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EXISTENCE OF OPTIMAL STOPPING RULES FOR REWARDS RELATED TO S_n/n

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1. Introduction and summary. Let $\{X_n, n \geqq 1\}$ be a sequence of independent and identically distributed random variables with mean 0 and finite νth moment for some $\nu \geqq 2$. Let $S_n = \sum_1^n X_i$. We observe the X's sequentially and must decide when to stop sampling. If we stop at time n we receive a reward of the form $h_n(S_n)$, and we are concerned with finding stopping rules t which maximize our expected reward, $E[h_t(S_t)]$. In particular we are concerned with showing that the so-called "functional equation rule" (FER) (for a definition, see Section 2) is such a (optimal) rule.

Chow and Robbins [2] have treated the *reward sequence* $n^{-1}S_n$ with $X_i = \pm 1$ each with probability $\frac{1}{2}$. Dvoretzky [4] considered reward sequences of the form $n^{-\alpha}S_n$ ($\alpha > \frac{1}{2}$) under the assumptions $EX_i = 0$, $EX_i^2 < \infty$. Teicher and Wolfowitz [7] considered sequences $c_nS_n^\beta$ ($\beta = 1, 2$) under the same assumptions on the distribution of X_i. They impose the conditions $c_n > 0$, $c_{n+1}^2 \leqq c_{n+2}c_n$, $(n+1)^\beta c_{n+1} \leqq n^\beta c_n$.

We establish certain principles which allow us to relate certain reward sequences h_n of a particularly simple form to others of a more complicated form and in the process conclude that the FER is optimal in the more complicated situation. Using the *basic reward sequence* $n^{-\alpha}|S_n|^\beta$ with $2\alpha > \beta > 0$ and assuming that $E|X_i|^{\max(2,\beta)} < \infty$, we examine the problem of optimality for reward sequences of the form $c_nS_n^+$, $c_n|S_n|^\beta$, $n^{-\alpha}\log|S_n|$, etc., where $c_1, c_2, \cdots$ are constants such that $\limsup_{n\to\infty} n^\alpha c_n < \infty$.

It seems somewhat customary in optimal stopping problems to try to verify that the reward sequence (in our case $h_n(S_n)$) is majorized by a non-negative random variable with finite expectation, and then to appeal to one of a class of general theorems in which this regularity condition appears. In the problems we consider we have found it easier to disregard this possibility and to use a direct approach.

We begin with a formal presentation of the problem and a development of preliminary results which show that the FER is optimal provided that it is a.s. finite. In later sections we develop and exploit machinery for relating two or more reward sequences and verify that the FER is a.s. finite for the basic reward sequence.

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2. Formulation of the problem and preliminary results. Let (Ω, F, P) be a probability space and let $\{Z_n, F_n\}_0^\infty$ be an integrable stochastic sequence, i.e., let (F_n) be an increasing sequence of sub-σ-algebras of F and for each $n = 0, 1, 2, \cdots$ let Z_n be a rv measurable with respect to F_n such that $E|Z_n| < \infty$. A *stopping rule* or *stopping variable* (sv) is a random variable t with values $0, 1, \cdots, +\infty$ such that

(i) $P\{t < \infty\} = 1$,

(ii) $\{t = n\} \varepsilon F_n, n = 0, 1, 2, \cdots$.

We would like to determine an optimal sv, i.e., defining $v = \sup_t EZ_t$, where the sup is taken over all sv's t such that EZ_t exists, we would like to determine a sv τ such that EZ_τ exists and equals v. We shall be interested mainly in the case that $Z_n = h_n(S_n)$, where S_n is the nth partial sum of a sequence of iid random variables.

Let $\gamma_n = \text{ess sup } E(Z_t \mid F_n)$, the ess sup being taken over all rules t such that $P(t \geqq n) = 1$ and EZ_t exists, $n = 0, 1, \cdots$. Then the FER is by definition (see, e.g., [3])

$$\sigma = \text{first } n \geqq 0 \quad \text{such that} \quad Z_n = \gamma_n$$
$$= \infty \quad \text{if} \quad Z_n < \gamma_n \quad \text{for all } n.$$

In general $P\{\sigma < \infty\} < 1$ and σ is not a sv. Siegmund [6] has shown, however, that if we enlarge our class of procedures by dropping the requirement that $P(t < \infty) = 1$, with the convention that $Z_t = Z_\infty = \limsup Z_n$ when $t = \infty$, then v is not increased and under the condition $E(\sup Z_n^+) < \infty$, σ is optimal in the class of extended sv's. In general there is no guarantee that the FER is a good procedure. However,

LEMMA 1. *If* $EZ_\sigma (= \int_{\sigma<\infty} Z_\sigma + \int_{\sigma=\infty} Z_\infty)$ *exists and*

$$\liminf_{n\to\infty} \int_{(\sigma>n)} \gamma_n \leqq \int_{(\sigma=\infty)} Z_\infty, \tag{1}$$

then σ *is optimal in the class of extended* sv's. *In order that* (1) *hold it suffices that*

$$\text{(a)} \quad P\{Z_\infty \geqq 0\} = 1, \qquad \text{(b)} \quad \lim_{n\to\infty} E\gamma_n^+ = 0. \tag{2}$$

PROOF. By Theorem 3 of [6] it suffices to show $EZ_\sigma \geqq v$. By Theorem 1 of [3], $v = E\gamma_0$ and $\gamma_n = \max(Z_n, E(\gamma_{n+1} \mid F_n))$, $n = 0, 1, \cdots$. It is easily seen by induction that

$$v = \int_{(\sigma\leqq n)} \gamma_\sigma + \int_{(\sigma>n)} \gamma_n = \int_{(\sigma\leqq n)} Z_\sigma + \int_{(\sigma>n)} \gamma_n$$

and the result follows on letting $n \to \infty$.

Suppose now that $X_1, X_2, \cdots$ are iid with $EX_1 = 0$. Let $F_0 = \{\phi, \Omega\}$, $S_0 = 0$; $F_n = \mathcal{B}(X_1, \cdots, X_n)$, $S_n = \sum_1^n X_i$, $n = 1, 2, \cdots$; and assume that $Z_n = h_n(S_n)$, $n = 0, 1, 2, \cdots$. We are in the stationary Markov case as defined in [6], and, putting

$$U_n(x) = \sup_t E[h_{n+t}(x + S_t)], \qquad -\infty < x < \infty,$$

we have by Theorem 6 of [6]

$$(3) \qquad \gamma_n = U_n(S_n), \qquad n = 0, 1, 2, \cdots,$$

A case of particular interest in what follows is

$$(4) \qquad h_n(x) = n^{-\alpha}|x|^{\beta}, \qquad n = 1, 2, \cdots \qquad (h_0 = 0)$$

where

$$(5) \qquad 0 < \beta < 2\alpha.$$

PROPOSITION 1. *Suppose that* (X_n), (F_n), (Z_n) *are as above and that* (4) *and* (5), *hold. If* $E|X_1|^{\max(2,\beta)} < \infty$, *then* $v < \infty$ *and the* FER *is optimal in the class of extended* sv's.

The proof is based on the following lemmas.

LEMMA 2. *There exists a constant* $A_\beta > 0$ *such that*

$$(6) \qquad EZ_n \leqq A_\beta n^{\beta/2-\alpha}, \qquad n = 1, 2, \cdots,$$

and hence

$$(7) \qquad \lim_{n\to\infty} EZ_n = 0.$$

PROOF. If $\beta \geqq 2$, (6) has been established by Brillinger [1]. If $0 < \beta < 2$, (6) follows from $EZ_n \leqq E^{\beta/2}(Z_n^{2/\beta}) = E^{\beta/2}(n^{-2\alpha/\beta}|S_n|^2)$.

LEMMA 3. *Let* $\beta \geqq 2$. *For* $n = 0, 1, \cdots$ *and all* sv's $t \geqq 1$

$$(8) \qquad E[(n + t)^{-\alpha}|S_t|^{\beta}] \leqq A_\beta \sum_1^\infty [k^{\beta/2} - (k-1)^{\beta/2}]/(n+k)^\alpha < \infty.$$

In particular there exists a $B_\beta > 0$ *such that for* $n = 1, 2, \cdots$

$$(9) \qquad E[(n + t)^{-\alpha}|S_t|^{\beta}] \leqq B_\beta n^{\beta/2-\alpha}.$$

PROOF. Since $(|S_n|^\beta)$ is a submartingale, we have for any sv $t \geqq 1$

$$(10) \qquad E[(n + t)^{-\alpha}|S_t|^{\beta}] \leqq \sum_{i=1}^\infty (n+i)^{-\alpha}\left[\int_{(t\leqq i)} |S_i|^\beta - \int_{(t\leqq i-1)} |S_{i-1}|^\beta\right],$$

and (8) follows from (6) and the proof of Lemma 2 of [4]. To prove (9) we write

$$\sum_1^\infty [k^{\beta/2} - (k-1)^{\beta/2}]/(n+k)^\alpha$$
$$\leqq \text{const} \sum_1^\infty k^{\beta/2-1}/(n+k)^\alpha \leqq \text{const} \int_0^\infty x^{\beta/2-1}/(n+x)^\alpha\, dx$$
$$\leqq \text{const} \int_0^\infty (n+x)^{-\alpha+\beta/2-1}\, dx = \text{const}\ n^{\beta/2-\alpha}.$$

PROOF OF PROPOSITION 1. That $v < \infty$ follows from (8) with $n = 0$ in the case $\beta \geqq 2$ and from (8) and

$$E(t^{-\alpha}|S_t|^\beta) \leqq E^{\beta/2}(t^{-2\alpha/\beta}|S_t|^2)$$

in the case $\beta < 2$.

To complete the proof it suffices by (2) and (3) to show

$$(11) \qquad \lim_{n\to\infty} EU_n(S_n) = 0.$$

Suppose that $\beta \geqq 2$. By the c_r inequality

$$(12) \quad U_n(x) \leqq c_\beta[U_n(0) + n^{-\alpha}|x|^\beta], \qquad -\infty < x < \infty, \quad n = 1, 2, \cdots.$$

It follows from (9) that $U_n(0) \to 0$, which combined with (12) and (7) proves (11).

If $\beta < 2$, we may write

$$E[(n + t)^{-\alpha}|x + S_t|^\beta] \leqq E^{\beta/2}[(n + t)^{-2\alpha/\beta}(x + S_t)^2]$$

and apply the above results. This proves the proposition.

REMARK. A trivial generalization of the above argument shows that for every real y and $m = 1, 2, \cdots$,

$$\sigma(m, y) = \text{first } n \geqq 0 \quad \text{such that} \quad (n + m)^{-\alpha}|y + S_n|^\beta \geqq U_{m+n}(y + S_n)$$

is optimal in the extended sense for the $RS(m + n)^{-\alpha}|y + S_n|^\beta$, $n = 0, 1, \cdots$.

3. Comparing reward sequences. Sometimes by having some knowledge about the behavior of a reward sequence Z_n one may infer something of the behavior of another reward sequence Y_n, defined on the same probability space relative to the same sequence of σ-algebras (F_n). We shall exploit two simply proved principles.

PRINCIPLE I. Suppose $Y_n \leqq Z_n$ for all n and for some ω point (in Ω) and some index m, $Y_m(\omega) = Z_m(\omega)$. If the FER for the Z_n sequence says to stop at time m with reward $Z_m(\omega)$ then the FER for the Y_n sequence will say to stop at time m with reward $Y_m(\omega)$.

PRINCIPLE II. Let $a > 0$ and b be two real numbers and $Y_n = aZ_n + b$. The functional equation rules for both reward sequences are equal.

Let $Z_n = h_n(S_n)$. We will say that the FER σ_Z stops at (n, y) if σ_Z says to stop at time n whenever $S_n = y$. Let

$$(13) \qquad A_m \equiv \{y : \sigma_Z \text{ stops at } (m, y)\}.$$

Let $Y_n = g_n(S_n)$ and define

$$(14) \quad B_m \equiv \{y : \exists\, a > 0, b \text{ such that } g_n(z) \leqq ah_n(z) + b \text{ for all } z \text{ and all } n \geqq m \text{ and } g_m(y) = ah_m(y) + b\}.$$

From an application of Principles I and II we find that the FER σ_Y (for the reward sequence Y_n) stops at (n, y) whenever $y \in A_n \cap B_n$. Clearly $\sigma_Y < \infty$ a.s. if

$$(15) \qquad P\{S_n \in A_n \cap B_n \text{ for some } n\} = 1.$$

LEMMA 4. *A sufficient condition for* (15) *to hold is that*

$$(16) \qquad \limsup P\{S_n \in A_n \cap B_n\} > 0.$$

PROOF.

$$\begin{aligned} P\{S_n \in A_n \cap B_n \text{ for some } n\} &\geqq P\{S_n \in A_n \cap B_n \text{ i.o.}\} = \lim_{n\to\infty} P\{S_m \in A_m \cap B_m \text{ for some } m \geqq n\} \\ &\geqq \limsup_{n\to\infty} P\{S_n \in A_n \cap B_n\} > 0. \end{aligned}$$

By the Hewitt-Savage 0 or 1 law [5] the second probability is 0 or 1, so the conclusion follows.

LEMMA 5. *Let* Y_n *be a reward sequence with* FER σ_Y *and* Z_n *a reward sequence such that*

$$Y_n \leqq aZ_n + b \quad \text{for some} \quad a > 0, b, \quad \text{for at least all large } n.$$

If $\sigma_Y < \infty$ a.s., EY_{σ_Y} *exists, and* (2b) *holds for reward sequence* Z_n, *then* σ_Y *is an optimal rule.*

PROOF. Under the assumptions, one may easily verify condition (1) for σ_Y.

Before considering applications we will state the following

BASIC THEOREM. *Let* $Z_n = n^{-\alpha}|S_n|^\beta$ *where* $2\alpha > \beta > 0$ *and* $E|X_i|^{\max(2,\beta)} < \infty$. *The* FER *stops and is optimal. Specifically,*

(i) *Conditions* (2a) *and* (2b) *hold.*

(ii) *There exists a* $K > 0$ *for which the* FER σ *stops at* (n, y) *whenever* $|y| > Kn^{\frac{1}{2}}$.

Part of the proof is in Section 2, the remainder (condition (ii)) will be shown in Section 5.

APPLICATION I. *Let* $Y_n = c_n(S_n^+)^\beta$ *where* $\limsup n^\alpha c_n < \infty$ *for some* $\alpha > \beta/2 > 0$, *and* $E|X_i|^{\max(2,\beta)} < \infty$. *The* FER σ_Y *is optimal.* ($c_n < 0$ *is allowed.*)

PROOF. Avoiding trivialities, we assume $c_n > 0$ i.o. There exists an $\alpha^* > \beta/2$ (call it α also) for which $n^\alpha c_n^+ \to 0$ as $n \to \infty$. Let $Z_n = n^{-\alpha}|S_n|^\beta$ and $a_m = \sup_{k \geqq m} k^\alpha c_k$. Referring to (13) and the Basic Theorem, we have

$$A_m \supset (-Km^{\frac{1}{2}}, Km^{\frac{1}{2}})^c$$

for $m = 1, 2, \cdots$. Now, infinitely often $a_m = m^\alpha c_m$. Therefore, identifying a_m with a for such m and 0 with b in (14), we have $B_m = [0, \infty)$ i.o.

By Lemma 4 and the central limit theorem, we conclude that $\sigma_Y < \infty$ a.s. Finally, since $Y_n \leqq (\sup_{k \geqq 1} k^\alpha c_k)Z_n$, we have, by the Basic Theorem and Lemma 5, the optimality of σ_Y.

REMARK. Using the same type of argument, we can show that the FER is optimal for reward sequences $c_n|S_n|^\beta$ and $c_n|S_n|^\beta$ sign (S_n) under essentially the same assumptions as in Application I.

APPLICATION II. *Let* $Y_n = g_n(S_n) = c_nP_k(S_n)$ *where* $c_n \geqq 0$, $\limsup n^\alpha c_n < \infty$ *for some* $\alpha > k/2$ *and where* P_k *is a polynomial of degree* k *with positive lead coefficient. Let* $E|X_1|^{\max(2,k)} < \infty$. *Then the* FER σ_Y *is optimal.*

PROOF. For some large r we can write

$$P_k(y) \equiv \xi_k(y - r)^K + \cdots + \xi_1(y - r) + \xi_0$$

with positive coefficients $\xi_0, \xi_1, \cdots, \xi_k$. Then let

$$h_n(y) = n^{-\alpha}\{\xi_k|y - r|^k + \cdots + \xi_1|y - r| + \xi_0\}$$

and $Z_n = h_n(S_n)$ where $\alpha > k/2$ is chosen so that $n^\alpha c_n \to 0$ as $n \to \infty$. Define $a_m = \sup_{n \geqq m} n^\alpha c_n$. Identifying a_m with a whenever $a_m = m^\alpha c_m$ and 0 with b in (14) we have, for infinitely many m, $B_m = [r, \infty)$. By virtue of the Basic Theorem and the following lemma, we have $A_m \supset (-Km^{\frac{1}{2}} - r, Km^{\frac{1}{2}} - r)^c$.

LEMMA 6. *Let* $2\alpha > \beta \geqq \beta' > 0$, $Z_n = n^{\alpha}|S_n|^{\beta}$ *and* $Z_n' = n^{\alpha}|S_n|^{\beta'}$ *with* FER σ_Z *and* $\sigma_{Z'}$, *respectively. Then* $\sigma_{Z'} \leqq \sigma_Z$.

PROOF OF LEMMA. Suppose σ_Z stops at (n, y) and t is any stopping rule. Then

$$E(n+t)^{-\alpha}|y+S_t|^{\beta'}$$
$$\leqq E^{\beta'/\beta}(n+t)^{-\alpha\beta/\beta'}|y+S_t|^{\beta} \leqq n^{-\alpha(\beta-\beta')/\beta}E^{\beta'/\beta}(n+t)^{-\alpha}|y+S_t|^{\beta}$$
$$\leqq n^{-\alpha(\beta-\beta')/\beta}\{n^{-\alpha}|y|^{\beta}\}^{\beta'/\beta} = n^{-\alpha}|y|^{\beta'}.$$

Thus $\sigma_{Z'} \leqq \sigma_Z$.

The remainder of the proof that σ_Y is optimal follows the same lines as in Application I.

APPLICATION III. *Let* $Y_n = g_n(S_n) = n^{-\alpha}\log^+|S_n|$, $\alpha > 0$, *and* $EX_1^2 < \infty$. *The* FER σ_Y *is optimal.*

PROOF. Choose a β satisfying $0 < \beta < \max(2, 2\alpha)$ and define $Z_n = h_n(S_n) = n^{-\alpha}|S_n|^{\beta}$. Then $A_m \supset (-Km^{\frac{1}{2}}, Km^{\frac{1}{2}})^c$. For $y > 1$, we identify $\beta^{-1}y^{-\beta}$ with a and $m^{-\alpha}(\log y - \beta^{-1})$ with b in (14), and find that $B_m \supset [e^{1/\beta}, \infty)$. The optimality of σ_Y follows as before.

REMARKS. (1) The success of the last application is closely related to the concavity of the log function. We will not try to state a general theorem but similar applications exist for many concave functions.

(2) A closely related and quite general result concerning concave functions can be verified. Namely, if Z_n is a reward sequence with optimal FER σ_Z and $Y_n = g(Z_n)$ where g is a non-decreasing concave function. Then the FER σ_Y of Y_n is optimal and $\sigma_Y \leqq \sigma_Z$.

4. Characterization of the stopping region. One is interested in characterizing the stopping region of the FER for a given reward sequence for at least two reasons:

(a) in order to construct close approximations to the optimal rule by backward induction;

(b) in order to prove that the FER stops with probability 1.

We shall use the following characterization in proving the Basic Theorem.

LEMMA 7. *Let* $c_n \geqq 0$ *and* $c_n > 0$ i.o. *Let* $\beta \geqq 1$. *Consider the reward sequence* $c_n|S_n|^{\beta}$. *There exist sequences* $\{a_n\}$ *and* $\{b_n\}$ *of positive numbers (possibly* $+\infty$*) such that the* FER *continues at time n when* $-a_n < S_n < b_n$ *and stops otherwise. If the* FER *is a legitimate stopping rule then* $a_n = b_n = \infty$ *if, and only if,* $c_{n+k} \geqq c_n$ *for some* $k \geqq 1$. *If* $c_n > 0$ *for all n,* $c_n \to 0$ *as* $n \to \infty$ *and the sequence* $\{\log c_n\}$ *is convex then* $\{a_n\}$ *and* $\{b_n\}$ *are monotone increasing sequences.*

PROOF. Suppose $c_{n+k} \geqq c_n$ for some $k \geqq 1$ (we may assume $c_{n+k} > 0$). Let $t \equiv k$. Then (using Jensen's inequality)

$$Ec_{n+t}|y+S_t|^{\beta} = c_{n+k}E|y+S_k|^{\beta} > c_{n+k}|y+ES_k|^{\beta} \geqq c_n|y|^{\beta}.$$

It follows that the FER will not stop at n for any y, i.e., $a_n = b_n = \infty$. Suppose

$c_{n+k} < c_n$ for all $k = 1, 2, \cdots$. Let $t \geqq 1$ be an arbitrary stopping rule. At time n with sum $y = S_n$ it is better to stop than to continue with rule t if, and only if,

(17) $$Ec_{n+t}|y + S_t|^\beta \leqq c_n|y|^\beta.$$

Let

$$g(z) \equiv Ec_{n+t}|1 + zS_t|^\beta - c_n .$$

(17) is equivalent to $g(1/y) \leqq 0$ when $y \neq 0$. We observe that $g(0) < 0$ and $g(z)$ is a convex function in z. Thus, if $0 < y_1 < y_2$ and $g(1/y_1) \leqq 0$ we have $g(1/y_2) \leqq 0$ and similarly for $0 > y_1 > y_2$. It follows that there exist constants $0 \leqq a_n(t), b_n(t) < \infty$ for which it is better to continue with t if, and only if, $-a_n(t) < S_n < b_n(t)$. Define $a_n \equiv \sup_t a_n(t)$ and $b_n = \sup_t b_n(t)$. Since $c_n > 0$ i.o. it is easy to show $a_n, b_n > 0$. If FER σ is a legitimate stopping rule then $a_n = a_n(\sigma)$, $b_n = b_n(\sigma)$ and the above argument shows that $a_n, b_n < \infty$. Finally, suppose that $c_n > 0$, $c_n \to 0$ as $n \to \infty$ and $\{\log c_n\}$ is convex. Let t be a favorable continuation at (n, y), i.e.,

$$Ec_{n+t}|y + S_t|^\beta > c_n|y|^\beta.$$

Then

$$Ec_{n+1+t}|y + S_t|^\beta = Ec_{n+t}c_{n+1+t}c_{n+t}^{-1}|y + S_t|^\beta \geqq c_{n+1}c_n^{-1}Ec_{n+t}|y + S_t|^\beta > c_{n+1}|y|^\beta.$$

It follows that $a_n \leqq a_{n+1}$ and $b_n \leqq b_{n+1}$.

REMARKS. (1) A similar characterization of the stopping region occurs for reward sequence $c_n(S_n^+)^\beta$.

(2) The precise character of the stopping region when $0 < \beta < 1$ is not known. It is difficult to believe the region is any more complicated than it is for $\beta \geqq 1$, but there exist times n and stopping rules t for which the region of favorable continuation (where one prefers to continue with t rather than stop at time n) is not an interval.

5. Proof of the Basic Theorem. The remainder of the proof of the basic theorem is given by Lemmas 8, 9, and 10 below, which are closely related to Lemmas 6, 7 and 8 of [4]. The proof of Lemma 9 is omitted; the reader is referred to [4] for details. For a proof of Lemma 8, we could again refer to [4]. However, Proposition 1, the subsequent remark, and Lemma 7 yield Lemma 8 immediately.

LEMMA 8. *Suppose that* $\beta \geqq 1$, $y \neq 0$, *and for some* $n = 1, 2, \cdots$

$$n^{-\alpha}|y|^\beta < U_n(y).$$

Then there exists an extended sv t *such that*

$$t < \infty \Rightarrow |S_t| \geqq \tfrac{1}{2}|y|$$

and

(18) $$n^{-\alpha}|\tfrac{1}{2}y|^\beta < E((n + t)^{-\alpha}|\tfrac{1}{2}y + S_t|^\beta).$$

LEMMA 9. *Suppose that* $\beta \geqq 1$, $y \neq 0$ *and for some* $n = 1, 2, \cdots$

$$n^{-\alpha}|y|^\beta < U_n(y).$$

Then there exists an extended sv t *such that* (18) *holds and*

$$E[(n + t)^{-\alpha}] \leqq (2n)^{-\alpha} + [n^{-\alpha} - (2n)^{-\alpha}]4\sigma^2 n/y^2. \tag{19}$$

LEMMA 10. *There exists a* $K > 0$ *such that* $|y| \geqq Kn^{\frac{1}{2}} \Rightarrow n^{-\alpha}|y|^{\beta} = U_n(y)$, $n = 1, 2, \cdots$.

PROOF. Suppose the contrary. Then, for every $K > 0$ there exists an $n = 1, 2, \cdots$ and an extended sv t which by Lemmas 8 and 9 may be assumed to satisfy (18) and (19) for some $|y| \geqq Kn^{\frac{1}{2}}$. First suppose that $\beta \geqq 2$ and consider the case $y > 0$. By Lemma 7 we may assume that $y = Kn^{\frac{1}{2}}$. From Lemmas 3 and 9 and the Minkowski inequality we have

$$n^{-\alpha/\beta+\frac{1}{2}}\tfrac{1}{2}K < \tfrac{1}{2}Kn^{\frac{1}{2}}E^{1/\beta}(n + t)^{-\alpha} + E^{1/\beta}[(n + t)^{-\alpha}|S_t|^{\beta}]$$
$$\leqq \tfrac{1}{2}Kn^{\frac{1}{2}}((2n)^{-\alpha/\beta} + (2^{\alpha} - 1)^{1/\beta}(2n)^{-\alpha/\beta}(4\sigma^2)^{1/\beta}n^{1/\beta}K^{-2/\beta}n^{-1/\beta}) + \text{const}\, n^{-\alpha/\beta+\frac{1}{2}}.$$

Hence

$$\tfrac{1}{2}Kn^{-\alpha/\beta+\frac{1}{2}}(1 - 2^{-\alpha/\beta}) \leqq \text{const}\, K^{1-2/\beta}n^{-\alpha/\beta+\frac{1}{2}} + \text{const}\, n^{-\alpha/\beta+\frac{1}{2}},$$

a contradiction for K sufficiently large. The case $y < 0$ is handled similarly, and the case $\beta < 2$ is an application of Lemma 6.

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