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ASYMPTOTICALLY OPTIMUM PROPERTIES OF CERTAIN SEQUENTIAL TESTS[1]

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1. Introduction and summary. Let $X_1, X_2, \cdots$ be independent and identically distributed random variables whose common distribution is of the one-parameter Koopman-Darmois type, i.e., the density function of X_1 relative to some σ-finite nondegenerate measure of F on the real line can be written as

$$f(x, \theta) = \exp(\theta x - b(\theta)),$$

where $b(\theta)$ is some real function of the parameter θ. Consider the hypotheses $H_0 = \{\theta \leqq \theta_0\}$ and $H_1 = \{\theta \geqq \theta_1\}$ where $\theta_0 < \theta_1$ and θ_0, θ_1 are in Ω, the natural parameter space. We want to decide sequentially between the two hypotheses. Suppose $l(\theta)$ is the loss for making a wrong decision when θ is the true parameter and assume $0 \leqq l(\theta) \leqq 1$ for all θ and $l(\theta) = 0$ if θ is in (θ_0, θ_1), i.e., (θ_0, θ_1) is an indifference zone. Let c be the cost of each observation. It is sufficient to let the decision depend on the sequence (n, S_n), $n \geqq 1$, where $S_n = X_1 + \cdots + X_n$. We shall consider the observed values of (n, S_n) as points in a (u, v) plane. Then, for any test, the region in the (u, v) plane where sampling does not stop is called the continuation region of the test. A test and its continuation region will be denoted by the same symbol.

Schwarz [4] introduced an *a priori* distribution W and studied the asymptotic shape of the Bayes continuation region, say $B_W(c)$, as $c \to 0$. He showed that $B_W(c)/\ln c^{-1}$ approaches, in a certain sense, a region B_W that depends on W only through its support. Whereas Schwarz's work is concerned with Bayes tests, in this paper the main interest is in characteristics of sequential tests as a function of θ. In particular, it is desired to minimize the expected sample size (uniformly in θ if possible) subject to certain bounds on the error probabilities. Our approach, like Schwarz's, is asymptotic, as $c \to 0$. It turns out that an asymptotically optimum test—in the sense indicated above, is $B_W \ln c^{-1}$ if W is a measure that dominates Lebesgue measure. Such a measure will be denoted by L (for Lebesgue dominating) from now on. Thus, Bayes tests, as a tool, will play a significant role in this paper.

In order to prove the optimum characteristic of $B_L \ln c^{-1}$, some other results, of interest in their own right, are established. For any W satisfying certain conditions that will be given later, we show that the stopping variable $N(c)$ of $B_W(c)$ approaches ∞ a.e. P_θ for every θ in Ω. This result together with Schwarz's

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result that $B_W(c) \ln c^{-1}$ approaches a finite region, leads to the following results: (i) for $B_W(c)$, $E_\theta N(c)/\ln c^{-1}$ tends to a constant for each θ in Ω and (ii) the same is true for the stopping variable of $B_W \ln c^{-1}$. Furthermore, it is shown that for $B_L \ln c^{-1}$ the error probabilities tend to zero faster than $c \ln c^{-1}$. Consequently, the contributions of the expected sample sizes of both $B_L \ln c^{-1}$ and $B_L(c)$ to their integrated risks, over any L-measure, approach 100%. Moreover $B_L \ln c^{-1}$ is asymptotically Bayes. The last result can be shown without (i) since it is sufficient to show (ii) and to apply the same argument used by Kiefer and Sacks [3] in the proof of their Theorem 1. But we show (i) because of its intrinsic interest and present a different proof using (i). Kiefer and Sacks assumed a more general distribution for X_1, constructed a procedure $\delta_c'^{I}$ and showed that it is asymptotically Bayes. Our $B_L \ln c^{-1}$ is somewhat more explicit than their $\delta_c'^{I}$. We would also like to point out that an example of $B_L \ln c^{-1}$, when the distribution is normal, is very briefly discussed in their work.

We shall restrict ourselves to *a priori* distribution W for which $\sup \pmod W)H_0 = \theta_0$, $\inf \pmod W)H_1 = \theta_1$ and $0 < W(H_0 \cup H_1) < 1$. The phrase "for any W" or "for every W" is to be understood in that sense. Any Lebesgue dominating measure satisfies these conditions and also the following type of W that will be used: the support of W consists of θ_0, θ_1 and a third point θ^*, $\theta_0 < \theta^* < \theta_1$. Such a W will be called a θ^*-measure, and the corresponding B_W denoted by B_{θ^*}. From Schwarz's equations for B_W it follows readily that $B_L \subset B_W$ for every W. In particular, $B_L \subset B_{\theta^*}$. As a consequence, the statement about the error probabilities as well as others concerning $B_L \ln c^{-1}$ in the last paragraph, remain true when L is replaced by θ^* or any W. Those geometric characteristics will be dealt with in Section 2. We shall also show there that ∂B_{θ^*}, the boundary of B_{θ^*} (which consists of line segments), is tangent to ∂B_L at some point, and that if θ^* is such that $b'(\theta^*) = (b(\theta_1) - b(\theta_0))/(\theta_1 - \theta_0)$ then $\max_{(u,v) \text{ in } B_L} u = \max_{(u,v) \text{ in } B_{\theta^*}} u$.

Let the ray through the origin and with slope equal to $E_\theta X_1$ intersect ∂B_L at $(m(\theta), m(\theta)E_\theta X_1)$. In Section 3, after proving $\lim_{c\to 0} N(c) = \infty$ a.e. P_θ, we show $\lim_{c\to 0} N(c)/\ln c^{-1} = m(\theta)$ a.e. P_θ and $\lim_{c\to 0} E_\theta N(c)/\ln c^{-1} = m(\theta)$. It is shown in Section 4 that $\sup_{\theta \text{ in } H_0 \cup H_1} P_\theta$ (error $| B_L \ln c^{-1}) = o(c \ln c^{-1})$.

The main results are given in Section 5. We first show that after dividing by $c \ln c^{-1}$, the difference of the integrated risks of $B_L \ln c^{-1}$ and $B_W(c)$, for any W, tends to zero. It follows from this result that $B_L \ln c^{-1}$ asymptotically minimizes the maximum (over θ in Ω) expected sample size in $\mathfrak{F}(c)$, a family of tests whose error probabilities are bounded by $\max_{i=0,1} P_{\theta_i}$ (error $| B_L \ln c^{-1}$). The precise statement is given in Theorem 5.1. A sharper result under a stronger hypothesis is given in Theorem 5.2 which states that $B_L \ln c^{-1}$ asymptotically minimizes the expected sample size $E_\theta N$ for each θ, $\theta_0 < \theta < \theta_1$, among all procedures of $\mathfrak{F}(c)$ for which $E_{\theta_0} N/\ln c^{-1}$ and $E_{\theta_1} N/\ln c^{-1}$ are bounded in c.

2. Geometric properties of asymptotic Bayes continuation regions. We shall first look into some geometric characteristics of ∂B_W, the boundary of B_W, for any given W, and then shall show some relations betwen B_L and B_{θ^*}.

Schwarz [4] has shown that $B_W = B_0 \cap B_1$ (both B_0 and B_1 also depend on W) where B_i, $i = 0, 1$, are first defined as

$$(2.1)\quad B_i = \{(u, v): s(v/u) \leqq u^{-1} + \sup_{\theta \text{ in } H_i} (\text{mod } W)(\theta v/u - b(\theta))\},$$

where for any k,

$$(2.2)\quad s(k) = \sup_{\theta \text{ in } \Omega} (\text{mod } W)(\theta k - b(\theta)).$$

Then he shows that, equivalently, B_i can be defined as

$$(2.3)\quad \begin{aligned} B_0 &= \{(u, v): v/u > k_0,\ s(v/u) \leqq u^{-1} + \theta_0 v/u - b(\theta_0)\},\\ B_1 &= \{(u, v): v/u < k_1,\ s(v/u) \leqq u^{-1} + \theta_1 v/u - b(\theta_1)\},\end{aligned}$$

where k_0 and k_1 are defined as follows:

$$(2.4)\quad \begin{aligned} k > k_0 &\text{ iff } s(k) > \sup_{\theta \text{ in } H_0} (\text{mod } W)(\theta k - b(\theta)),\\ k < k_1 &\text{ iff } s(k) > \sup_{\theta \text{ in } H_1} (\text{mod } W)(\theta k - b)\theta)).\end{aligned}$$

If $\theta^0 = \inf\ (\text{mod } W)(\Omega - H_0)$, $\theta' = \sup\ (\text{mod } W)(\Omega - H_1)$, it can be easily shown $k_i = [b(\theta^i) - b(\theta_i)]/[\theta^i - \theta_i]$, $i = 0, 1$. Let

$$(2.5)\quad k^* = [b(\theta_1) - b(\theta_0)]/[\theta_1 - \theta_0].$$

The strict convexity of b implies

$$(2.6)\quad k_0 < k^* < k_1 .$$

It is clear that if W is an L-measure

$$(2.7)\quad k_0 = b'(\theta_0), \qquad k_1 = b'(\theta_1).$$

And that if W is a θ^*-measure

$$(2.8)\quad k_0 = [b(\theta^*) - b(\theta_0)]/[\theta^* - \theta_0], \qquad k_1 = [b(\theta_1) - b(\theta^*)]/[\theta_1 - \theta^*].$$

For any fixed k, let $\theta(k)$ be any number satisfying

$$(2.9)\quad s(k) = \theta(k) - b(\theta(k)).$$

$\theta(k)$ depends on W and may not be unique. If it is not, there are exactly two possible values. Some of the properties of $\theta(k)$ that we shall use are stated in the following lemma.

LEMMA 2.1. *If* $l_1 < l_2$ *then*

(i) $\theta(l_1) = \theta(l_2)$

(ii) *for any* $\alpha < \theta(l_1)$,

$$[\theta(l_2)l_2 - b(\theta(l_2)) - \alpha l_2 + b(\alpha)] - [\theta(l_1)l_1 - b(\theta(l_1)) - \alpha l_1 + b(\alpha)] > 0\cdot$$

PROOF. (i) Suppose $\theta' > \theta$ and let $q(k) \Rightarrow (\theta' - \theta)k - b(\theta') + b(\theta)$ so that $q(k)$ is strictly increasing in k. Thus, if $l_2 > l_1$, then $(\theta' - \theta)l_1 - b(\theta') + b(\theta) \geqq 0$ implies $(\theta' - \theta)l_2 - b(\theta') + b(\theta) > 0$. That is, if $\theta' l_1 - b(\theta') \geqq \theta l_1 - b(\theta)$, then $\theta' l_2 - b(\theta') > \theta l_2 - b(\theta)$. It follows $\theta(l_1) \leqq \theta(l_2)$.

(ii) It follows readily from the definition of $\theta(k)$,

$$\theta(l_2)l_2 - b(\theta(l_2)) \geqq \theta(l_1)l_2 - b(\theta(l_1)),$$

That is

$$\delta = \theta(l_2)l_2 - \theta(l_1)l_2 - b(\theta(l_2)) + b(\theta(l_1)) \geqq 0.$$

We have

$$[\theta(l_2)l_2 - b(\theta(l_2)) - \alpha l_2 + b(\alpha)] - [\theta(l_1)l_1 - b(\theta(l_1)) - \alpha l_1 + b(\alpha)]$$
$$= (\theta(l_1) - \alpha)(l_2 - l_1) + \delta > 0.$$

This completes the proof of the lemma.

It follows from (2.3) that ∂B_W consists of two curves given by:

$$\text{(2.10)} \qquad v/u > k_0, \quad s(v/u) = u^{-1} + \theta_0 v/u - b(\theta_0),$$
$$v/u < k_1, \quad s(v/u) = u^{-1} + \theta_1 v/u - b(\theta_1);$$

or

$$\text{(2.11)} \qquad v/u > k_0, \quad u = [s(v/u) - \theta_0 v/u + b(\theta_0)]^{-1},$$
$$v/u < k_1, \quad u = [s(v/u) - \theta_1 v/u + b(\theta_1)]^{-1},$$

where $s(\cdot)$ is defined in (2.2). The first curve in (2.10) or (2.11), denoted by ∂B_0, is called the upper boundary, the other ∂B_1 the lower boundary. Define ray(k) as the ray through the origin of the (u, v) plane and making a slope k with the positive u-axis. Since for each k, $s(k)$ is unique, so if $k > k_0$ (respectively $k < k_1$) ray(k) intersects ∂B_0 (respectively ∂B_1) at a unique point, say $(u_0(k), ku_0(k))$ (respectively $(u_1(k), ku_1(k))$), where using (2.11),

$$u_i(k) = [s(k) - \theta_i k + b(\theta_i)]^{-1}, \qquad i = 0, 1.$$

By (2.4) and (2.9), if $k > k_0$, $\theta(k) > \theta_0$, so by Lemma 2.1 (ii), $s(k) - \theta_0 k + b(\theta_0)$ is increasing in k. Thus $u_0(k)$ is decreasing in k. Similarly $u_1(k)$ is increasing in k. Consequently, the two boundaries meet at a point (m^*, v^*) that satisfies

$$\text{(2.12)} \qquad \max_{(u,v) \text{ in } B_W} u = m^*.$$

Solving the two equations of (2.10), we find

$$\text{(2.13)} \qquad v^*/m^* = k^*$$

where k^* is given by (2.5). So

$$\text{(2.14)} \qquad m^* = u_0(k^*) = u_1(k^*).$$

It then follows

$$\text{(2.15)} \qquad u_0(k) \leqq u_1(k) \quad \text{if} \quad k \geqq k^* \quad \text{and} \quad u_0(k) \geqq u_1(k) \quad \text{if} \quad k \leqq k^*.$$

If we let m_k be the u-coordinate of the intersection of ray(k) with ∂B_W, then $m_k = \min(u_0(k), u_1(k))$; so we have from (2.11) and (2.15)

$$
\begin{aligned}
(2.16)\qquad m_k &= u_0(k) = [s(k) - \theta_0 k + b(\theta_0)]^{-1} \quad \text{if} \quad k \geqq k^* \\
&= u_1(k) = [s(k) - \theta_1 k + b(\theta_1)]^{-1} \quad \text{if} \quad k \leqq k^*.
\end{aligned}
$$

Also, there will be no loss of generality if we rewrite (2.3) as

$$
\begin{aligned}
(2.17)\qquad B_0 &= \{(u, v): v/u \geqq k^*, s(v/u) \leqq u^{-1} + \theta_0 v/u - b(\theta_0)\}, \\
B_1 &= \{(u, v): v/u \leqq k^*, s(v/u) \leqq u^{-1} + \theta_1 v/u - b(\theta_1)\}.
\end{aligned}
$$

If W is an L-measure, it can be seen that $b'(\theta(k)) = k$. It is known that b'' is a positive function so the inverse of b' exists and is $\theta(\cdot)$, where $\theta(\cdot)$ is defined in (2.9). Thus, in this case if $b(\theta)$ is given, (2.10), the equations of the boundaries, can be expressed explicitly.

Next we shall consider the case where W is a θ^*-measure. By (2.4) and (2.6)

$$
\begin{aligned}
s(v/u) = \sup_{\theta=\theta_0,\theta^*,\theta_1} (\theta v/u - b(\theta)) &= \theta_0 v/u - b(\theta_0) \quad \text{if} \quad v/u \leqq k_0 \\
(2.18)\qquad &= \theta^* v/u - b(\theta^*) \quad \text{if} \quad k_0 \leqq v/u \leqq k_1 \\
&= \theta_1 v/u - b(\theta_1) \quad \text{if} \quad v/u \geqq k_1.
\end{aligned}
$$

Thus ∂B_0 consists of the two following line segments:

$$
\begin{aligned}
l_1 : \theta_1 v/u - b(\theta_1) &= u^{-1} + \theta_0 v/u - b(\theta_0), \qquad v/u > k_1, \\
l_2 : \theta^* v/u - b(\theta^*) &= u^{-1} + \theta_0 v/u - b(\theta_0), \qquad k_0 < v/u \leqq k_1,
\end{aligned}
$$

and ∂B_1 consists of

$$
\begin{aligned}
l_3 : \theta_0 v/u - b(\theta_0) &= u^{-1} + \theta_1 v/u - b(\theta_1), \qquad v/u < k_0, \\
l_4 : \theta^* v/u - b(\theta^*) &= u^{-1} + \theta_1 v/u - b(\theta_1), \qquad k_0 \leqq v/u < k_1.
\end{aligned}
$$

For any L-measure L and arbitrary W, as

$$
\begin{aligned}
\sup_{\theta \text{ in } \Omega} (\text{mod } L)(\theta v/u - b(\theta)) &= \sup_{\theta \text{ in } \Omega} (\theta v/u - b(\theta)) \\
&\geqq \sup_{\theta \text{ in } \Omega} (\text{mod } W)(\theta v/u - b(\theta)),
\end{aligned}
$$

it follows from (2.3) that the following theorem holds.

THEOREM 2.1. *If L is an L-measure, for any W $B_L \subset B_W$. In particular, $B_L \subset B_{\theta^*}$.*

COROLLARY 2.1. $\max_{(u,v) \text{ in } B_L} u \leqq \max_{(u,v) \text{ in } B_{\theta^*}} u$.

Let the u-coordinates of the intersection of ray(k) with ∂B_L and ∂B_{θ^*} be $u(k)$ and $m(k)$ respectively. By Theorem 2.1, $u(k) \leqq m(k)$. The following theorem tells when the equality holds.

THEOREM 2.2. *If $k = b'(\theta^*)$, then*

$$u(k) = m(k).$$

PROOF. Assume $k = k^*$. The case $k < k^*$ is similar. By the strict convexity of b and (2.8), $k < k_1$. By (2.6), $k \geqq k^*$ implies $k > k_0$. Thus, it follows from (2.18) and (2.16),

$$m(k) = [\theta^* k - b(\theta^*) - \theta_0 k + b(\theta_0)]^{-1}.$$

For an L-measure, $k = b'(\theta^*)$ implies $\theta(k) = \theta^*$, so from (2.16) and (2.9) we have

$$u(k) = [\theta^* k - b(\theta^*) - \theta_0 k + b(\theta_0)]^{-1}.$$

Hence the theorem is established.

Corollary 2.2. *There exists a* θ^*, $\theta_0 < \theta^* < \theta_1$ *such that* $b'(\theta^*) = k^*$ *where* k^* *is given by* (2.5). *Furthermore, for such* θ^*,

$$\max_{(u,v) \text{ in } B_L} u = \max_{(u,v) \text{ in } B_{\theta^*}} u.$$

Proof. Since $b'(\theta_0) < k^* < b'(\theta_1)$ and b' is continuous, there exists a θ^* such that $\theta_0 < \theta^* < \theta_1$ and $b'(\theta^*) = k^*$. The second assertion follows from Theorem 2.2 and (2.14).

Corollary 2.3. *For any* $\theta_0 < \theta' < \theta_1$, $\partial B_{\theta'}$ *is tangent to* ∂B_L *at some point that lies on* ray(k), *where* $k = b'(\theta')$.

Proof. Again we assume $k \geqq k^*$. From Theorem 2.2, we know $\partial B_{\theta'}$, ∂B_L and ray(k) meet at a common point, say, (u', v'). From the proof of Theorem 2.2 we also know that (u', v') lies on the upper boundaries of both B_L and $B_{\theta'}$ and that l_2 is the line segment of $\partial B_{\theta'}$ on which (u', v') lies. It can be easily seen that the slope of l_2 is $[b(\theta') - b(\theta_0)]/[\theta' - \theta_0]$. To establish the corollary, we only have to show that ∂B_0 of B_L has the same slope at (u', v'). Multiplying the first equation of (2.10) by u and then differentiating with respect to u and noting that $b'(\theta(v/u)) = v/u$, we obtain

$$\theta(v/u)\, dv/du + v\, d\theta(v/u)/du - b(\theta(v/u)) - ub'(\theta(v/u))\, d\theta(v/u)/du = \theta_0\, dv/du - b(\theta_0).$$

Thus,

$$dv/du = [b(\theta(v/u)) - b(\theta_0)]/[\theta(v/u) - \theta_0].$$

Since $v'/u' = b'(\theta')$ and $\theta(b'(\theta')) = \theta'$, at (u', v'),

$$dv/du = [b(\theta') - b(\theta_0)]/[\theta' - \theta_0].$$

This completes the proof.

3. Some asymptotic properties of the sample size. Let u, v be any values of n, S_n respectively, where $S_n = X_1 + \cdots + X_n = n\bar{X}_n$. Define, as in Schwarz, for $i = 0, 1$,

$$R_i(u, v) = \int_{H_i} \exp[\theta v - ub(\theta)] l(\theta) W(d\theta) \Big/ \int_\Omega \exp[\theta v - ub(\theta)] W(d\theta), \tag{3.1}$$

$$C_i(c) = \{(u, v) : R_i(u, v) = c\}, \tag{3.2}$$

$$C(c) = C_0(c) \cap C_1(c). \tag{3.3}$$

Schwarz has shown that for any W

$$C(c) \supset B_W(c) \supset C(dc \ln c^{-1}), \tag{3.4}$$

where d is some positive constant not depending on W. We shall apply this result to show in this section that for each θ in Ω $\lim_{c\to 0} N(c) = \infty$ a.e. P_θ, $\lim_{c\to 0} N(c)/\ln c^{-1} = m_{\mu_\theta}$ a.e. P_θ and $\lim_{c\to 0} E_\theta N(c)/\ln c^{-1} = m_{\mu_\theta}$, where $N(c)$ is the stopping variable of $B_W(c)$ and m_{μ_θ} will be defined later. The following assumption is needed in proving Lemmas 3.1 and 3.2.

ASSUMPTION A. If Ω has a finite endpoint, say α, then $b(\theta)$ tends to ∞ as θ tends to α. (It can be easily seen that the Ω's for normal, Bernoulli and Poisson distributions have no finite end points and that geometric and exponential distributions have finite endpoints but satisfy the assumption.)

LEMMA 3.1. *If* $\inf_\theta b'(\theta)$ *is finite, then* Ω *is unbounded on the left.*

PROOF. Suppose that Ω has a finite left endpoint, say α. Then by Assumption A, $b(\theta) \to \infty$ as $\theta \downarrow \alpha$. So $b'(\theta) \to -\infty$ as $\theta \downarrow \alpha$, contradicting our hypothesis.

LEMMA 3.2. *If* $\inf_\theta b'(\theta) = a$, *finite, then for each* θ,

$$P_\theta(\bar{X}_n < a) = 0, \qquad n = 1, 2, \cdots,$$

Furthermore, either $P_\theta(\bar{X}_n = a) = 0$ *for* $n = 1, 2, \cdots$, *or* > 0 *for* $n = 1, 2, \cdots$.

PROOF. It is sufficient to show the assertions for X_1. By Lemma 3.1, Ω is unbounded on the left. Since $b'(\theta)$ is strictly increasing and bounded below there exists a sequence $\theta_n \downarrow -\infty$ $(n = 1, 2, \cdots)$ such that $\lim_{n\to\infty} b''(\theta_n) = 0$. As $E_\theta X_1 = b'(\theta)$ and $V_\theta X_1 = b''(\theta)$, by Chebyshev's inequality, for $n = 1, 2, \cdots$, and any positive ϵ

$$P_{\theta_n}(|X_1 - b'(\theta_n)| \geqq \epsilon) \leqq b''(\theta_n)/\epsilon^2.$$

Thus $\lim_{n\to\infty} P_{\theta_n}(|X_1 - b'(\theta_n)| \geqq \epsilon) = 0$. But

$$\begin{aligned} P_\theta(|X_1 - b'(\theta)| \geqq \epsilon) &\geqq P_\theta(X_1 - b'(\theta) \leqq -\epsilon) \\ &= P_\theta(X_1 \leqq b'(\theta) - \epsilon) \\ &\geqq P_\theta(X_1 \leqq a - \epsilon). \end{aligned}$$

So $\lim_{n\to\infty} P_{\theta_n}(X_1 \leqq a - \epsilon) = 0$, for any $\epsilon > 0$. Since X_1 has monotone likelihood ratio property, for any x, $P_{\alpha_1}(X_1 \leqq x) \geqq P_{\alpha_2}(X_1 \leqq x)$ whenever $\alpha_1 < \alpha_2$, it then follows that for every θ,

$$P_\theta(X_1 \leqq a - \epsilon) \leqq \lim_{n\to\infty} P_{\theta_n}(X_1 \leqq a - \epsilon) = 0,$$

implying $P_\theta(X_1 < a) = 0$. The second part of the lemma is an immediate consequence of the first part.

We shall say a ray intersects some curve if the ray intersects the curve at a point different from the origin. A ray (k) may or may not intersect ∂B (B for B_W).

LEMMA 3.3 *If* k *satisfies*

$$\sup_{\theta \text{ in } \Omega} (\theta k - b(\theta)) < \infty, \tag{3.5}$$

then ray(k) *intersects* ∂B.

PROOF. By the definition (2.2) of $s(\cdot)$, (3.5) inplies $s(k) < \infty$. As

$$s(k) > \sup_{\theta \text{ in } H_i} (\text{mod } W)\,(\theta k - b(\theta)) = s_i(k), \quad \text{say},$$

for $i = 0$ or 1, there exists u_0, $0 < u_0 < \infty$, such that $s(k) \leqq u_0^{-1} + s_i(k)$, $i = 0, 1$, and the equality holds for at least one i. Thus it follows from (2.1) that (u_0, ku_0) is on ∂B, and the lemma is therefore established.

COROLLARY 3.1. *If k is in the range of b', then ray(k) intersects ∂B.*

PROOF. This follows from the fact that $\theta k - b(\theta)$ attains its maximum at the θ satisfying $b'(\theta) = k$.

COROLLARY 3.2. *If for some α, $P_\alpha(X_1 = k) > 0$, then ray(k) intersects ∂B.*

PROOF. $P_\alpha(X_1 = k) > 0$ implies $F(\{k\}) > 0$. Since for any θ

$$\int \exp(\theta x - b(\theta))F(dx) = 1,$$

we have $\exp(\theta k - b(\theta)) \leqq 1/F(\{k\})$ so that $\sup_{\theta \text{ in } \Omega} (\theta k - b(\theta)) < \infty$.

REMARK. If $\sup_\theta b'(\theta)$ is finite, results analogous to Lemmas 3.1 and 3.2 can be established in a similar way.

Let $\mathfrak{X}$ denote the probability space on which $X_1, X_2, \cdots$ are defined, and $\mathcal{Y} = \{\omega : \omega \text{ in } \mathfrak{X}$ and for some positive integer n, ray$(\bar{X}_n(\omega))$ does not intersect $\partial B\}$.

LEMMA 3.4. *$P_\theta(\mathcal{Y}) = 0$ for all θ.*

PROOF. *Case* 1. $\inf_\theta b'(\theta) = -\infty$ and $\sup_\theta b'(\theta) = \infty$. By Corollary 3.1, $\mathcal{Y}$ is an empty set.

Case 2. $\inf_\theta b'(\theta) = a$, finite, and $\sup_\theta b'(\theta) = \infty$. The only rays which might not intersect ∂B are rays with slopes less than or equal to a.

$$P_\theta(\omega : \bar{X}_n(\omega) < a \quad \text{for some} \quad n) \leqq \sum_{n=1}^\infty P_\theta(\omega : \bar{X}_n(\omega) < a) = 0, \qquad \text{by Lemma 3.2.}$$

$$P_\theta(\omega : \bar{X}_n(\omega) = a \quad \text{for some} \quad n) \leqq \sum_{n=1}^\infty P_\theta(\omega : \bar{X}_n(\omega) = a) = 0 \quad \text{if} \quad P_\theta(\omega : X_1(\omega) = a) = 0,$$

by Lemma 3.2. Thus if $P_\theta(\omega : X_1(\omega) = a) = 0$, then $P_\theta(\mathcal{Y}) = P_\theta(\omega : \bar{X}_n(\omega) \leqq a$ for some $n) = 0$. If $P_\theta(\omega : X_1(\omega) = a) > 0$, then by Corollary 3.2 ray(a) intersects ∂B, and so

$$P_\theta(\mathcal{Y}) = P_\theta(\omega : \bar{X}_n(\omega) < a \quad \text{for some} \quad n) = 0$$

Case 3. $\inf b'(\theta) = -\infty$ and $\sup b'(\theta)$ is finite. This case is similar to Case 2.

Case 4. Both $\inf b'(\theta)$ and $\sup b'(\theta)$ are finite. The proof for this case follows from Cases 2 and 3.

Schwarz's main result is concerned with the asymptotic shape of $B(c)$ and $C(c)$. It can be described as follows: If ray(k) intersects ∂B at the point A then ray(k) also intersects both $\partial B(c)$ and $\partial C(c)$ when c is small, say, at points $P(c)$ and $Q(c)$ respectively. As c tends to zero, the coordinates of $P(c)$ and $Q(c)$, divided by $\ln c^{-1}$, converge to the corresponding coordinates of A. This result will be applied in proving some of the following lemmas and theorems. $P(c)$ and $Q(c)$ are not necessarily unique, but the uniqueness will not be required in the following work and we shall therefore assume for simplicity that the intersection points are unique.

THEOREM 3.1. *If $N(c)$ is the stopping variable for $B(c)$ then for each θ in Ω,*

$$\lim_{c\to 0} N(c) = \infty \text{ a.e. } P_\theta .$$

PROOF. Suppose ω is not in $\mathcal{Y}$, then ray$(\bar{X}_n(\omega))$, where n is any fixed positive integer, intersects ∂B and therefore also intersects $\partial B(c)$ where c is small. Let m and $u(c)$ be respectively the u-coordinates of the intersections. Applying Schwarz's result, we have

$$\lim_{c\to 0} u(c)/\ln c^{-1} = m.$$

Thus for any $\epsilon > 0$ there exists $c_0 > 0$ such that for $c < c_0$

$$(m - \epsilon) \ln c^{-1} \leqq u(c) \qquad \text{and} \qquad n < (m - \epsilon) \ln c^{-1}.$$

Consequently $n < u(c)$ for $c < c_0$. So the point $(n, S_n(\omega))$ is in $B(c)$ for $c < c_0$. This result can be readily generalized to: given any positive integer M, there exists a $c_M > 0$ such that for $c < c_M$, the points $(n, S_n(\omega))$, $n = 1, 2, \cdots, M$, are all in $B(c)$. Therefore for $c < c_M$, $N(c) > M$ for the fixed ω. Using Lemma 3.4 completes the proof.

In the following work, we shall restrict ourselves to the values of k such that ray(k) intersects ∂B. Let ray(k) intersect ∂B_0 and $\partial C_0(c)$ at points with u-coordinates equal to m_k and $u_k(c)$ respectively. Schwarz showed that $\lim_{c\to 0} u_k(c)/\ln c^{-1} = m_k$. The speed of convergence may depend on k, but we shall show in the following lemma that the convergence is uniform in some sense.

LEMMA 3.5. *Let $k > k_0$ be fixed, where k_0 is defined by* (2.4). *Given $\epsilon > 0$, there exist $c_1 > 0$, $c_2 > 0$ such that*

(i) *for $c < c_1$, $u_{k'}(c)/\ln c^{-1} < m_k + \epsilon$ for all $k' \geqq k$,*

(ii) *for $c < c_2$, $m_k - \epsilon < u_{k'}(c)/\ln c^{-1}$ for all $k_0 < k' \leqq k$.*

PROOF. (i) From (2.16), $m_k = [s(k) - \theta_0 k + b(\theta_0)]^{-1}$. It follows from (3.1) and (3.2) that ray(k) intersects $\partial C_0(c)$ at a point whose u-coordinate satisfies

$$c = \int_{H_0} \exp [(\theta k - b(\theta))u] l(\theta) W(d\theta) / \int_\Omega \exp [(\theta k - b(\theta))u] W(d\theta). \tag{3.6}$$

Wlog assume $c < 1$ so that $\ln c^{-1} > 0$.

Case 1. Suppose $W[\theta', \theta(k)) > 0$ for all $\theta' < \theta(k)$. Choose α such that $\theta_0 < \alpha < \theta(k)$ and

$$|[\alpha k - b(\alpha) - \theta_0 k + b(\theta_0)]^{-1} - m_k| < \epsilon/2. \tag{3.7}$$

By the definition of $\theta(k)$, θ in $[\alpha, \theta(k))$ implies $\theta k - b(\theta) \geqq \alpha k - b(\alpha)$, so

$$\begin{aligned} \int_\Omega \exp [(\theta k - b(\theta))u] W(d\theta) &\geqq \int_{[\alpha,\theta(k))} \exp [(\theta k - b(\theta))u] W(d\theta) \\ &\geqq \exp [(\alpha k - b(\alpha))u] W[\alpha, \theta(k)). \end{aligned} \tag{3.8}$$

Also,

$$\int_{H_0} \exp [(\theta k - b(\theta))u] l(\theta) W(d\theta) \leqq \exp [(\theta_0 k - b(\theta_0))u]. \tag{3.9}$$

(3.6), (3.8) and (3.9) yield

$$0 \leqq \exp [(\theta_0 k - b(\theta_0) - \alpha k + b(\alpha))u] \rho(k),$$

where $\rho(k) = 1/W[\alpha, \theta(k)) > 1$ (and $< \infty$). So

$$\ln c - \ln \rho \leqq (\theta_0 k - b(\theta_0) - \alpha k + b(\alpha))u,$$

$$\ln c^{-1} + \ln \rho \geqq (\alpha k - b(\alpha) - \theta_0 k + b(\theta_0))u.$$

Since $\alpha k - b(\alpha) - \theta_0 k + b(\theta_0) > 0$ by the choice of α,

$$u(\ln c^{-1} + \ln \rho)^{-1} \leqq (\alpha k - b(\alpha) - \theta_0 k + b(\theta_0))^{-1}. \tag{3.10}$$

As u depends on k and c, and ρ depends on k, (3.10) should be written as

$$u_k(c)(\ln c^{-1} + \ln \rho(k))^{-1} \leqq (\alpha k - b(\alpha) - \theta_0 k + b(\theta_0))^{-1}. \tag{3.11}$$

Suppose $k' \geqq k$, then by Lemma 2.1 (*i*), $\theta(k') \geqq \theta(k)$, so $\theta_0 < \alpha < \theta(k)$ implies $\theta_0 < \alpha < \theta(k')$. Clearly, $W[\alpha, \theta(k)) \leqq W[\alpha, \theta(k'))$, so that $\rho(k) \geqq \rho(k')$. Also. $\rho(k') > 1$. Thus

$$0 < \ln c^{-1} + \ln \rho(k') \leqq \ln c^{-1} + \ln \rho(k). \tag{31.12}$$

It can be easily seen that k in (3.11) can be replaced by k', thus

$$u_{k'}(c)(\ln c^{-1} + \ln \rho(k'))^{-1} \leqq (\alpha k' - b(\alpha) - \theta_0 k' + b(\theta_0))^{-1},$$

$$u_{k'}(c)/\ln c^{-1} \leqq (\alpha k' - b(\alpha) - \theta_0 k' + b(\theta_0))^{-1}(1 + \ln \rho(k')/\ln c^{-1}).$$

Clearly, $\alpha k' - b(\alpha) - \theta_0 k' + b(\theta_0) \leqq \alpha k - b(\alpha) - \theta_0 k + b(\theta_0)$, and by (3.12)

$$1 + \ln \rho(k')/\ln c^{-1} \leqq 1 + \ln \rho(k)/\ln c^{-1},$$

so that

$$u_{k'}(c)/\ln c^{-1} \leqq (\alpha k - b(\alpha) - \theta_0 k + b(\theta_0))^{-1}(1 + \ln \rho(k)/\ln c^{-1}). \tag{3.13}$$

There exists $c_1 > 0$ such that for $c < c_1$

$$(\alpha k - b(\alpha) - \theta_0 k + b(\theta_0))^{-1}(1 + \ln \rho(k)/\ln c^{-1}) < (\alpha k - b(\alpha) - \theta_0 k + b(\theta_0))^{-1} + \epsilon/2. \tag{3.14}$$

Hence for $c < c_1$, (3.7), and (3.14) yield

$$u_{k'}(c)/\ln c^{-1} < m_k + \epsilon.$$

Case 2. The proof for this case is analogous to that of Case 1, with only some minor changes.

(ii) Choose β such that $\beta < \theta_0$ and

$$m_k - \epsilon/2 < (s(k) - \beta k + b(\beta))^{-1}. \tag{3.15}$$

Suppose $k_0 < k' \leqq k$. $k' > k_0$ implies for θ in $[\beta, \theta_0]$, $\theta k' - b(\theta) \geqq \beta k' - b(\beta)$, so

$$\int_{H_0} \exp [(\theta k' - b(\theta))u]l(\theta)W(d\theta) \geqq \int_{[\beta,\theta_0]} \exp [(\theta k' - b(\theta))u]l(\theta)W(d\theta)$$
$$\geqq \exp [(\beta k' - b(\beta))u]\gamma,$$

where $\gamma = W[\beta, \theta_0] > 0$, since sup (mod $W)H_0 = \theta_0$. Clearly,

$$\int_\Omega \exp [(\theta k' - b(\theta))u]W(d\theta) \leqq \exp [s(k')u].$$

So by (3.6)

$$c \geqq \exp [(\beta k' - b(\beta) - s(k'))u]\gamma.$$

Thus

$$\ln c - \ln \gamma \geqq (\beta k' - b(\beta) - s(k'))u,$$
$$\ln c^{-1} + \ln \gamma \leqq (s(k') - \beta k' + b(\beta))u.$$

Choose $c' > 0$ such that for $c < c'$, $\ln c^{-1} + \ln \gamma > 0$. Clearly

$$s(k') - \beta k' + b(\beta) > 0.$$

Then for $c < c'$,

$$u(\ln c^{-1} + \ln \gamma)^{-1} \geqq (s(k') - \beta k' + b(\beta))^{-1}.$$

Since u depends on k' and c, we have

$$u_{k'}(c)(\ln c^{-1} + \ln \gamma)^{-1} \geqq (s(k') - \beta k' + b(\beta))^{-1}.$$

By Lemma 2.1 (ii), $u_{k'}(c)(\ln c^{-1} + \ln \gamma)^{-1} \geqq (s(k) - \beta k + b(\beta))^{-1}$, implying

$$u_{k'}(c)/\ln c^{-1} \geqq (s(k) - \beta k + b(\beta))^{-1}(1 + \ln \gamma/\ln c^{-1}). \tag{3.16}$$

Choose $c'' > 0$ such that for $c < c''$

$$(s(k) - \beta k + b(\beta))^{-1}(1 + \ln \gamma/\ln c^{-1}) > (s(k) - \beta k + b(\beta))^{-1} - \epsilon/2. \tag{3.17}$$

Let $c_2 = \min (c', c'')$. Then for $c < c_2$, (3.15), (3.16) and (3.17) imply

$$m_k - \epsilon < u_{k'}(c)/\ln c^{-1}.$$

Hence (ii) and the lemma are proved.

Replacing k_0, B_0, and $C_0(c)$ by k_1, B_1, and $C_1(c)$ respectively, the following lemma can be proved in a similar way.

LEMMA 3.6 *Let $k < k_1$ be fixed. Given $\epsilon > 0$ there exist c_3, $c_4 > 0$ such that*
(i) *for $c < c_3$ $u_{k'}(c)/\ln c^{-1} < m_k + \epsilon$ for all $k' \leqq k$,*
(ii) *for $c < c_4$ $m_k - \epsilon < u_{k'}(c)/\ln c^{-1}$ for all $k \leqq k' < k_1$.*

REMARK. To distinguish the intersections of ray(k) with different boundaries, let $u_k^0(c)$, $u_k^1(c)$, $u_k^*(c)$, $u_k(c)$, m_k^0, m_k^1 and m_k be respectively the u-coordinates of the intersections of ray(k) with boundaries of $C_0(c)$, $C_1(c)$, $C(c)$, $B(c)$, B_0, B_1 and B. According to this notation, $u_{k'}(c)$ in Lemmas 3.5 and 3.6 should be replaced by $u_{k'}^0(c)$ and $u_{k'}^1(c)$ and m_k by m_k^0 and m_k^1 respectively. As $C(c) = C_0(c) \cap C_1(c)$, $u_k^*(c) = \min (u_k^0(c), u_k^1(c))$.

LEMMA 3.7 *Given $\epsilon > 0$, there exists $c_1 > 0$ such that for $c < c_1$*
(i) *$u_k^0(c)/\ln c^{-1} - u_k^*(c)/\ln c^{-1} < \epsilon$ for all $k \geqq k^*$.*
(ii) *$u_k^1(c)/\ln c^{-1} - u_k^*(c)/\ln c^{-1} < \epsilon$ for all $k \leqq k^*$.*

PROOF. Only (i) will be proved as the other follows analogously. As we have seen in Section 2 that $m_{k^*}^0 = m_{k^*}^1$ and m_k^0, m_k^1 are both continuous in k, given $\epsilon > 0$ there exists l such that $k_1 > l > k^*$ and

$$|m_l^1 - m_l^0| = \epsilon/2.$$

But since m_k^0 is decreasing in k if $k \geq k_0$ and m_k^1 is increasing in k if $k \leqq k_1$, as shown in Section 2, we have

$$m_l^1 - m_l^0 = \epsilon/2. \tag{3.18}$$

By Lemmas 3.5 and 3.6, there exists $c_1 > 0$ such that for $c < c_1$

$$m_l^0 - \epsilon/4 < u_k^0(c)/\ln c^{-1} < m_{k^*} + \epsilon/4,$$

$$m_{k^*} - \epsilon/4 < u_k^1(c)/\ln c^{-1} < m_l^1 + \epsilon/4 \quad \text{for all} \quad k^* \leqq k \leqq l.$$

Hence for $c < c_1$,

$$|u_k^1(c)/\ln c^{-1} - u_k^0(c)/\ln c^{-1}| < \epsilon \quad \text{for all} \quad k^* \leqq k \leqq l. \tag{3.19}$$

Also by Lemmas 3.5 and 3.6, there exists $c_2 > 0$ such that for $c < c_2$,

$$u_k^0(c)/\ln c^{-1} < m_l^0 + \epsilon/4, \qquad \text{and}$$

$$u_k^1(c)/\ln c^{-1} > m_l^1 - \epsilon/4, \quad \text{for all} \quad l < k \leqq k_1 .$$

So it follows from (3.18) that if $c < c_2$

$$u_k^0(c)/\ln c^{-1} < u_k^1(c)/\ln c^{-1} \quad \text{for all} \quad l < k \leqq k_1 . \tag{3.20}$$

If $k > k_1$, it follows from the definition of k_1 that ray(k) intersects ∂B_0 and is contained in B_1. Thus when $c < c_3$ for some $c_3 > 0$, ray(k) intersects $\partial C_0(c)$ and is contained in $C_1(c)$. As $C(c) = C_0(c) \cap C_1(c)$,

$$u_k^*(c) = u_k^0(c) \quad \text{if} \quad k > k_1 . \tag{3.21}$$

Hence (3.19), (3.20) and (3.21) imply that if $c' = \min(c_1, c_2, c_3)$ then for $c < c'$

$$u_k^0(c)/\ln c^{-1} - u_k^*(c)/\ln c^{-1} < \epsilon \quad \text{for} \quad k \geqq k^*.$$

Lemma 3.8. *Given $\epsilon > 0$ and $l_2 > l_1$, there exists $c' > 0$ such that for $c < c'$ and $l_1 \leqq k \leqq l_2$,*

(i) $l_1 \geqq k^*$ *implies* $m_{l_2} - \epsilon < u_k(c)/\ln c^{-1} < m_{l_1} + \epsilon$,

(ii) $k^* \geqq l_2$ *implies* $m_{l_1} - \epsilon < u_k(c)/\ln c^{-1} < m_{l_2} + \epsilon$,

(iii) $l_2 > k^* > l_1$ *implies* $m_{l'} - \epsilon < u_k(c)/\ln c^{-1} < m_{k^*} + \epsilon$,

where $m_{l'} = \min(m_{l_1}, m_{l_2})$.

Proof. Only (i) will be proved as the others are similar to (i). By Lemmas 3.5 and 3.7, there exists $c_1 > 0$ such that for $c < c_1$, d some positive constant,

$$m_{l_2} - \epsilon/2 < u_k^*(dc \ln c^{-1})/\ln (dc \ln c^{-1})^{-1}, \tag{3.22}$$

$$u_k^*(c)/\ln c^{-1} < m_{l_1} + \epsilon, \quad \text{for} \quad l_1 \leqq k \leqq l_2 . \tag{3.23}$$

By Schwarz's Theorem 1, there exists $c_2 > 0$ such that for $c < c_2$, $C(dc \ln c^{-1}) \subset B(c) \subset C(c)$, so

$$u_k^*(dc \ln c^{-1}) \leqq u_k(c) \leqq u_k^*(c). \tag{3.24}$$

By Lemmas (3.5) and (3.7) there exists $c_3 > 0$ such that for $c < c_3$, $u_k^*(c) \ln c^{-1} < m_k^* + \epsilon$, for $k \geqq k^*$, so there exists $0 < c_4 < c_3$ such that for $c < c_4$, $k \geqq k^*$, (3.25) $$|u_k^*(dc \ln c^{-1})/\ln (dc \ln c^{-1})^{-1} - u_k^*(dc \ln c^{-1})/\ln c^{-1}| < \epsilon/2$$

Let $c' = \min (c_1, c_2, c_4)$, then for $c < c'$, (3.22), (3.23), (3.24) and (3.25) yield

$$m_{l_2} - \epsilon < u_k(c)/\ln c^{-1} < m_{l_1} + \epsilon \quad \text{for} \quad l_1 \leqq k \leqq l_2 .$$

THEOREM 3.2 *Let* $\mu_\theta = E_\theta X$. *Then*

$$\lim_{c\to 0} N(c)/\ln c^{-1} = m_{\mu_\theta} \text{ a.e. } P_\theta .$$

PROOF. Assume $\mu_\theta > k^*$, the proofs for $\mu_\theta < k^*$ and $\mu_\theta = k^*$ being similar. Given $\epsilon > 0$, choose $\epsilon' > 0$ such that $\mu_\theta - \epsilon' > k^*$ and

(3.26) $$m_{\mu_\theta - \epsilon'} - m_{\mu_\theta + \epsilon'} < \epsilon/2.$$

From Theorem 3.1 and the strong law of large numbers, we have that if ω is not in $\mathcal{Y} \cup \mathcal{Z}$, where $\mathcal{Z} \subset \mathcal{X}$, $P_\theta(\mathcal{Z}) = 0$ so that $P_\theta(\mathcal{Y} \cup \mathcal{Z}) = 0$, then (in the following, $N(c)$ is a function of ω but for simplicity we suppress the argument ω)

$$\lim_{c\to 0} S_{N(c)} / N(c) = \mu_\theta .$$

Thus, there exists $c_1 > 0$ such that for $c < c_1$,

$$|S_{N(c)} / N(c) - \mu_\theta| < \epsilon', \text{ or}$$

(3.27) $$\inf_{\mu_\theta - \epsilon' < k < \mu_\theta + \epsilon'} u_k(c) - 1 < N(c) < \sup_{\mu_\theta - \epsilon' < k < \mu_\theta + \epsilon'} u_k(c) + 1.$$

By Lemma 3.8 (i), there exists $c_2 > 0$ such that for $c < c_2$,

(3.28) $$m_{\mu_\theta + \epsilon'} - \epsilon/4 < u_k(c)/\ln c^{-1} < m_{\mu_\theta - \epsilon'} + \epsilon/4,$$

for $\mu_\theta - \epsilon' < k < \mu_\theta + \epsilon'$. Choose $c_3 > 0$ such that for $c < c_3$,

(3.29) $$(\ln c^{-1})^{-1} < \epsilon/4.$$

Let $c' = \min (c_1, c_2, c_3)$, then for $c < c'$, (3.27), (3.28) and (3.29) imply

$$m_{\mu_\theta + \epsilon'} - \epsilon/2 < N(c)/\ln c^{-1} < m_{\mu_\theta - \epsilon'} + \epsilon/2.$$

Hence, by (3.26) and the fact that $m_{\mu_\theta + \epsilon'} < m_{\mu_\theta} < m_{\mu_\theta - \epsilon'}$, if $c < c'$,

$$|N(c)/\ln c^{-1} - m_{\mu_\theta}| < \epsilon.$$

LEMMA 3.9. *Given* $\epsilon > 0$, *there exists* $c' > 0$ *such that for* $c < c'$ *and any* k,

$$u_k(c)/\ln c^{-1} < m_{k^*} + \epsilon.$$

PROOF. It follows readily from Lemma 3.8 (i) and (ii).

COROLLARY 3.3. *Given* $\epsilon > 0$, *there exists* $c' > 0$ *such that for* $c < c'$ *and all* θ,

$$N(c)/\ln c^{-1} < m_{k^*} + \epsilon, \quad \text{and}$$

$$E_\theta N(c)/\ln c^{-1} < m_{k^*} + \epsilon.$$

PROOF. The proof follows immediately from the definition of $u_k(c)$ and Lemma 3.9.

THEOREM 3.3. $\text{Lim}_{c\to 0} E_\theta N(c)/\ln c^{-1} = m_{\mu_\theta}$ *for each* θ.

PROOF. By Corollary 3.3, $N(c)/\ln c^{-1}$ is uniformly bounded when c is small. The theorem is established by applying Theorem 3.2 and Lebesgue dominated convergence theorem.

Let $B \ln c^{-1}$ be the homothetic transform of B when the multiplication factor is $\ln c^{-1}$. Let $N(B \ln c^{-1})$ denote the stopping variable for a procedure whose continuation region is $B \ln c^{-1}$, and let $m_k(c)$ denote the u-coordinate of the intersection point of ray(k) with $\partial B \ln c^{-1}$. It is clear that for each c, $m_k(c)/\ln c^{-1} = m_k$. Thus Lemmas 3.8 and 3.9 obviously hold also for $m_k(c)$ and Theorems 3.1 and 3.2 hold for $N(B \ln c^{-1})$. Hence the following theorem, an analogy to Theorem 3.3, can be proved in a similar way.

THEOREM 3.4. $\text{Lim}_{c\to 0} E_\theta N(B \ln c^{-1})/\ln c^{-1} = m_{\mu_\theta}$ *for each* θ.

4. Error probabilities. Let $B_W \ln c^{-1}$ denote the procedure whose continuation region is $B_W \ln c^{-1}$ and which accepts H_1 (respectively H_0) if $(n, S_n) = (u, v)$ lies above (respectively below) $\partial B_0 \ln c^{-1}$ (respectively $\partial B_1 \ln c^{-1}$). By Theorem 2.1, for any $W, B_L \subset B_W$ so that $B_L \ln c^{-1} \subset B_W \ln c^{-1}$. Consequently, for each θ, $P_\theta(\text{error} \mid B_L \ln c^{-1}) \geqq P_\theta(\text{error} \mid B_W \ln c^{-1})$. To show for any W, $P_\theta(\text{error} \mid B_W \ln c^{-1}) = o(c \ln c^{-1})$, it will be sufficient to show $P_\theta(\text{error} \mid B_L \ln c^{-1}) = o(c \ln c^{-1})$, which will be done in this section.

From (2.17), we have

$$(4.1)\quad \begin{aligned} B_0 \ln c^{-1} &= \{(u, v): v/u \geqq k^*, s(v/u) \leqq u^{-1} \ln c^{-1} + \theta_0 v/u - b(\theta_0)\}, \\ B_1 \ln c^{-1} &= \{(u, v): v/u \leqq k^*, s(v/u) \leqq u^{-1} \ln c^{-1} + \theta_1 v/u - b(\theta_1)\}. \end{aligned}$$

If W is an L-measure, then $\theta(k)$, defined in (2.9), satisfies $b'(\theta(k)) = k$. Since b' is strictly increasing, $\theta(k)$ is strictly increasing, so if $k \geqq k^* > k_0$, $\theta(k) > \theta_0$. Thus, by Lemma 2.1 (ii), if $k \geqq k^*$, $s(k) - \theta_0 k$ is strictly increasing in k, i.e. for $v/u \geqq k^*$, $v/u \leqq k$ iff $s(v/u) - \theta_0 v/u \leqq s(k) - \theta_0 k$. Define $k_0(u)$ such that $s(k_0(u)) = u^{-1} \ln c^{-1} + \theta_0 k_0(u) - b(\theta_0)$. Then for $v/u \geqq k^*$, $v/u \leqq k_0(u)$ iff $s(v/u) \leqq u^{-1} \ln c^{-1} + \theta_0 v/u - b(\theta_0)$. A similar result holds for θ_1. Hence (4.1) can be written as

$$(4.2)\quad \begin{aligned} B_0 \ln c^{-1} &= \{(u, v): u \leqq m^* \ln c^{-1}, k^* \leqq v/u \leqq k_0(u)\}, \\ B_1 \ln c^{-1} &= \{(u, v): u \leqq m^* \ln c^{-1}, k_1(u) \leqq v/u \leqq k^*\}, \end{aligned}$$

where m^* is defined in (2.12).

Before we show that P_θ (error $\mid B_L \ln c^{-1}$) is $o(c \ln c^{-1})$ we shall show a lemma whose proof follows immediately from the result of Bahadur and Rao [1].

LEMMA 4.1. *Let* θ *in* Ω, *if* a_1 *and* a_2 *are in the range of* b' *such that* $a_2 > a_1 > b'(\theta)$, *then for* $a_1 \leqq a \leqq a_2$,

$$P_\theta(\bar{X}_n > a) < [m(a)]^n D/n^{\frac{1}{2}},$$

where D *does not depend on* a *or* n *and*

$$(4.3)\quad m(a) = \exp[-\theta(a)a + b(\theta(a)) + \theta a - b(\theta)].$$

PROOF. Bahadur and Rao [1] showed

$$(4.4) \qquad P_\theta(\bar{X}_n \geqq a) = I^n b_n(1 + o(1))/(2\pi n)^{\frac{1}{2}},$$

where $\ln b_n = O(1)$, b_n depends on a, and

$$(4.5) \qquad I = I(a) = \inf_t \exp[-at]M_\theta(t) = \inf_t \exp[-at]E_\theta \exp[tX].$$

Putting

$$(4.6) \qquad H(t) = \exp[-at]M_\theta(t) = \exp[-at + b(\theta + t) - b(\theta)]$$

we compute

$$\begin{aligned} dH/dt &= H(t)(-a + b'(\theta + t)), \\ d^2H/dt^2 &= H(t)[(-a + b'(\theta + t)^2 + b''(\theta + t)] \\ &> 0, \quad \text{since} \quad b''(\theta) > 0 \quad \text{for all} \quad \theta. \end{aligned}$$

Therefore H is strictly concave. Setting $dH/dt = 0$ we obtain

$$(4.7) \qquad t = \theta(a) - \theta.$$

Hence, by (4.3), (4.5), (4.6) and (4.7),

$$(4.8) \qquad I(a) = \inf_t H(t) = m(a).$$

It can easily be seen in the work of Bahadur and Rao that when $a_1 \leqq a \leqq a_2$, b_n is bounded uniformly both in a and n. Thus for all $a_1 \leqq a \leqq a_2$, and any positive integer n,

$$b_n[1 + o(1)]/(2\pi)^{\frac{1}{2}} < D, \qquad \text{say.}$$

This together with (4.4) and (4.8) establishes the lemma.

We have defined $k_0(u)$ such that $s(k_0(u)) = u^{-1} \ln c^{-1} + \theta_0 k_0(u) - b(\theta_0)$. So $k_0(u)$ is the slope of the ray the u-coordinate of whose intersection with $\partial B_0 \ln c^{-1}$ is u. Let u' be any fixed positive number. Clearly,

$$u' \ln c^{-1} \leqq u \leqq m^* \ln c^{-1} \qquad \text{iff} \qquad k^* \leqq k_0(u) \leqq k_0(u' \ln c^{-1}).$$

If $[x]$ denotes the largest integer less than or equal to x, then

$$(4.9) \quad [u' \ln c^{-1}] \leqq u \leqq [m^* \ln c^{-1}] \qquad \text{implies} \qquad k^* \leqq k_0(u) \leqq k_0(u' \ln c^{-1}) + \epsilon,$$

where ϵ is some fixed positive number small enough so that $k_0(u' \ln c^{-1}) + \epsilon$ is in the range of b'.

COROLLARY 4.1. *Let $0 < u' < m^*$ and assume c small enough so that $[u' \ln c^{-1}] > 1$. Then*

$$\begin{aligned} p(c, u') &= \sum_{n=[u' \ln c^{-1}]}^{[m^* \ln c^{-1}]} P_{\theta_0}(\bar{X}_n > k_0(n)) \\ &= O(c(\ln c^{-1})^{\frac{1}{2}}) = o(c \ln c^{-1}). \end{aligned}$$

PROOF. By the definitions of $k_0(u)$ and $s(k)$, we have

$$\begin{aligned} &\theta(k_0(u))k_0(u) - b[\theta(k_0(u))] - \theta_0 k_0(u) + b(\theta_0) \\ &\qquad = s(k_0(u)) - \theta_0 k_0(u) + b(\theta_0) = u^{-1} \ln c^{-1}. \end{aligned}$$

Thus, it follows from (4.3) using $\theta = \theta_0$

(4.10) $$[m(k_0(n))]^n = \exp[-(n^{-1} \ln c^{-1})n] = c.$$

By (4.9), (2.6) and (2.7), we can apply Lemma 4.1 to get

$$p(c, u') \leqq cDm^* \ln c^{-1}/(u' \ln c^{-1})^{\frac{1}{2}}.$$

This completes the proof.

In Lemma 4.1, a has to be bounded. We shall use Chernoff's result [2] to find a bound for $P(\bar{X}_n > a)$ without the restriction on a. But the bound that we shall obtain is larger than the one given in Lemma 4.1 when n is large.

LEMMA 4.2. *If $a > b'(\theta)$, then*

$$P_\theta(\bar{X}_n > a) \leqq [m(a)]^n.$$

PROOF. According to Theorem 1 in Chernoff's paper [2],

$$P_\theta(\bar{X}_n > a) = [I(a)]^n,$$

where $I(a)$ is defined in (4.5). Using (4.8) completes the proof.

COROLLARY 4.2. *If $u' > 0$ and c small enough so that $[u' \ln c^{-1}] \geqq 1$, then*

$$\sum_{n=1}^{[u' \ln c^{-1}]} P_{\theta_0}(\bar{X}_n > k_0(n)) \leqq cu' \ln c^{-1}.$$

PROOF. The proof follows readily from (4.10) and Lemma 4.2.

THEOREM 4.1.

$$\sup_{\theta \text{ in } H_i} P_\theta(\mathit{error} \mid B_L \ln c^{-1}) = o(c \ln c^{-1}), \qquad i = 0, 1.$$

PROOF. We shall show only the case where $i = 0$, as the other is analogous. It follows from (4.2)

$$\begin{aligned} P_{\theta_0}&(\text{error} \mid B_L \ln c^{-1}) \\ &= P_{\theta_0}(\omega: \bar{X}_n(\omega) > k_0(n) \geqq k^* \quad \text{for some } n, 1 \leqq n \leqq m^* \ln c^{-1}) \\ &\leqq \sum_{n=1}^{[m^* \ln c^{-1}]} P_{\theta_0}(\bar{X}_n > k_0(n)). \end{aligned}$$

Given $\epsilon > 0$, choose $u' > 0$ such that $u' < \epsilon$. Then by Corollaries 4.1 and 4.2,

$$\lim_{c \to 0} P_{\theta_0}(\text{error} \mid B_L \ln c^{-1})/c \ln c^{-1} \leqq 0 + u' < \epsilon,$$

so that

$$P_{\theta_0}(\text{error} \mid B_L \ln c^{-1}) = o(c \ln c^{-1}).$$

The fact that X_1 has the monotone likelihood ratio property concludes the proof.

COROLLARY 4.3. *For any W, in particular, θ^*-measure,*

$$\sup_{\theta \text{ in } H_i} P(\text{error} \mid B_W \ln c^{-1}) = o(c \ln c^{-1}), \qquad i = 0, 1.$$

PROOF. The proof follows from Theorems 2.1 and 4.1.

Although Corollary 4.3 is sufficient for Section 5, we shall prove a stronger result for θ^*-measures in the next theorem because of its intrinsic interest. Let $\theta^* = \theta_2$, $\theta_0 < \theta_2 < \theta_1$, and define $R_{ij}(n, S_n)$ as follows:

(4.11) $$R_{ij}(u, v) = (\theta_i - \theta_j)v - u(b(\theta_i) - b(\theta_j)), \qquad i, j = 0, 1, 2.$$

Set

$$(4.12) \qquad r_{ij}(x_1, \cdots, x_n) = \prod_{k=1}^{n} f(x_k, \theta_i) / \prod_{k=1}^{n} f(x_k, \theta_j),$$

then $R_{ij}(n, S_n) = \ln r_{ij}(X_1, \cdots, X_n)$. It follows from (2.17), (2.18) and (4.11)

$$B_0 = \{(u, v): R_{10}(u, v) \leqq 1, R_{20}(u, v) \leqq 1, v/u \geqq k^*\},$$

$$B_1 = \{(u, v): R_{10}(u, v) \geqq -1, R_{12}(u, v) \geqq -1, v/u \leqq k^*\},$$

which yields

$$B_0 \ln c^{-1} = \{(u, v): R_{10}(u, v) \leqq \ln c^{-1}, R_{20}(u, v) \leqq \ln c^{-1}, v/u \geqq k^*\},$$

$$B_1 \ln c^{-1} = \{(u, v): R_{10}(u, v) \geqq -\ln c^{-1}, R_{12}(u, v) \geqq -\ln c^{-1}, v/u \leqq k^*\}.$$

Thus

$$(4.13) \quad P_{\theta_0}(\text{error} \mid B_{\theta^*} \ln c^{-1}) \leqq P_{\theta_0}(R_{10} > \ln c^{-1}) + P_{\theta_0}(R_{20} > \ln c^{-1}),$$

$$P_{\theta_1}(\text{error} \mid B_{\theta^*} \ln c^{-1}) \leqq P_{\theta_1}(R_{10} < -\ln c^{-1}) + P_{\theta_1}(R_{12} < -\ln c^{-1}).$$

THEOREM 4.2.

$$P_{\theta_i}(\textit{error} \mid B_{\theta^*} \ln c^{-1}) \leqq 2c, \qquad i = 0, 1.$$

PROOF. Suppose we carry on simultaneously three sequential probability ratio tests (SPRT's), namely θ_0 vs. θ_1, θ_2 vs. θ_1 and θ_0 vs. θ_2, the first based on the sequence $r_{10}(X_1, \cdots, X_n)$, the second r_{12} and the third r_{20}; and the bounds for each test are c and c^{-1}. Then by using the fundamental relations among the error probabilities and the bounds for a SPRT [5], we have:

$$\text{for} \quad \theta_0 \text{ vs. } \theta_1, \qquad P_{\theta_0}(\text{error}) = P_{\theta_0}(R_{10} > \ln c^{-1}) \leqq 1/c^{-1} = c,$$

$$P_{\theta_1}(\text{error}) = P_{\theta_1}(R_{10} < -\ln c^{-1}) \leqq c;$$

$$\text{for} \quad \theta_2 \text{ vs. } \theta_1, \qquad P_{\theta_1}(\text{error}) = P_{\theta_1}(R_{12} < -\ln c^{-1}) \leqq c;$$

$$\text{for} \quad \theta_0 \text{ vs. } \theta_2, \qquad P_{\theta_0}(\text{error}) = P_{\theta_0}(R_{20} > \ln c^{-1}) = 1/c^{-1} = c.$$

These results and (4.13) establish the theorem.

COROLLARY 4.4.

$$\sup\nolimits_{\theta \text{ in } H_i} (\textit{error} \mid B_{\theta^*} \ln c^{-1}) \leqq 2c, \qquad i = 0, 1.$$

PROOF. This corollary is an immediate consequence of Theorem 4.2 and the fact that X_1 has the monotone likelihood ratio property.

5. Asymptotic optimal properties. We have seen in Section 3 that the expected sample sizes of the Bayes test $B_W(c)$ are asymptotically equal to those of $B_W \ln c^{-1}$ and in Section 4 that the error probabilities of $B_W \ln c^{-1}$ are $o(c \ln c^{-1})$. As shown in Lemma 5.1, these results imply that for any W the integrated risk of $B_W \ln c^{-1}$ or $B_L \ln c^{-1}$ is asymptotically equal to that of $B_W(c)$. Our main results, concerning the optimal characteristics of the expected sample sizes of $B_L \ln c^{-1}$, are the consequences of Lemma 5.1. Let $r_W(\delta)$ and

$N(\delta)$ be respectively the integrated risk (over W) and the stopping variable of a procedure δ. (Note: in this notation, $N(B(c))$ is the same as $N(c)$ used in previous sections.)

LEMMA 5.1. (i) $\lim_{c\to 0} [r_W(B_W \ln c^{-1}) - r_W(B_W(c))]/c \ln c^{-1} = 0$,

(ii) $\lim_{c\to 0} [r_W(B_L \ln c^{-1}) - r_W(B_W(c))]/c \ln c^{-1} = 0$.

PROOF. (i)

$$\begin{aligned}[r_W(B_W \ln c^{-1}) - r_W(B_W(c))]/c \ln c^{-1} &= \int_{H_0 \cup H_1} [P_\theta \text{ (error} \mid B_W \ln c^{-1})/c \ln c^{-1}]W(d\theta) \\ &\quad - \int_{H_0 \cup H_1} [P_\theta \text{ (error} \mid B_W(c))/c \ln c^{-1}]W(d\theta) \\ &\quad + \int_\Omega [E_\theta N(B_W \ln c^{-1})/\ln c^{-1}]W(d\theta) \\ &\quad - \int_\Omega [E_\theta N(B_W(c))/\ln c^{-1}]W(d\theta) \\ &= a_1(c) - a_2(c) + a_3(c) - a_4(c) \geqq 0.\end{aligned}$$

It is sufficient to prove $a_1(c) \to 0$, $a_3(c) - a_4(c) \to 0$. By Corollary 4.3, $a_1(c) \to 0$. Since for all c, $N(B_W \ln c^{-1}) \leqq m^* \ln c^{-1}$, where m^* is defined in (2.12), $E_\theta N(B_W \ln c^{-1})/\ln c^{-1}$ is uniformly bounded for all θ and c. So by Theorem 3.4,

$$\lim_{c\to 0} a_3(c) = \int_\Omega m_{\mu_\theta} W(d\theta). \tag{5.1}$$

By Corollary 3.3, when c is small $E_\theta N(B_W(c))/\ln c^{-1}$ is uniformly bounded. So by Theorem 3.3,

$$\lim_{c\to 0} a_4(c) = \int_\Omega m_{\mu_\theta} W(d\theta). \tag{5.2}$$

Hence (5.1) and (5.2) imply $a_3(c) - a_4(c) \to 0$, which concludes the proof for (i).

(ii) Let

$$b_1(c) = \int_{H_0 \cup H_1} [P_\theta \text{ (error} \mid B_L \ln c^{-1})/c \ln c^{-1}]W(d\theta)$$

$$b_3(c) = \int_\Omega [E_\theta N(B_L \ln c^{-1})/\ln c^{-1}]W(d\theta).$$

Then,

$$\begin{aligned}[r_W(B_L \ln c^{-1}) - r_W(B_W(c))]/c \ln c^{-1} \\ = b_1(c) - a_2(c) + b_3(c) - a_4(c) \geqq 0,\end{aligned} \tag{5.3}$$

where a_2 and a_4 are defined in (i). It is sufficient to show $b_1(c) \to 0$ and $\lim (b_3(c) - a_4(c)) \leqq 0$. By Corollary 4.3, $b_1(c) \to 0$. m_{μ_θ}, as defined in Section 3, depends on W through its support. Let m'_{μ_θ} denote m_{μ_θ} when W is an L-measure. As in (i) it can be easily shown

$$\lim_{c\to 0} [b_3(c) - a_4(c)] = \int_\Omega (m'_{\mu_\theta} - m_{\mu_\theta}) W(d\theta).$$

Using Theorem 2.1, we have

$$\lim (b_3(c) - a_4(c)) \leqq 0, \tag{5.4}$$

which completes the proof.

REMARK. By (5.3) and (5.4), $W\{\theta: m'_{\mu_\theta} < m_{\mu_\theta}\} = 0$, or $W\{\theta: m'_{\mu_\theta} = m_{\mu_\theta}\} = 1$. Theorem 2.2 can be considered as a special case of this result.

Let $\alpha(c)$ be a function of c such that $\lim_{c\to 0} \alpha(c)/c \ln c^{-1} = 0$ and $\sup_{\theta \text{ in } H_0 \cup H_1} P(\text{error } B_L \ln c^{-1}) \leqq \alpha(c)$. Such an $\alpha(c)$ exists as we can take $\alpha(c) = \max_{i=0,1} P_{\theta_i}(\text{error} \mid B_L \ln c^{-1})$. Let $\mathfrak{F}(c)$ be the family of procedure δ that satisfy:

$$\sup_{\theta \text{ in } H_0 \cup H_1} P(\text{error} \mid \delta) \leqq \alpha(c).$$

$\mathfrak{F}(c)$ depends on the choice of α.

THEOREM 5.1. *As* $c \to 0$

$$\max_\theta E_\theta N(B_L \ln c^{-1})/\ln c^{-1} - \inf_{\delta \text{ in } \mathfrak{F}(c)} \max_\theta E_\theta N(\delta)/\ln c^{-1} \to 0.$$

PROOF. If not, there exists $\epsilon > 0$, a sequence $c_i \downarrow 0$ and a sequence of procedures δ_{c_i} in $\mathfrak{F}(c_i)$ such that

$$\max_\theta E_\theta N(B_L \ln c_i^{-1})/\ln c_i^{-1} - \max_\theta E_\theta N(\delta_{c_i})/\ln c_i^{-1} > \epsilon/2. \tag{5.5}$$

Let θ' be such that $m_{\mu_{\theta'}} = m^*$. From Section 2 we know $\theta_0 < \theta' < \theta_1$. Clearly, for all c,

$$\max_\theta E_\theta N(B_L \ln c^{-1})/\ln c^{-1} \leqq m^*.$$

Thus, by (5.5) for all c_i

$$\max_\theta E_\theta N(\delta_{c_i})/\ln c_i^{-1} < m^* - \epsilon/2,$$

which yields, for any W,

$$\int [E_\theta N(\delta_{c_i})/\ln c_i^{-1}] W(d\theta) < m^* - \epsilon/2. \tag{5.6}$$

Since

$$\lim_{c\to 0} E_{\theta'} N(B_L \ln c^{-1})/\ln c^{-1} = m^* > \lim_{c\to 0} E_{\theta_i} N(B_L \ln c^{-1})/\ln c^{-1},$$

for $i = 0, 1$, there exists $d_1 > 0$ and a θ'-measure W' that assigns sufficient weight on θ' such that for $c < d_1$,

$$\int [E_\theta N(B_L \ln c^{-1})/\ln c^{-1}] W'(d\theta) > m^* - \epsilon/4. \tag{5.7}$$

(5.6) and (5.7) imply that if $c_i < d_1$

$$\int [E_\theta N(B_L \ln c_i^{-1})/\ln c_i^{-1}] W'(d\theta) - \int [E_\theta N(\delta_{c_i})/\ln c_i^{-1}] W'(d\theta) > \epsilon/4. \tag{5.8}$$

Choose $d_2 > 0$ such that for $c < d_2$, $\alpha(c)/c \ln c^{-1} < \epsilon/16$. Thus for $c_i < d_2$,

$$\int_{H_0 \cup H_1} [P_\theta(\text{error} \mid \delta_{c_i})/\ln c_i^{-1}] W'(d\theta) < \epsilon/16. \tag{5.9}$$

By Lemma 5.1 (ii), there exists $d_3 > 0$ such that for $c < d_3$,

$$r_{W'}(B_L \ln c^{-1})/c \ln c^{-1} - r_{W'}(B_{W'}(c))/c \ln c^{-1} < \epsilon/16. \tag{5.10}$$

Hence if $c_i < \min(d_1, d_2, d_3)$ it follows from (5.10), (5.8) and (5.9)

$$r_{W'}(B_{W'}(c_i))/c_i \ln c_i^{-1} > r_{W'}(B_L \ln c_i^{-1})/c_i \ln c_i^{-1} - \epsilon/16$$

$$> \int [E_\theta N(B_L \ln c_i^{-1})/\ln c_i^{-1}] W'(d\theta) - \epsilon/16$$
$$> \int [E_\theta N(\delta_{c_i})/\ln c_i^{-1}] W'(d\theta) + 3\epsilon/16$$
$$> r_{W'}(\delta_{c_i})/c_i \ln c_i^{-1} + \epsilon/8,$$

which is a contradiction since $B_{W'}(c_i)$ is Bayes (W'). Hence the theorem is proved.

Let M be any fixed positive number. Define $\mathfrak{F}'(c)$ as a subfamily of $\mathfrak{F}(c)$, consisting of procedures δ that satisfy: for all c,

$$E_{\theta_i} N(\delta)/\ln c^{-1} < M, \qquad i = 0, 1. \tag{5.11}$$

$\mathfrak{F}'$ depends on M.

THEOREM 5.2. *For each θ, $\theta_0 < \theta < \theta_1$, as $c \to 0$*

$$E_\theta N(B_L \ln c^{-1})/\ln c^{-1} - \inf_{\delta \text{ in } \mathfrak{F}'(c)} E_\theta N(\delta)/\ln c^{-1} \to 0.$$

PROOF. If not, there exists $\epsilon > 0$, $\theta_0 < \theta_2 < \theta_1$, $c_i \downarrow 0$ and δ_{c_i} in $\mathfrak{F}'(c_i)$ such that

$$E_{\theta_2} N(B_L \ln c_i^{-1})/\ln c_i^{-1} - E_{\theta_2} N(\delta_{c_i})/\ln c_i^{-1} > \epsilon. \tag{5.12}$$

From Theorem 2.1 we have $N(B_L \ln c^{-1}) \leqq N(B_{\theta_2} \ln c^{-1})$ so that (5.12) implies

$$E_{\theta_2} N(B_{\theta_2} \ln c_i^{-1})/\ln c_i^{-1} - E_{\theta_2} N(\delta_{c_i})/\ln c_i^{-1} > \epsilon. \tag{5.13}$$

It can be shown that (5.13) leads to a contradiction. The proof is analogous to that of Theorem 5.1.

REMARK. Theorem 5.2 holds for $\theta_0 < \theta < \theta_1$, but it can be generalized to hold for any θ in Ω if we replace $\mathfrak{F}'(c)$ by $\mathfrak{F}''(c)$, a subfamily of $\mathfrak{F}'(c)$ consisting of procedures δ that satisfy: for all c, $E_{\theta'} N(\delta)/\ln c^{-1} < M$, where θ' is any fixed value between θ_0 and θ_1. The proof is analogous to that given for Theorem 5.2.

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