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SIMULTANEOUS TESTS FOR THE EQUALITY OF COVARIANCE MATRICES AGAINST CERTAIN ALTERNATIVES

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1. Introduction and summary. In many situations, it is of interest to test for the equality of variances or covariance matrices against certain alternatives. Hartley [6] considered the problem of testing for the equality of variances against the alternative that at least one variance is different from the other. Gnanadesikan [3] considered the problem of testing for the equality of variances against the alternative that at least one variance is not equal to the standard. Recently, Krishnaiah [12] considered testing for the equality of variances against the alternative that at least one variance is not equal to the next. In the above procedures, it was assumed that the underlying populations are univariate normal. In this paper, we consider multivariate generalizations of the above test procedures. The procedures proposed in this paper are based upon expressing the total hypothesis as the intersection of some elementary hypotheses and testing these elementary hypotheses by using conditional distributions. In the two sample case, our procedures are similar (but not equivalent) to the procedure proposed by Roy [16]; the test statistics used by him in testing some of the elementary hypotheses are different from those used in this paper.

2. Preliminaries and statement of problems. Let $S_i = (s_{iqr})$ denote ith sample sums of squares and cross products (SP) matrix and let $n_i + 1$ denote ith sample size. Let Σ_{ij} denote top $j \times j$ left hand corner of $\Sigma_i = (\sigma_{iqr})$ and let S_{ij} denote the top $j \times j$ left hand corner of $S_i = (s_{iqr})$. Also, let

$$\boldsymbol{\beta}_{ij} = \begin{pmatrix} \beta_{ij1} \\ \vdots \\ \beta_{ijj} \end{pmatrix} = \Sigma_{ij}^{-1} \begin{pmatrix} \sigma_{i1,j+1} \\ \vdots \\ \sigma_{ij,j+1} \end{pmatrix}$$

$$\mathbf{b}_{ij} = \begin{pmatrix} b_{ij1} \\ \vdots \\ b_{ijj} \end{pmatrix} = S_{ij}^{-1} \begin{pmatrix} s_{i1,j+1} \\ \vdots \\ s_{ij,j+1} \end{pmatrix},$$

$s_{i,j+1}^2 = |S_{i,j+1}|/|S_{ij}|$, $\sigma_{i\ j+1}^2 = |\Sigma_{i,j+1}|/|\Sigma_{ij}|$ for $j = 1, 2, \cdots, (p-1)$, $s_{i1}^2 = s_{i11}$, $\sigma_{i1}^2 = \sigma_{i11}$, $s_{\cdot,j+1}^2 = \sum_i s_{i,j+1}^2$, $S_{ij}^{-1} = (s_{ijuv}^*)$, $D_{imtu} = (b_{itu} - b_{mtu})/(s_{ituu}^* + s_{mtuu}^*)^{\frac{1}{2}}$, $N = \sum_{i=1}^k n_i$, $F_{imt} = s_{it}^2(n_m - t + 1)/s_{mt}^2(n_i - t + 1)$, $F_{imtu} = (N - kt)D_{imtu}^2/s_{\cdot,t+1}^2$.

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In addition, we need the following notation:

$$H_{j1}:\sigma_{1j}^2 = \cdots = \sigma_{kj}^2, \qquad H_{j2}:\beta_{1j} = \cdots = \beta_{kj},$$
$$A_{1j1} = \bigcup_{i=1}^{k-1}[\sigma_{ij}^2 \neq \sigma_{i+1,j}^2], \qquad A_{1j2} = \bigcup_{i=1}^{k-1}[\beta_{ij} \neq \beta_{i+1,j}],$$
$$A_{2j1} = \bigcup_{i=1}^{k-1}[\sigma_{ij}^2 \neq \sigma_{kj}^2], \qquad A_{2j2} = \bigcup_{i=1}^{k-1}[\beta_{ij} \neq \beta_{kj}],$$
$$A_{3j1} = \bigcup_{i\neq i'=1}^{k}[\sigma_{ij}^2 \neq \sigma_{i'j}^2], \qquad A_{3j2} = \bigcup_{i\neq i'=1}^{k}[\beta_{ij} \neq \beta_{i'j}].$$

In this paper, we consider the problem of testing H against A_1, A_2 and A^3 where $H:\Sigma_1 = \cdots = \Sigma_k$, $A_1 = \bigcup_{j=1}^{p} A_{1j1} \bigcup_{j'=1}^{p-1} A_{1j'2}$, $A_2 = \bigcup_{j=1}^{p} A_{2j1} \bigcup_{j'=1}^{p-1} A_{2j'}{}^2$ and $A_3 = \bigcup_{j=1}^{p} A_{3j1} \bigcup_{j'=1}^{p-1} A_{3j'2}$. In the sequel, we assume that σ_{0j}^2 is the common (unknown) value of σ_{ij}^2 when $\sigma_{1j}^2 = \cdots = \sigma_{kj}^2$.

The test procedures considered in this paper are based on the following method. We first test H_{11} against the alternative of interest. If H_{11} is rejected, we declare that H is rejected. If H_{11} is accepted, we proceed further and test H_{21} and H_{12} holding the first variate fixed. If $H_{12} \cap H_{21}$ is accepted, we proceed further and test H_{31} and H_{22} holding the second variate fixed. We continue this procedure until H is accepted or rejected. Here we note that $\bigcap_{j=1}^{r} H_{j1} \bigcap_{j=1}^{r-1} H_{j2}$ is equivalent to the hypothesis that

$$\Sigma_{1r} = \cdots = \Sigma_{kr}.$$

We need the following known results (see [16]) in the sequel:

When S_{ij} is fixed, the distribution of $\mathbf{b}_{ij}$ is independent of the distribution of $s_{i,j+1}^2$; the distribution of $\mathbf{b}_{ij}$ is j-variate normal with mean vector β_{ij} and covariance matrix $\sigma_{i,j+1}^2 S_{ij}^{-1}$, and $s_{i,j+1}^2/\sigma_{i,j+1}^2$ is distributed as χ^2 with $(n_i - j)$ degrees of freedom.

3. Test for H against A_1.

The following lemma is needed in the sequel.

LEMMA 3.1. *If $x_1, x_2, \cdots, x_k$ are distributed independently as central chi-square variates with $m_1, m_2, \cdots, m_k$ degrees of freedom, then*

$$f(F_{12}, F_{23}, \cdots, F_{k-1,k}) = \Gamma\left(\textstyle\sum_{j=1}^{k} m_j/2\right) \prod_{i=1}^{k-1}[(m_i/m_k)^{m_i/2} F_{i,i+1}^{(m_1+\cdots+m_i-2)/2}]$$
$$\cdot [\textstyle\prod_{j=1}^{k}\Gamma(m_j/2)[1 + m_k^{-1}\sum_{j=1}^{k-1} m_j \prod_{i=j}^{k-1} F_{i,i+1}]^{\Sigma m_j/2}]^{-1}$$

where $F_{ij} = x_i m_j/x_j m_i$.

The above lemma was proved in [12].

We will first consider the problem of testing H_{j1} against the alternative A_{1j1} when the first $(j-1)$ variates are held fixed (with the understanding that no variate is held fixed when H_{11} is tested). In this case, we accept H_{j1} if and only if

$$\lambda_{ij} \leqq F_{i,i+1,j} \leqq \mu_{ij} \tag{3.1}$$

where λ_{ij} and μ_{ij} are chosen such that

$$P[\lambda_{ij} \leqq F_{i,i+1,j} \leqq \mu_{ij};\ i = 1, 2, \cdots, k-1 \mid H_{j1}] = P_j. \tag{3.2}$$

When H_{j1} is true, $s_{1j}^2/\sigma_{0j}^2, \cdots, s_{kj}^2/\sigma_{0j}^2$ are independently distributed as chi-square variates with $(n_1 - j + 1), \cdots, (n_k - j + 1)$ degrees of freedom. So, using Lemma 3.1, we can write down the joint distribution of $F_{12}, F_{23}, \cdots, F_{k-1,k}$ when $H_{j1+1,1}$ is true. We will now discuss about a procedure for testing H_{j2} against A_{1j2} when $H_{j+1,1}$ is true and when the first j variates are held fixed.

When $H_{j+1,1}$ is true and the first j variates are held fixed, we accept H_{j2} if and only if

$$(3.3) \qquad F_{i,i+1,ju} \leqq c_{ij}, \qquad u = 1, 2, \cdots, j;\ i = 1, 2, \cdots, k-1,$$

where c_{ij}'s are chosen such that

$$(3.4) \quad P[F_{i,i+1,ju} \leqq c_{ij};\ u = 1, 2, \cdots, j, i = 1, 2, \cdots, k-1 \mid H_{j+1,1} \cap H_{j2}] = P_j'.$$

When $H_{j+1,1}$ is true, $s_{\cdot,j+1}^2/\sigma_{0,j+1}^2$ is distributed as a chi-square variate with $(N - kj)$ degrees of freedom and it is distributed independently of $D_{i,i+1,ju}^2$ for $i = 1, 2, \cdots, k-1$ and $u = 1, 2, \cdots, j$. Also, when $H_{j+1,1} \cap H_{j2}$ is true, the joint distribution of $(D_{12j1}, \cdots, D_{k-1,kj1}, D_{12j2}, \cdots, D_{k-1,kj2}, \cdots, D_{12jj}, \cdots, D_{k-1,kjj})$ is the central multivariate normal with zero mean vector and with Ω^j as the covariance matrix where

$$\Omega^j = \begin{bmatrix} \Omega_{11}^j & \Omega_{12}^j & \cdots & \Omega_{1j}^j \\ \Omega_{21}^j & \Omega_{22}^j & \cdots & \Omega_{2j}^j \\ \vdots & \vdots & & \vdots \\ \Omega_{j1}^j & \Omega_{j2}^j & \cdots & \Omega_{jj}^j \end{bmatrix},$$

$$\Omega_{ii'}^j = (\omega_{ii'vw}^j)\sigma_{0,j+1}^2,$$

$$\begin{aligned} \omega_{ii'vw}^j &= 0, \qquad |w - v| > 1, \\ &= [s_{vjii'}^* + s_{v+1,jii'}^*][(s_{vjii}^* + s_{v+1,jii}^*)(s_{vji'i'}^* + s_{v+1,ji'i'}^*)]^{-\frac{1}{2}}, \qquad w = v, \\ &= -s_{tjii'}^*[(s_{vjii}^* + s_{v+1,jii}^*)(s_{wji'i'}^* + s_{w+1,ji'i'}^*)]^{-\frac{1}{2}}, \qquad |w - v| = 1, \end{aligned}$$

and $t = \max.(w, v)$. So, the joint distribution of

$$(F_{12j1}, \cdots, F_{k-1,kj1}, \cdots, F_{12jj}, \cdots, F_{k-1,kjj})$$

is a multivariate F distribution with $(1, N - kj)$ degrees of freedom and with Ω^j as the covariance matrix of the "accompanying" multivariate normal. For various details on the multivariate F distribution, the reader is referred to [9], [10], [11].

Now, combining (3.1), (3.2), (3.3) and (3.4) we use the following procedure for testing H against A_1.

Accept H against A_1 if and only if

$$(3.5) \quad \begin{aligned} &\lambda_{ij}^* \leqq F_{i,i+1,j} \leqq \mu_{ij}^*; && i = 1, 2, \cdots, k-1;\ j = 1, 2, \cdots, p; \\ &F_{i,i+1,ju} \leqq c_{ij}^*; && u = 1, 2, \cdots, j;\ j = 1, 2, \cdots, (p-1); \\ & && i = 1, 2, \cdots, k-1, \end{aligned}$$

where λ_{ij}^*, μ_{ij}^* and c_{ij}^* are chosen such that the probability of (3.5) holding good, when H is true, is $(1 - \alpha)$. But this probability is equal to $\prod_{j=1}^{p} q_j \prod_{j=1}^{p-1} q_j'$ where

$$q_j = P[\lambda_{ij}^* \leqq F_{i,i+1,j} \leqq \mu_{ij}^*; \quad i = 1, 2, \cdots, k-1 \mid H],$$

$$q_j' = P[F_{i,i+1,ju} \leqq c_{ij}^*; u = 1, 2, \cdots, j; i = 1, 2, \cdots, (k-1) \mid H].$$

The optimum choice of the critical values is not known. For practical purposes' we impose the following restrictions.

$$q_1 = \cdots = q_p = q_1' = \cdots = q_{p-1}' = (1 - \alpha)^{1/(2p-1)}, \qquad c_{ij}^* = c_j^*.$$

In addition, we impose the restriction that the test associated with testing H_{j1} is locally unbiased.

The $(1 - \alpha)$ % simultaneous confidence intervals associated with the above test procedure are given by

$$\lambda_{ij}^* s_{i+1,j}^2 (n_i - j + 1)/s_{ij}^2 (n_{i+1} - j + 1)$$
$$\leqq \sigma_{i+1,j}^2/\sigma_{ij}^2 \leqq \mu_{ij}^* (n_i - j + 1)(n_{i+1} - j + 1)^{-1} s_{i+1,j}^2/s_{ij}^2;$$
$$i = 1, 2, \cdots, k-1; \quad j = 1, 2, \cdots, p,$$

and

$$|b_{iju} - b_{i+1,ju} - \beta_{iju} + \beta_{i+1,ju}| \leqq \{c_{ij}^* s_{\cdot,j+1}^2 (s_{ijuu}^* + s_{i+1,juu}^*)(N - kj)^{-1}\}^{\frac{1}{2}}$$
$$u = 1, 2, \cdots, j; j = 1, \cdots, (p-1); i = 1, 2, \cdots, k-1.$$

4. Tests for H against A_2 and A_3. When H is tested against A_2, we accept H if and only if

$$a_{ij} \leqq F_{ikj} \leqq b_{ij}; \quad i = 1, 2, \cdots, k-1; \quad j = 1, 2, \cdots, p,$$

$$F_{ikju} \leqq c_{ij}; \quad u = 1, 2, \cdots, j; \quad j = 1, 2, \cdots, p; \quad i = 1, 2, \cdots, k-1,$$

where a_{ij}, b_{ij} and c_{ij} are chosen such that

$$\prod_{j=1}^{p} Q_j \prod_{j=1}^{p-1} Q_j' = 1 - \alpha,$$

and

$$Q_j = P[a_{ij} \leqq F_{ikj} \leqq b_{ij}; i = 1, 2, \cdots, k-1; j = 1, 2, \cdots, p \mid H],$$

$$Q_j' = P[F_{ikju} \leqq c_{ij}; u = 1, 2, \cdots, j; i = 1, 2, \cdots, k-1 \mid H].$$

We can evaluate $Q_1, \cdots, Q_p$ by using the methods (or their modifications) discussed in [1], [4], [5], [8], [13]; also $Q_1', \cdots, Q_{p-1}'$ can be evaluated since the joint distribution of the test statistics F_{ikju}, $(u = 1, 2, \cdots, j; i = 1, 2, \cdots, k-1)$, is a central multivariate F distribution when H is true. The optimum choice (in terms of increasing power of the test) of the critical values is not known. But, for practical purposes, we can choose them by imposing restrictions similar to those imposed in the previous section.

We will now propose a procedure to test H against A_3. According to this procedure, we accept H if and only if

$$\mu_{ii'j} \leqq F_{ii'j} \leqq \lambda_{ii'j}; \qquad i \neq i' = 1, 2, \cdots, k,$$

$$F_{ii'ju} \leqq c_j; \qquad i \neq i' = 1, 2, \cdots, k; \ u = 1, 2, \cdots, j,$$

where $\mu_{ii'j}$, $\lambda_{ii'j}$ and c_j are chosen such that

$$\prod_{j=1}^{p} R_j \prod_{j=1}^{p-1} R_j' = (1 - \alpha),$$

and

$$R_j = P[\mu_{ii'j} \leqq F_{ii'j} \leqq \lambda_{ii'j}; \ i \neq i' = 1, 2, \cdots, k \mid H],$$

$$R_j' = P[F_{ii'ju} \leqq c_j; i = 1, 2, \cdots, i' - 1; i' = 2, 3, \cdots, k; u = 1, 2, \cdots, j \mid H].$$

The critical values are chosen such that

$$R_1 = \cdots = R_p = R_1' = \cdots = R_{p-1}' = (1 - \alpha)^{1/(2p-1)}.$$

Also, when the sample sizes are equal, we impose the restriction that $\lambda_{ii'j} = \lambda_j$ and $\mu_{ii'j} = 1/\lambda_j$. Using Ramachandran's result [14], we know that this restriction will achieve the unbiasedness of the test associated with H_{j1} against A_{3j1}. We will now discuss about the evaluation of $R_1, \cdots, R_p, R_1', \cdots, R_{p-1}'$.

When the sample sizes are unequal, we can use Bonferroni's inequality ([2], p. 100) to compute lower bounds on $R_1, \cdots, R_p$. When the sample sizes are equal, we can use the method discussed in [6] to evaluate $R_1, \cdots, R_p$. The exact evaluation of $R_1', \cdots, R_{p-1}'$ is complicated since, when H is true and s_{ij} are fixed, the statistics $F_{ii'ju}$ are jointly distributed as a singular multivariate F distribution. But, using the results of Khatri [7], we obtain the following lower bounds:

$$(4.1) \quad R_j' \geqq \prod_{i'=2}^{k} \prod_{i=1}^{i'-1} \prod_{u=1}^{j} P[F_{ii'ju} \leqq c_j \mid H]; \qquad j = 1, 2, \cdots, p - 1.$$

Also, using Bonferroni's inequality, we obtain the following alternative bound on R_j':

$$R_j' \geqq 1 - \sum_{i'=2}^{k} P(E_{i'}^*)$$

where

$$P(E_{i'}^*) = 1 - P[F_{ii'ju} \leqq c_j; i = 1, 2, \cdots, i' - 1; u = 1, 2, \cdots, j \mid H].$$

Here we note that lower bounds similar to (4.1) can be obtained on q_j' and Q_j' using the results of [7]; upper bounds on R_j', q_j' and Q_j' can be also obtained by making use of Poincaré's formula which is sometimes referred to as inclusion-exclusion formula.

5. Test for H against a general alternative. Let $(j_1, j_2, \cdots, j_p)$ be any permutation of $(1, 2, \cdots, p)$ and $(j_1', j_2', \cdots, j_{p-1}')$ be any permutation of $(1, 2, \cdots, p - 1)$. In addition, let

$$A_{(g_1, g_2, g_3, g_4)} = \left(\bigcup_{g=1}^{g_1} A_{1j_g1}\right) \cup \left(\bigcup_{h=g_1+1}^{g_1+g_2} A_{2j_h1}\right) \cup \left(\bigcup_{l=g_1+g_2+1}^{p} A_{3g_l1}\right) \cup \left(\bigcup_{\delta=1}^{g_3} A_{1j_\delta'2}\right)$$
$$\cup \left(\bigcup_{\partial=g_3+1}^{g_3+g_4} A_{2j_\partial'2}\right) \cup \left(\bigcup_{a=g_3+g_4+1}^{p-1} A_{3j_a'2}\right).$$

Then it is of some interest to test H against the alternative $A_{(g_1,g_2,g_3,g_4)}$. A procedure for testing H against $A_{(g_1,g_2,g_3,g_4)}$ can be proposed by combining the methods used in Sections 3 and 4; the details are omitted for the sake of brevity.

One would intuitively expect the power of the test for H against the alternative $A_{(g_1,g_2,g_3,g_4)}$ to be greater, in some directions, than the powers of the tests proposed in sections 3 and 4. This is a motivation behind testing H against $A_{(g_1,g_2,g_3,g_4)}$ sometimes. Also, when H is rejected, the experimenter may be interested in making different kinds of comparisons among populations on different sets of variates. In some of these situations, one should test H against the alternatives of the form $A_{(g_1,g_2,g_3,g_4)}$.

6. General remarks. Roy [16] proposed a procedure, based on conditional distributions, for testing the equality of two covariance matrices. But, the lengths of the confidence intervals associated with the procedures proposed in this paper are at least as short as the lengths of the corresponding confidence intervals associated with the procedure by Roy [16]. In the univariate case, the procedures proposed in this paper for testing H against A_1 and A_2 are respectively equivalent to the procedures considered by Krishnaiah [12] and Gnanadesikan [3], and the test for H against A_3 for the case of equal sample sizes is equivalent to Hartley's test [6]. The procedures proposed in this paper are based upon union-intersection principle [15]. The simultaneous confidence intervals associated with tests for H against A_2, A_3 and $A_{(g_1,g_2,g_3,g_4)}$ can be obtained easily.

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