; \omega) + b(\omega)\varphi(\sigma(t; \omega))$ with $\sigma(t; \omega) = \langle C(t; \omega)X(t; \omega)\rangle$;

where $X(t;\omega)$ is an $n \times 1$ vector whose elements are random variables, $A(\omega)$ is an $n \times n$ matrix whose elements are measurable functions, $b(\omega)$ and $C(t;\omega)$ are $n \times 1$ vectors whose elements are measurable functions, $\sigma(t;\omega)$ and $f(t;\omega)$ are scalar functions and $\langle C\, X\rangle$ denotes the scalar product in the Euclidean space. This system can be easily reduced into the stochastic integral equation of the convolution type of the form $\sigma(t;\omega) = h(t;\omega) + \int_0^t k(t-\tau;\omega)\varphi(\sigma(\tau;\omega))\,d\tau$. The objective of this paper is to investigate the stochastic absolute stability properties of system (1) utilizing V. M. Popov's frequency response method [V. M. Popov, "On the absolute stability of nonlinear control systems", *Avtom. i Telemekh.*, **22**, 8 (1961)]. The results in this paper are more general than the recent work of T. Morozan, [Morozan, T., The method of V. M. Popov for control systems with random parameters. *J. Math. Anal. Appl.* **16** (1966) 201–215]. (Received 3 May 1968.)

23. Density estimation by orthogonal series. GEOFFRY S. WATSON, The Johns Hopkins University. (By title)

Given a random sample $x_1 \cdots x_n$ from the density $f(x) = \sum \alpha_m\varphi_m(x)$ where $\{\varphi_m(x)\}$ is an orthogonal basis, the estimator $f_n^*(x) = \sum \lambda_m(n)a_m\varphi_m(x)$ where $a_m = n^{-1}\sum_{k=1}^n \varphi_m(x_k)$ is suggested. $f_n^*(x)$ will be a minimum integrated mean square error estimator in its class

$$\lambda_m(n) = \alpha^2/\alpha_m^2 + \text{var}\,(\varphi_m(x))/n.$$

These estimators are related to the kernel estimators discussed by Watson and Leadbetter (*Ann. Math. Statist.* (1963) **34**, 480–491). (Received 31 May 1968.)

24. Sufficient conditions for the almost admissibility of formal Bayes estimators under squared error loss. JAMES B. ZIDEK, University of British Columbia.

Explicit conditions for the admissibility of formal Bayes estimators (which are obtained like Bayes estimators, but respect to a σ-finite prior, Π) are known in only a few special cases in which are prescribed the form of the underlying distributions and the function of the parameter being estimated. The author has obtained sufficient conditions for the almost admissibility (admissibility a.e. [Π]) of such estimators in the problem freed of these restrictions where the parameter space is a subinterval (joining θ_1 and θ_2, say) of the real line and loss is squared error. In essence, these conditions assert the almost admissibility of the estimator in question provided $\int d\theta/[\pi(\theta)h(\theta)] = \infty$ if $\int \pi(\theta)\rho(\theta)\,d\theta = \infty$ whenever these integrals are over (c, θ_2) or over (θ_1, c), where $c \in (\theta_1, \theta_2)$, ρ is the risk of the estimator, Π has density π, and h is a function summarizing the relevant aspects of the problem. Although h depends on Π, in special cases considered, it is bounded by a function which is independent of the prior. These special cases include that of Karlin, "Admissibility for estimation with quadratic loss", *Ann. Math. Statist.* **29**, 411–415, of the estimation of the mean of the one dimensional exponential family (when, as is usually the case, his estimator is formal Bayes), where the bound is 1, and that of estimating a single unknown location parameter. These results are a refinement, in the case considered, of an argument of Stein, "Approximation of improper prior measures by prior probability measures", *Bernoulli, Bayes, Laplace, Anniversary Volume* (1965) Springer-Verlag, New York. (Received 24 May 1968.)

(Abstracts of papers to be presented at the European meeting, Amsterdam, Netherlands, September 2–7, 1968. Additional abstracts appeared in the April and June issues.)

5. Bayesian stratified two-phase sampling results: k characteristics. NORMAN DRAPER and IRWIN GUTTMAN, University of Wisconsin.

The authors have obtained some results concerning the optimum allocation of sampling effort among k strata at the second phase of a two-phase sampling procedure, using information obtained from the first phase (see Draper and Guttman, *Biometrika* **55** (1968)). One variable (or characteristic) was involved. Two different approaches were employed: A Bayesian posterior analysis and a Bayesian preposterior analysis. Two different allocation methods were derived and illustrated with some numerical examples, for cases where some or all of the nuisance parameters were unknown. In this paper we extend our results above to the case of k characteristics. We propose three criterions and use them to solve the allocation problem, again using both posterior and preposterior analysis. The cases of stratum weights known and unknown is also discussed. (Received 12 May 1968.)

6. Automatic classification of samples using an electronic computer (preliminary report). ERWIN FABER, Deutsches Rechenzentrum, Darmstadt.

By automatic classification we understand the division of a sample into g different groups using an automatic procedure. Let a sample of m objects with n different variables be given. The question is, from how many populations the sample is taken, and how many elements belong to a certain population. A heuristic method, based on a procedure of P. Ihm, will be presented which separates the sample into g different groups. This is accomplished with the aid of an appropriate function which is attached to every element of the sample space. The maxima of the sum of these functions characterize the groups. The procedure to trace out the sample groups requires extensive calculations, therefore a FORTRAN-program for a digital computer has been written. To reduce the number of variables of a problem, low-dimensional sub-spaces are used that change from iteration to iteration. Experiences with this program will be described. (Received 3 May 1968.)

7. Some numerical results to R. Borges' approximation of the binomial distribution. FRIEDRICH GEBHARDT, Deutsches Rechenzentrum, Darmstadt.

R. Borges has shown that a certain beta-transform of the binomial distribution yields a normal distribution with an error term $O(1/n)$ while the commonly used transformations (arc sin-transform, log-transform, no transformation) have error terms $O(n^{-\frac{1}{2}})$ (except for $p = 0.5$ with $O(1/n)$). Numerical calculations show the superiority of Borges' approximation even for $p = 0.5$. For large n and p not too small, it is also better than the Camp-Paulson approximation. For given n and $p \leqq 0.5$, the maximal error e is essentially a function of np (for $np \geqq 5$, $e \approx 0.012/np$). A table is computed to facilitate the beta-transformation. (Received 3 May 1968.)

8. Asymptotic properties of the optimal restricted Bayesian double sampling plan. SØREN JOHANSEN, Institute for Matematisk Statistik, H. C. Ørsted Institut, Copenhagen.

We consider double sampling plans for attributes with specified linear cost function and prior distribution, and define the optimal plan, as the one that minimises the expected regret function R. Under suitable conditions on the prior distribution and the cost function, we prove that R is of the order of $N^{2/5}(\ln N)^{1/5}$, where N, the batch size, tends to infinity. (Received 3 May 1968.)

9. On estimating the accumulative distribution. MOHAMED A. H. TAHA, University of Alexandria.

Having N observations of a random variable X, the emperical distribution function $F_N(x) = i/N$ where i is the number of observations less than or equal to x, is used mostly to estimate the unknown distribution function $F(x)$. Two minimax procedures are used to estimate this function, and both of them lead to the estimate $S_N(x) = (2i + 1)/(2N + 2)$. Some of the optimal properties of these estimate are discussed. (Received 31 May 1968.)

(Abstracts of papers not connected with any meeting of the Institute.)

1. Continuous time Markovian sequential control processes. S. S. CHITGOPEKAR, The Florida State University.

We consider a stochastic system with a finite state space and a finite action space. Between actions, the waiting time to transition is a random variable with a continuous distribution function depending only on the current state and the action taken. There are positive costs of taking actions and the system earns at a rate depending upon the state of the system and the action taken. In contrast to R. A. Howard [*Proc. Internat. Statist. Inst.* (1963) 625–652], we allow actions to be taken between transitions. A policy for which there is a positive probability of an action between transitions involves "hesitation". For any policy S, the criterion of interest is $I(S)$, given by

$$I(S) = \liminf_{N\to\infty} \sum_{n=1}^{N} E(i_n(S))/\sum_{n=1}^{N} E(T_n(S))$$

where $i_n(S)$ is the income earned under the nth action and $T_n(S)$ is the time spent under the nth action. It is shown that there exists a non-randomized stationary policy that is optimal in the class of all policies for which the actions taken form a sequence. "Hesitation" can be eliminated if the waiting time distributions are exponential. Howard's policy improvement method can be used to obtain an optimal policy. The costless actions case was reported, ibid., Abstract No. 2, 970 **38** (1967). The costly observations case will be reported later. (Received 9 May 1968.)

2. A characterization of the normal law on Hilbert space. MORRIS L. EATON and P. K. PATHAK, University of Chicago and University of Illinois.

Let $\hat{\mu}$ be the characteristic function of a probability measure μ on a real separable Hilbert space H. Suppose that $\hat{\mu}$ satisfies the functional equation $\hat{\mu}(y) = \prod_1^k [\hat{\mu}(B_i y)]^{a_i}$ where $a_i > 0$, B_i is a bounded Hermitian operator on H with a bounded inverse, and suppose there exists a λ_0 such that $0 < \lambda_0 < 1$ and $-\lambda_0 I \leqq B_i \leqq \lambda_0 I$, $i = 1, \cdots, k$ ($\leqq$ is in the sense of positive definiteness). In this paper it is shown that: (i) the characteristic function $\hat{\mu}$ is infinitely divisible; (ii) if $\sum_1^k a_i B_i^2 \geqq I$, then $\hat{\mu}$ corresponds to the normal distribution on H; (iii) if $\sum_1^k a_i \leqq 1$, then $\hat{\mu}$ corresponds to the distribution degenerate at $0 \in H$. In the special case when $k = 1$, some representation results are obtained for non-normal situations. (Received 1 April 1968.)

3. On a scheduling problem in sequential analysis. SYLVAIN EHREFELD, New York University and University of California, Berkeley.

The usual sequential decision problem of choosing between two simple hypotheses H_0 and H_1, in terms of i, i, r, v, is reconsidered when there is a time delay, assumed to have a known exponential distribution, in obtaining observations. The problem, at any time is whether to stop and choose between H_0 and H_1, or to continue and decide on how many further tests to schedule. Cost assumptions are made involving testing, time until a final decision and the usual losses due to decision errors. Bayes procedures are studied. The information, at any time, can be described by (n, π) where π is the current posterior probability of H_0 and n the number of results outstanding from tests already scheduled. It is shown that possible decision changes should be made only when test results are obtained. When the stopping rule is a SPRT procedure there is a bounded function $z(\pi)$ such that the optimal testing schedule quantity, $y(n, \pi) = \max [0, z(\pi) - n]$. In the general case, various results about the optimal stopping region, in the (n, π) plane, are derived, and it is proven that the optimal procedure terminates with probability one. Also, it is shown that there exists functions $z_1(\pi) \leqq z_2(\pi) \leqq M < \infty$ such that if (n, π) is a continuation point $y(n, \pi) = z_1(\pi) - n$ if $n \leqq z_1(\pi)$ and $y(n, \pi) = 0$ if $n \geqq z_2(\pi)$. (Received 27 May 1968.)

4. A fiducial distribution in survey-sampling. V. P. GODABME, University of Waterloo.

Let a unit i be drawn at *random* from a population of N units $i = 1, \cdots, N$, with *equal* probability of drawing for each unit. Now if X_i, $i = 1, \cdots, N$, denote the *unknown* variate values associated with the units $i = 1, \cdots, N$ respectively, our problem is to make certain inference about the *unknown* $\mathbf{X} = (X_1, \cdots, X_N)$ on the basis of the *data* (i, x), where i as before is the unit drawn and x is the observed value of X associated with the unit i. We can write the likelihood function as $\text{Prob}((i, x) \mid X_1, \cdots, X_N) = 1/N$ if $X_i = x$ and 0 otherwise, for $i = 1, \cdots, N$. (I). Thus the vector $\mathbf{X} = (X_1, \cdots, X_N)$ plays the role of a *parameter*. If nothing is known the parametric space is entire Euclidean space R_N, $\mathbf{X} \varepsilon R_N$. Now let $\bar{X}(\mathbf{X}) = \sum_1^N X_i/N$. We *assume* the parameter $\mathbf{X}$ belongs to a subset R_N^* ($R_N^* \subset R_N$) defined by $R_N^* = \{\mathbf{X}: -\infty < \bar{X}(\mathbf{X}) < \infty, X_i = \bar{X} + Y_i$, where $(Y_p, \cdots, Y_N)$ is any permutation of a *fixed* vector $(Y_1^*, \cdots, Y_N^*)\}$. (II). It is shown that the likelihood function in (I) above implies that $\text{Prob}(x = X_i) = 1/N$ $i = 1, \cdots, N$, and hence if $\mathbf{X} = (X_1, \cdots, X_N) \varepsilon R_N^*$ in (II) above, we have the *fiducial distribution* of $\bar{X}$ given the data (i, x), as $\text{Prob}(\bar{X} = x - Y_i^*) = 1/N, i = 1, \cdots, N$. (III). This *fiducial distribution* would obviously be valid *if and only if* the unit i was drawn at *random* with equal probability of drawing to all the units. On the other hand the distribution given by (III) above can also be obtained as a *Bayes posterior* on the basis of the data (i, x) when the *prior* on the parametric space R_N^* in (II) above is given by $\xi(\mathbf{X}) = \phi(\bar{X}).\ \psi(\mathbf{Y}), \phi$ being the uniform distribution on $(-\infty, \infty)$ and $\psi(\mathbf{Y}) = 1/N!$ for all the permutations of the fixed vector $(Y_1, \cdots, Y_N^*)$. However this *Bayes posterior unlike the fiducial distribution* in above paragraph is *independent of randomisation*, i.e. the Bayes Posterior would be the *same* whether the unit i is chosen at random with equal probabilities to all units or it is chosen any other way. The same result is generalised for arbitrary size sample, using ancillary statistic. (Received 12 April 1968.)

5. On symmetry of the product of two random variables and their independence. D. V. GOKHALE, University of Poona.

Let $\mathbf{S}(0)$ be the class of random variables (rv's) having a distribution symmetric about zero. Consider two rv's X and Y such that one, say X, belongs to $\mathbf{S}(0)$. Then independence of X and Y implies that the product XY belongs to $\mathbf{S}(0)$. The converse is not true in general. However, it is shown to be true under some conditions, in many models for dependence like the bivariate normal distribution, the regression model and models proposed by Konijn [*Ann. Math. Statist.* **27**, (1956) 500–515], Gokhale [Ph.D. Thesis (1966), University of California, Berkeley] and Farlie [*Biometrika* **50**, (1963) 499–504]. Thus if X belongs to $\mathbf{S}(0)$, a test of symmetry of XY is a conservative test of independence of X and Y, while the two are equivalent under suitable conditions. In particular, distribution-free tests of symmetry can be used. Asymptotic efficiencies of such tests are studied. The sign-test, for example, is simple and controls the probability of false rejection even when the distributions of (X_i, Y_i) in a sample depend on i. Examples are given where this sign-test has larger asymptotic efficiency than the usual tests of independence based on the product-moment or rank correlations. (Received 23 May 1968.)

6. Asymptotically nonparametric tests of symmetry in univariate populations. SHULAMITH T. GROSS, Harvard University.

In this paper a class of rank tests for the hypothesis of symmetry in univariate populations against the alternative of positive or negative skewness is proposed. It includes Gupta's test (*Ann. Math. Statist.* (1968)) as a particular case. The test statistics are of the form $\sum_{i=1}^{n} J(R_i/N+1)$ where $R_1 \cdots R_n$ denote the ranks of the n positive sample deviations from the median among the N absolute sample deviations from the median. The sample is of size N from a continuous cumulative distribution function F. The score function J on (0, 1) is assumed to satisfy regularity conditions of Govindarajulu et al.'s type (Fifth Berkeley Symposium). When the population median is not known, the deviations are taken from a consistent estimate of the population median. In that case the tests are not asymptotically distribution free under the hypothesis but can be made so by proper studentization. Asymptotic comparisons of these tests with Gupta's test and the skewness test based on the coefficient of skewness are made under different models for the alternative. Some small sample calculations of power and efficiency using Monte Carlo methods are in progress. Extensions to the analogous multivariate problem are being considered. (Received 8 April 1968.)

7. Distribution of Wilks' Λ in the noncentral linear case. A. K. GUPTA, Purdue University and University of Arizona.

In this paper, the exact distribution of Wilks' likelihood ratio criterion, Λ, in the noncentral linear case, i.e. when the alternative hypothesis is of unit rank, has been obtained giving explicit expressions for the same for $p = 2(1)5$ and general f_1 and f_2. Earlier, K. C. S. Pillai and A. K. Gupta, [to appear in *Biometrika*], using convolution techniques, have derived the exact distribution of Λ for $p = 3(1)6$ in a finite series except when p and f_2 are both odd, in which case it is given in infinite series form. Using the same techniques the results of the present paper have been derived. (Received 29 April 1968.)

8. On characterizations of the gamma distribution. W. J. HALL and GORDON SIMONS, Stanford University.

Let S_n be the cumulative sum of n iid random variables, and $\mathcal{S}_n$ the σ-field generated by $\{S_n, S_{n+1}, \cdots\}$. If the rv's (or their negatives) have a gamma distribution then $\{Z_n \equiv S_n^r/E(S_n^r), \mathcal{S}_n; s \geqq 1\}$ is a *reverse martingale sequence* for any positive r. That is, $E(Z_n \mid \mathcal{S}_{n+1}) = Z_{n+1}$ a.s. for $n \geqq 1$. These reverse martingales find applications in sequential analysis. In this paper the converse is proved for any integer $r > 1$, and this provides a characterization of the gamma distribution; in fact, it is sufficient that the reverse martingale sequence have finite length r. Another characterization is also proved, extending the case $r = 2$ to non-identically distributed rv's. Roughly stated, it asserts that if $X^2/(X+Y)^2$ and $Y^2/(X+Y)^2$ each have constant regression on $X + Y$, then the independent rv's X and Y (or else $-X$ and $-Y$) have gamma distributions with common scale parameter. (Received 20 May 1968.)

9. Estimating partial derivatives of an unconditional multivariate density (preliminary report). Pi-Erh Lin, Columbia University.

Let a sample space $\mathfrak{X}$ and parameter space Ω each be subsets of R^p, Euclidean p-space. We make the usual Empirical Bayes assumptions of the sequence $(X_1, \theta_1), \cdots, (X_n, \theta_n)$, and assume further that, given θ_n, X_n has a multivariate density function of the form $f(x \mid \theta) = \beta(\theta)h(x)e^{\theta' x}$. Denote by $f(x)$ the unconditional density of X_n. The existence and continuity of the mth partial derivatives of $f(x)$ with respect to the ith component of x, denoted by $d_i^{(m)}f(x)$, is established. By a method related to that used by Parzen (*Ann. Math. Statist.* **32,** (1962) 1065–1076) for the univariate case, consistent estimators of the $d_i^{(m)}f(x)$ are derived. Their rates of convergence, in terms of bias and mean squared error, and their asymptotic multivariate distributions are obtained. It is also shown that for any $0 < \epsilon < 1$, it is possible to construct estimators which have rate of convergence of the order of $n^{-(1-\epsilon)}$, in terms of mean squared error. As an application, let X be a p-variate normal random vector with mean $\theta \varepsilon \Omega$ and known positive definite covariance matrix Σ. The hypothesis considered is $H: \theta'\Sigma^{-1}\theta \leqq c$, c being a constant. Assume that θ has an unknown priori distribution $\pi(\theta)$. With an appropriate loss function, the test function obtained is shown to be asymptotically optimal in the sense of Robbins (*Ann. Math. Statist.* **35,** (1964) 1–20). (Received 24 April 1968.)

10. Random caps on a sphere. Roger E. Miles, Australian National University.

Prob $\{N$ independent uniformly distributed equal circular caps, of angular radius ϕ, on the surface of a sphere *have at least one common point*$\}$ $= 1 -$ Prob $\{N$ independent uniformly distributed equal circular caps, of angular radius $\pi - \phi$, on the surface of a sphere *completely cover the surface*$\}$ $= \binom{N}{2} \int_0^\phi (\sin^2 \tfrac{1}{2}\theta)^{N-2} \sin 2\theta \, d\theta + \tfrac{3}{4}\binom{N}{3} \int_0^\phi (\sin^2 \tfrac{1}{2}\theta)^{N-3} \cdot \sin^3\theta \, d\theta$ $(N = 2, 3, \cdots; 0 \leqq \phi \leqq \pi/2)$. This extends the known result for $\phi = \pi/2$, but leaves open the case $\pi/2 < \phi < \pi$ (see Gilbert [*Biometrika* **52,** (1965) 323–330]). The final expression is also equal to $1 -$ Prob $\{N$ independent isotropically distributed equal "thick great circles", of angular thickness $\pi - 2\phi$, on the surface of a sphere *completely cover the surface*$\}$ $(N = 2, 3, \cdots; 0 \leqq \phi \leqq \pi/4)$, leaving open the case $\pi/4 < \phi < \pi/2$. (Received 16 May 1968.)

11. Delaunay triangles and probabilities of coverage and concentration for Poisson discs. Roger E. Miles, Australian National University.

Consider the Poisson point process of intensity ρ in E^2. The (a.s. ergodic) pdf for the associated Delaunay triangles (cf. Rogers [*"Packing and Covering"*, Cambridge Univ. Press (1964), Ch. 8]) is (1) $f(r, \theta, \phi) = (16\pi/3)\rho^2 r^3 \exp(-\pi\rho r^2) \sin\theta \sin\phi \sin(\theta + \phi)$ $(r \geqq 0; \theta \geqq 0, \phi \geqq 0, \theta + \phi \leqq 2\pi)$, where r, θ, ϕ are the circum-radius and 2 interior angles. Suppose a disc of radius R is constructed about each particle as centre. Then (1) and extreme value theory yield, for $X \subset E^2$, the asymptotic values as $|X| \to \infty$ of (2) $P\{$*every* point of X is covered by *at least* i discs$\}$ $(i = 1, 2, \cdots)$, and (3) $\{$*no* point of X is covered by *more than* j discs$\}$ $(j = 1, 2, \cdots)$. The substitution $\rho = N/|X|$ yields approximations for N independent uniform discs in X; the additional substitution $|X| = 4\pi$ extends validity to circular caps of small angular radius R on a sphere. Despite significant dependence, edge and other effects, comparison of (2) with $i = 1$ with Gilbert's [*Biometrika* **52,** (1965) 330] simulation taking X as a disc of radius $3R$ shows only roughly 10% discrepancies in N value for his 5 coverage probability estimates. (3) with $j = 1$ agrees with Efron [*Ann. Math. Statist.* **38,** (1967) 298]. Extensions are immediate to 1(!), 3 or more dimensions. (Received 24 May 1968.)

12. Some new results in the mathematical theory of phage-reproduction. PREM S. PURI, Purdue University.

In the theory of phage reproduction, the mathematical models considered thus far (see Gani, J. (1965), *J. Appl. Prob.* **2**, 225–268) assume that the bacterial burst occurs a fixed time after infection, or after a fixed number of generations of phage-multiplications, or when the number of mature bacteriophages reaches a fixed threshold. In the present paper, such hypotheses of fixed thresholds are abandoned in favour of a more realistic assumption: Given that until the time t the bacterial burst has not yet taken place, the occurrence of the burst between t and $t + \Delta t$ is treated as a random event, the probability of which is $f(\cdot \mid t)\Delta t + o(\Delta t)$, where f is a non-negative and non-decreasing function of the number $X(t)$ of vegetative phages and of $Z(t)$, the number of mature bacteriophages at time t. More specifically it is assumed that $f = b(t)X(t) + c(t)Z(t)$ with $b(t), c(t) \geqq 0$. Here $X(t)$ is assumed to be a linear birth and death process and $Z(t)$ corresponds to the number of deaths until time t. One of the problems considered here is the joint distribution of X_T and Z_T, the numbers at burst of vegetative and mature bacteriophages respectively. The distribution of Z_T is then fitted to observed data due to Delbrück [*J. Bacteriology* **50**, 131–35 (1945)]. (Received 8 April 1968.)

13. Discrete dynamic programming with sensitive optimality criteria (preliminary report). ARTHUR F. VEINOTT, JR., Stanford University.

Consider a discrete time parameter Markovian decision process with finite action set A_s available in state $s(=1, \cdots, S)$. Set $F = \times_{s=1} A_s$. Let $r(f)$ and $P(f)$ be respectively the one period S-vector of rewards and $S \times S$ sub-stochastic transition matrix when $f \varepsilon F$ is used in a period. For $g, f_1, f_2, \cdots$ in F, let $\pi = (f_1, f_2, \cdots)$ and $g^\infty = (g, g, \cdots)$ be respectively a policy and a stationary policy. Let $P^N(\pi) = P(f_1) \cdots P(f_N)$ and $\rho > -1$ be the interest rate. Define $V_\rho(\pi) = \sum_{j=0}^\infty (1 + \rho)^{-j} P^j(\pi) r(f_{j+1})$ (for $|\rho|$ small enough) and $V_{n-1}^{\pm N}(\pi) = \sum_{j=0}^{N-1} \phi_n^{\pm N}(j) P^j(\pi) r(f_{j+1})$ where $\phi_n^{\pm N}(j) = \sum_{k=0}^n \binom{N+n}{n-k}\binom{j+k}{k}(\mp 1)^k$, $N, n \geqq 0$. It is shown that $\sum_{N=0}^\infty P^N(\pi) < \infty$ for all π if and only if $\sum_{N=0}^\infty P(g)^N < \infty$ for all $g \varepsilon F$. This is assumed in the sequel in discussions of $\mathfrak{D}_n^-$ and $\mathfrak{A}_n^-$. Let

$$\mathfrak{D}_n^\pm = \{f: \liminf_{\rho \to 0\pm} |\rho|^{-n} [V_\rho(f^\infty) - V_\rho(\pi)] \geqq 0 \text{ all } \pi\}$$

and $\mathfrak{A}_n^\pm = \{f: \liminf_{N \to \infty} N^{-1}[V_n^{\pm N}(f^\infty) - V_n^{\pm N}(\pi)] \geqq 0 \text{ all } \pi\}$, $n = -1, 0, 1, \cdots$, and $\mathfrak{D}_\infty^\pm = \{f: V_\rho(f^\infty) - V_\rho(\pi) \geqq 0, \text{ all } \pi \text{ and } 0 < \pm\rho < \rho^* \text{ for some } \rho^*\}$. It is shown here that $\mathfrak{D}_n^\pm = \mathfrak{A}_n^\pm$ and $\mathfrak{D}_n^\pm \supset \mathfrak{D}_{n+1}^\pm$ for $n = -1, 0, 1, \cdots$, and $\mathfrak{D}_{2S-1}^\pm = \mathfrak{D}_{2S}^\pm = \cdots = \mathfrak{D}_\infty^\pm$ with all these sets being nonempty. A generalized policy improvement method is devised for finding an element of $\mathfrak{D}_n^\pm$, $n = -1, 0, 1, \cdots, 2S - 1$. The proofs exploit the following new representations: $V_\rho(f^\infty) = (1 + \rho)\sum_{n=-1}^\infty \rho^n y_n(f)$ for all small enough $\rho > 0$ and

$$V_{n-1}^N(f^\infty) = \sum_{i=-1}^n \binom{N+n}{n-i} y_i(f) - P(f)^{N+n} y_n(f),$$

where

$y_{-1}(f) = P^*(f)\, r(f)$, $\quad P^*(f) = \lim_{N \to \infty} (N+1)^{-1} \sum_{i=0}^N P(f)^i$, $\quad y_n(f) = (-1)^n H(f)^{n+1}$ for $n \geqq 0$, and $H(f) = [I - P(f) + P^*(f)]^{-1} - P^*(f)$. Analogous results are developed for the continuous time parameter case. (Received 25 April 1968.)