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INDEPENDENT SEQUENCES WITH THE STEIN PROPERTY[1]

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1. Introduction. Throughout this note $Z = (Z_1, Z_2, \cdots)$ is an independent sequence of complex valued random variables on a probability space $(\Omega, \mathfrak{A}, P)$.

It is convenient to say that Z has the *Stein property* if there is a number $b > 0$ such that if A is in $\mathfrak{A}$ and $P(A) > 0$, then there is a positive integer n such that if $a, a_1, a_2, \cdots$ are complex numbers and the series $\sum_{k=1}^{\infty} a_k Z_k$ converges almost everywhere, then

$$b(\sum_{k=n}^{\infty} |a_k|^2)^{\frac{1}{2}} \leqq \text{ess sup}_{\omega \varepsilon A} |a + \sum_{k=1}^{\infty} a_k Z_k(\omega)|.$$

For $b > 0$ and k a positive integer, let $\pi_k(b)$ be the least upper bound of $P(|a + Z_k| < b)$ for a complex. Let $\pi(b) = \lim \sup_{k \to \infty} \pi_k(b)$. The function π_k is essentially Lévy's "fonction de concentration" for the random variable Z_k ([5], [6], [4]).

We show here that the Stein property holds if and only if $\pi(b) < 1$ for some $b > 0$ (Theorem 3). A bound on a related conditional probability, giving more precise information in one direction, is contained in Theorem 1. Theorems 2 and 4 contain information about the constant b. For example, if $|Z_k(\omega)| = 1$, $\omega \varepsilon \Omega$, and $-Z_k$ has the same distribution as Z_k, $k \geqq 1$, then Z has the Stein property with $b = 1$.

These theorems have obvious Hilbert space analogues of two kinds: (I) Keeping Z complex, one may suppose that $a, a_1, a_2, \cdots$ are elements of a Hilbert space H and interpret $|\cdot|$, where appropriate, as the norm function on H. The a in the definition of π_k remains a complex number. (II) With $a_1, a_2, \cdots$ complex, one may suppose that $Z = (Z_1, Z_2, \cdots)$ is an independent sequence of strongly measurable functions from $(\Omega, \mathfrak{A}, P)$ into H. Here a must be in H.

For both cases, (I) and (II), Theorem 1, 2, 3, 4 remain true and our proofs go through with little or no change.

Theorem 3 contains Stein's Lemma 2 [10], which asserts that the sequence $r = (r_1, r_2, \cdots)$ of Rademacher functions, defined on the Lebesgue unit interval $[0, 1)$ by $r_k(\omega) = 1$ if ω belongs to $[2j/2^k, (2j + 1)/2^k)$ for some $j = 0, \cdots, 2^{k-1} - 1$, $r_k(\omega) = -1$ otherwise, has the Stein property slightly modified: The constant b may depend on the set A and a is always taken to be 0. Our Theorem 4 indicates that b may always be taken to be 1 in the Rademacher case. Stein's lemma and various generalizations have been proved independently by A. M. Garsia, R. F. Gundy, P. A. Meyer, S. Sawyer, G. Weiss, the present author, and probably by others. Our proof, obtained in 1962, uses the method of Theorem 1. No proofs have heretofore been published although the lemma, or some immediate

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consequence thereof, is a basic tool in several papers ([10], [1], [9]). A different proof of a related result has recently appeared [3].

2. Independent sequences with the Stein property. For $b > 0$ and n a positive integer, let $p_n(b) = p_n^Z(b)$ be the least upper bound of $P(|a + \sum_{k=n}^N a_k Z_k| < b(\sum_{k=n}^N |a_k|^2)^{\frac{1}{2}})$ for $a, a_1, a_2, \cdots$ complex numbers and $N \geqq n$. Note that $1 \geqq p_1(b) \geqq p_2(b) \geqq \cdots$. Let $p(b) = p^Z(b) = \lim_{n\to\infty} p_n(b)$. If $(Z_n, Z_{n+1}, \cdots)$ has the same distribution as Z, $n \geqq 1$, then $p(b) = p_1(b)$. In particular, $p^r(b) = p_1^r(b)$ for any sequence r having the same distribution as the Rademacher sequence.

THEOREM 1. *Suppose that A is in $\mathcal{A}$, $P(A) > 0$, $b > 0$, and $\epsilon > 0$. Then there is a positive integer n such that if $a, a_1, a_2, \cdots$ are complex numbers and the series $\sum_{k=1}^\infty a_k Z_k$ converges almost everywhere, then*

$$P(|a + \textstyle\sum_{k=1}^\infty a_k Z_k| < b(\sum_{k=n}^\infty |a_k|^2)^{\frac{1}{2}} \mid A) < p(b) + \epsilon.$$

Here $P(B \mid A) = P(A \cap B)/P(A)$ for B in $\mathcal{A}$. A common notation for conditional probabilities and expectations relative to the σ-fields generated by particular families of random variables will also be used. The proof rests on standard facts about conditional expectations.

PROOF. Let $0 < \delta < \epsilon P(A)/3$ and $U = P(A \mid Z)$. There is an integer $n > 1$ such that $p_n(b) < p(b) + \delta$ and $V = P(A \mid Z_1, \cdots, Z_{n-1})$ satisfies $E|U - V| < \delta$. Let $a, a_1, a_2, \cdots$ be complex numbers such that the series $\sum_{k=1}^\infty a_k Z_k$ converges almost everywhere. Let W be the indicator function of the set $B = \{\omega: |a + \sum_{k=1}^\infty a_k Z_k(\omega)| < b(\sum_{k=n}^\infty |a_k|^2)^{\frac{1}{2}}\}$. Then

$$\begin{aligned} P(A \cap B) &= EUW < EVW + \delta \\ &= E[VE(W \mid Z_1, \cdots, Z_{n-1})] + \delta. \end{aligned}$$

Now, by independence,

$$E(W \mid Z_1, \cdots, Z_{n-1}) = \varphi(a + \textstyle\sum_{k=1}^{n-1} a_k Z_k)$$

where

$$\begin{aligned} \varphi(a') &= P(|a' + \textstyle\sum_{k=n}^\infty a_k Z_k| < b(\sum_{k=n}^\infty |a_k|^2)^{\frac{1}{2}}) \\ &\leqq \liminf_{N\to\infty} P(|a' + \textstyle\sum_{k=n}^N a_k Z_k| < b(\sum_{k=n}^N |a_k|^2)^{\frac{1}{2}}) \\ &\leqq p_n(b). \end{aligned}$$

Thus,

$$\begin{aligned} P(A \cap B) &< EVp_n(b) + \delta \leqq (EU + \delta)p_n(b) + \delta \\ &\leqq P(A)p_n(b) + 2\delta \leqq P(A)p(b) + 3\delta < P(A)[p(b) + \epsilon], \end{aligned}$$

the desired inequality.

THEOREM 2. *Let $b > 0$. The sequence Z has the Stein property with constant b if and only if $p(b) < 1$.*

PROOF. Suppose that $p(b) < 1$. Choose $\epsilon > 0$ so that $p(b) + \epsilon < 1$. For A in

$\mathcal{A}$ with $P(A) > 0$, choose n as in Theorem 1. Then, if $a, a_1, a_2, \cdots$ are complex numbers such that the series $\sum_{k=1}^{\infty} a_k Z_k$ converges almost everywhere, we have, by Theorem 1, that the set of ω in A satisfying $|a + \sum_{k=1}^{\infty} a_k Z_k(\omega)| \geqq b(\sum_{k=n}^{\infty} |a_k|^2)^{\frac{1}{2}}$ has strictly positive probability. The Stein property with constant b follows.

Now suppose that $p(b) = 1$. Then, by the definition of $p(b)$, there exist $Y_j = a_j + \sum_{k=1}^{\infty} a_{jk} Z_k$ satisfying $a_{jk} \neq 0$ for only finitely many k, $a_{jk} = 0$ if $k \leqq j$, $\sum_{k=1}^{\infty} |a_{jk}|^2 = 1$, and $P(|Y_j| < b) > 1 - 4^{-j}$, implying, by left continuity, the existence of $0 < b_j < b$ such that $P(|Y_j| < b_j) > 1 - 4^{-j}$, $j \geqq 1$. Let $A = \bigcap_{j=1}^{\infty} \{\omega: |Y_j(\omega)| < b_j\}$. Then $P(A) > 0$ and

$$b(\textstyle\sum_{k=j}^{\infty} |a_{jk}|^2)^{\frac{1}{2}} = b > b_j \geqq \operatorname{ess\,sup}_{\omega \varepsilon A} |Y_j(\omega)|$$

for all $j \geqq 1$. Therefore the Stein property does not hold here with the constant b. This completes the proof.

THEOREM 3. *The sequence Z has the Stein property if and only if $\pi(b) < 1$ for some $b > 0$.*

This follows at once from Theorem 2 and Lemma 3 below.

LEMMA 1. (Paley and Zygmund) *Let $\alpha > \beta \geqq 0$. If X is a real random variable satisfying $EX \geqq \alpha$ and $EX^2 = 1$, then*

$$P(X \geqq \beta) \geqq (\alpha - \beta)^2.$$

This standard fact—see Lemma 19 of [8], Lemma α of [7], and Lemma V. 8.26 of [11]—follows at once from

$$\alpha \leqq EXY + EX(1 - Y) \leqq E^{\frac{1}{2}}Y^2 + \beta,$$

where $Y(\omega) = 1$ if $X(\omega) \geqq \beta$, $= 0$ otherwise.

LEMMA 2. *Let $\delta > 0$ and $\epsilon > 0$. If $Y_1, \cdots, Y_n$ are nonnegative random variables such that $P(Y_k \geqq \delta) \geqq \epsilon$, $1 \leqq k \leqq n$, then*

$$P(\textstyle\sum_{k=1}^{n} a_k Y_k \geqq \gamma\delta\epsilon) \geqq (1 - \gamma)^2 \epsilon$$

for all $0 < \gamma < 1$ and all nonnegative numbers $a_1, \cdots, a_n$ with $\sum_{k=1}^{n} a_k = 1$.

PROOF. In the proof, we may and do assume that the probability space $(\Omega, \mathcal{A}, P)$ is nonatomic. Let $A_k \varepsilon \mathcal{A}$, $A_k \subset \{\omega: Y_k(\omega) \geqq \delta\}$, $P(A_k) = \epsilon$, $X_k(\omega) = \delta$ if $\omega \varepsilon A_k$, $= 0$ otherwise, $1 \leqq k \leqq n$. Finally, let $X = \sum_{k=1}^{n} a_k X_k$. Then $EX = \delta\epsilon$ and $EX^2 \leqq E \sum_{k=1}^{n} a_k X_k^2 = \delta^2\epsilon$, implying that α, defined by $EX = \alpha E^{\frac{1}{2}} X^2$, satisfies $\alpha^2 \geqq \epsilon$. Let $\beta = \alpha\gamma$. Using Lemma 1, we have that $P(\sum_{k=1}^{n} a_k Y_k \geqq \gamma\delta\epsilon) \geqq P(X \geqq \gamma EX) = P(X \geqq \beta E^{\frac{1}{2}} X^2) \geqq (\alpha - \beta)^2 = (1 - \gamma)^2\alpha^2 \geqq (1 - \gamma)^2\epsilon$.

LEMMA 3. *$p(b) < 1$ for some $b > 0$ if and only if $\pi(b) < 1$ for some $b > 0$.*

PROOF. It is clear from the definitions that $\pi(b) \leqq p(b)$, $b > 0$, implying the "only if" part.

The "if" part is deeper, although several special cases, one of which we shall use in the proof, are known. Let $r = (r_1, r_2, \cdots)$ be an independent sequence of random variables satisfying $P(r_k = -1) = P(r_k = 1) = \frac{1}{2}$, $k \geqq 1$. Then there is a number $b_r > 0$ such that $p^r(b_r) < 1$. (See Lemma 19 of [8] or Lemma V. 8.27 of

[11]. By our Theorem 4, b_r may be taken to be 1.) Another special case, containing this one, has been studied by Marcinkiewicz and Zygmund [7]. They assume that the independent sequence Z satisfies $EZ_k = 0$ and $E|Z_k|^2 = 1, k \geqq 1$. Their Theorem 2 and Lemma α imply the "if" part of Lemma 3 for real random variables and real $a, a_1, a_2, \cdots$ satisfying $a = 0$. Their methods apply just as easily to the complex case (or to the Hilbert (I) case) with a unrestricted. For the martingale version of Theorem 2 of Marcinkiewicz and Zygmund, not amenable to their approach, see Lemma 4 below.

Now suppose that $\delta > 0$ satisfies $\pi(\delta) < 1$. Let $0 < b < b_r\delta[1 - \pi(\delta)]^{\frac{1}{2}}/2$. Then $p(b) < 1$ as we now show. Write $4b^2/b_r^2 = \gamma\delta^2\epsilon$ where $0 < \gamma < 1$ and $0 < \epsilon < 1 - \pi(\delta)$. Then there is a positive integer n such that $\pi_k(\delta) \leqq 1 - \epsilon$, $k \geqq n$. Assume, as we may, the existence and independence of three sequences r, Z, and Z', where r is as above, Z is the given sequence, and $Z' = (Z_1', Z_2', \cdots)$ is a sequence with the same distribution as Z. Let $\tilde{Z}_k = Z_k - Z_k', k \geqq 1$, and $\tilde{Z} = (\tilde{Z}_1, \tilde{Z}_2, \cdots)$. Then $(r_1\tilde{Z}_1, r_2\tilde{Z}_2, \cdots)$ and $\tilde{Z}$ have the same distribution. Let $N \geqq n$ and $a_1, a_2, \cdots$ satisfy $\sum_{k=n}^{N} |a_k|^2 = 1$. Then

$$\begin{aligned} P(|a + \textstyle\sum_{k=n}^{N} a_k Z_k| < b)^2 &= P(|a + \textstyle\sum_{k=n}^{N} a_k Z_k| < b,\ |a + \textstyle\sum_{k=n}^{N} a_k Z_k'| < b) \\ &\leqq P(|\textstyle\sum_{k=n}^{N} a_k \tilde{Z}_k| < 2b) \\ &= P(|\textstyle\sum_{k=n}^{N} a_k r_k \tilde{Z}_k| < 2b) \\ &= EP(|\textstyle\sum_{k=n}^{N} a_k r_k \tilde{Z}_k| < 2b \mid \tilde{Z}) \\ &\leqq Ep^r(2b(\textstyle\sum_{k=n}^{N} |a_k \tilde{Z}_k|^2)^{-\frac{1}{2}}) \end{aligned}$$

where $p^r(\infty)$ is defined to be 1. Since $p^r(c) \leqq p^r(b_r) < 1, 0 < c \leqq b_r$, to finish proving $p(b) \leqq p_n(b) < 1$, it remains only to notice that

$$\begin{aligned} P(2b(\textstyle\sum_{k=n}^{N} |a_k\tilde{Z}_k|^2)^{-\frac{1}{2}} \leqq b_r) &= P(\textstyle\sum_{k=n}^{N} |a_k|^2 |\tilde{Z}_k|^2 \geqq \gamma\delta^2\epsilon) \\ &\geqq (1 - \gamma)^2\epsilon, \end{aligned}$$

which follows from Lemma 2 and the fact that

$$\begin{aligned} P(|\tilde{Z}_k| < \delta) &= EP(|Z_k - Z_k'| < \delta \mid Z_k') \\ &\leqq E\pi_k(\delta) \leqq 1 - \epsilon, \qquad k \geqq n. \end{aligned}$$

This completes the proof of Lemma 3. Theorem 3 is established.

COROLLARY 1. *If Z satisfies $\pi(b) < 1$ for some $b > 0$ and $a_1, a_2, \cdots$ are complex numbers such that the series $\sum_{k=1}^{\infty} a_k Z_k$ converges almost everywhere, then* (i) $\sum_{k=1}^{\infty} |a_k|^2 < \infty$, *and* (ii) *in case $a_k \neq 0$ for infinitely many k, $P(\sum_{k=1}^{\infty} a_k Z_k = a) = 0$ for all complex a.*

Many special cases of (i) are known; for example, see [7]. A special case of (ii) for the Rademacher sequence is presented in [3]. Corollary 1 is an immediate consequence of Theorem 3.

COROLLARY 2. *Suppose that Z satisfies $\sup_k \pi_k(b) < 1$ for some $b > 0$ and that $a_{11}, a_{12}, \cdots, a_{21}, \cdots$ are complex numbers such that for each positive integer n the* *series* $\sum_{k=1}^{\infty} a_{nk} Z_k$ *converges almost everywhere. Let* $Y_n = \sum_{k=1}^{\infty} a_{nk} Z_k$ *and suppose that the sequence* $Y = (Y_1, Y_2, \cdots)$ *converges almost everywhere. Then there are complex numbers* $a_1, a_2, \cdots$ *such that*

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} |a_{nk} - a_k|^2 = 0.$$

Furthermore,

$$\sum_{k=1}^{\infty} |a_k|^2 \leqq \liminf_{n\to\infty} \sum_{k=1}^{\infty} |a_{nk}|^2 \leqq \sup_n \sum_{k=1}^{\infty} |a_{nk}|^2 < \infty.$$

PROOF. Let k be a positive integer. Then $(a_{1k}, a_{2k}, \cdots)$ is a convergent sequence: Write $Y_n = a_{nk} Z_k + W_n$ and note that Z_k is independent of $W = (W_1, W_2 \cdots)$. Let $\epsilon > 0$. Choose $b > 0$ such that $\pi_k(b) < 1$ and let $\delta = b\epsilon$. Since Y converges almost everywhere there is a positive integer N such that if $N < m < n$ then

$$\begin{aligned} \pi_k(b) &< P(|Y_n - Y_m| < \delta) \\ &= EP(|(a_{nk} - a_{mk})Z_k + W_n - W_m| < \delta \mid W) \\ &\leqq E\pi_k(\delta/|a_{nk} - a_{mk}|), \end{aligned}$$

using independence and the definition of π_k with $\pi_k(\infty)$ defined to be 1. Therefore, since π_k is nondecreasing, $|a_{nk} - a_{mk}| < \delta/b = \epsilon$, $N < m < n$, implying convergence. Let a_k denote the limit of $(a_{1k}, a_{2k}, \cdots)$.

We now show that $\lim_{n\to\infty} \sum_{k=1}^{\infty} |a_{mk} - a_k|^2 = 0$. Let $\epsilon > 0$. By Theorem 3, there is a $b > 0$ such that Z has the Stein property with constant b. Here let $\delta > 0$ satisfy $\delta + \delta^2/b^2 < \epsilon$. There is a positive integer M such that

$$A = \{\omega : \sup_{M<m<n} |Y_n(\omega) - Y_m(\omega)| < \delta\}$$

satisfies $P(A) > 0$. By the Stein property, there exists an integer $K > 1$ such that if $M < m < n$ then

$$b\left(\sum_{k=K}^{\infty} |a_{nk} - a_{mk}|^2\right)^{\frac{1}{2}} \leqq \operatorname{ess\,sup}_{\omega\epsilon A} |Y_n(\omega) - Y_m(\omega)| \leqq \delta.$$

There is an integer $N > M$ such that $\sum_{k=1}^{K-1} |a_{nk} - a_{mk}|^2 < \delta$, $N < m < n$. Therefore, if $N < m < n$, then

$$\sum_{k=1}^{\infty} |a_{nk} - a_{mk}|^2 < \delta + \delta^2/b^2 < \epsilon.$$

By Fatou, if $N < m$, then

$$\sum_{k=1}^{\infty} |a_k - a_{mk}|^2 \leqq \liminf_{n\to\infty} \sum_{k=1}^{\infty} |a_{nk} - a_{mk}|^2 \leqq \epsilon,$$

implying the desired convergence to 0.

By Corollary 1, $\sum_{k=1}^{\infty} |a_{nk}|^2 < \infty$, $n \geqq 1$. From this fact and the above it follows that $\sup_n \sum_{k=1}^{\infty} |a_{nk}|^2 < \infty$, and another application of Fatou completes the proof of Corollary 2.

LEMMA 4. *To each* $\delta > 0$ *corresponds an* $\alpha > 0$ *with the following property: If* $f = (f_1, f_2, \cdots)$ *is a martingale and* $E|d_k| \geqq \delta E^{\frac{1}{2}}|d_k|^2$, $k \geqq 1$, *where* $d_1 = f_1$, $d_2 = f_2 - f_1$, $d_3 = f_3 - f_2, \cdots$, *then* $E|f_n| \geqq \alpha E^{\frac{1}{2}}|f_n|^2$, $n \geqq 1$.

This result, which will be used in the proof of Theorem 4, is the martingale version of Theorem 2 of Marcinkiewicz and Zygmund [7]. Our proof is necessarily different since here $\{d_j \bar{d}_k : 1 \leqq j < k\}$ is not always an orthogonal family.

PROOF. Let $\delta > 0$. Suppose that f is a martingale satisfying $E|d_k| \geqq \delta c_k$, where $c_k = E^{\frac{1}{2}}|d_k|^2$ may be assumed to be strictly positive, $k \geqq 1$. Necessarily, $\delta \leqq 1$. By Lemma 1, $P(|d_k| \geqq \beta c_k) \geqq \beta$, $k \geqq 1$, if β is chosen to satisfy $0 < \beta < \delta$ and $\beta = (\delta - \beta)^2$. Let $S_n(f) = (\sum_{k=1}^n |d_k|^2)^{\frac{1}{2}}$. By Lemma 2,

$$P(S_n(f) \geqq \beta^2 E^{\frac{1}{2}}|f_n|^2) = P(\textstyle\sum_{k=1}^n c_k^2 |d_k/c_k|^2 \geqq \beta^4 \sum_{k=1}^n c_k^2) \geqq (1 - \beta)^2 \beta$$
$$\geqq (\delta - \beta)^2 \beta = \beta^2.$$

Therefore, by Schwarz's inequality and Theorem 9 of [2],

$$E^{\frac{1}{2}}|f_n| E^{\frac{1}{2}}|f_n|^2 \geqq E|f_n|^{\frac{3}{2}} \geqq M_{\frac{3}{2}} E S_n(f)^{\frac{3}{2}} \geqq M_{\frac{3}{2}} (\beta^2 E^{\frac{1}{2}}|f_n|^2)^{\frac{3}{2}} \beta^2,$$

implying that $E|f_n| \geqq \alpha E^{\frac{1}{2}}|f_n|^2$ for $\alpha = M_{\frac{3}{2}}^2 \beta^{10}$.

THEOREM 4. *If the sequence Z is such that $-Z_k$ has the same distribution as Z_k and $|Z_k(\omega)| = 1$, $\omega \varepsilon \Omega$, $k \geqq 1$, then $p_1(1) < 1$.*

PROOF. In the proof we may and do assume the existence of an independent sequence $r = (r_1, r_2, \cdots)$ on $(\Omega, \mathcal{A}, P)$, independent of Z, such that $P(r_k = -1) = P(r_k = 1) = \frac{1}{2}$, $k \geqq 1$. Then $p_1^Z(1) \leqq p^r(1)$, since, if $\sum_{k=1}^n |a_k|^2 = 1$, then also $\sum_{k=1}^n |a_k Z_k|^2 = 1$, and

$$\begin{aligned} P(|a + \textstyle\sum_{k=1}^n a_k Z_k| < 1) &= P(|a + \textstyle\sum_{k=1}^n a_k r_k Z_k| < 1) \\ &= EP(|a + \textstyle\sum_{k=1}^n a_k r_k Z_k| < 1 \mid Z) \\ &\leqq E p^r(1) = p^r(1), \end{aligned}$$

using the fact that $(r_1 Z_1, r_2 Z_2, \cdots)$ has the same distribution as Z.

We now show that $p^r(1) < 1$. Let $a, a_1, a_2, \cdots$ be complex numbers and define $a_0 = a$ and $r_0 = 1$. Then

$$\begin{aligned} P(|a + \textstyle\sum_{k=1}^n a_k r_k| < (\sum_{k=1}^n |a_k|^2)^{\frac{1}{2}}) &\leqq P(|\textstyle\sum_{k=0}^n a_k r_k|^2 < \sum_{k=0}^n |a_k|^2) \\ &= P(\textstyle\sum_{j=0}^n \sum_{k=0}^n a_j \bar{a}_k r_j r_k < \sum_{k=0}^n |a_k|^2) \\ &= P(\textstyle\sum_{k=1}^n (\sum_{j=0}^{k-1} a_{jk} r_j) r_k < 0) \\ &= P(f_n < 0) \end{aligned}$$

where $a_{jk} = 2 \operatorname{Re} a_j \bar{a}_k$ and $f = (f_1, f_2, \cdots)$ is the real martingale with $d_k = (\sum_{j=0}^{k-1} a_{jk} r_j) r_k$, $k \geqq 1$. Since $|d_k| = |\sum_{j=0}^{k-1} a_{jk} r_j|$ almost everywhere, the existence of a $\delta > 0$ such that $E|d_k| \geqq \delta E^{\frac{1}{2}}|d_k|^2$ follows from well known facts, or, alternatively, is a consequence of Lemma 4. Therefore, by Lemma 4, $E|f_n| \geqq \alpha E^{\frac{1}{2}}|f_n|^2$, where $\alpha > 0$ satisfies the inequality for all n and all $a, a_1, a_2, \cdots$. Since $Ef_n = 0$, we have that $Ef_n^+ = E|f_n|/2 \geqq (\alpha/2) E^{\frac{1}{2}}|f_n|^2 \geqq (\alpha/2) E^{\frac{1}{2}}|f_n^+|^2$, implying, by Lemma 1, that

$$P(f_n \geqq 0) \geqq P(f_n^+ \geqq \beta E^{\frac{1}{2}}|f_n^+|^2) \geqq [(\alpha/2) - \beta]^2, \qquad 0 < \beta < \alpha/2.$$

Therefore,

$$p^r(1) \leqq 1 - \alpha^2/4.$$

This completes the proof of Theorem 4.

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