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THE ESTIMATION OF VARIANCES AFTER USING A GAUSSIANATING TRANSFORMATION

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1. Introduction and summary. Neyman and Scott [12] considered the problem of estimating the mean of a distribution after some fixed Gaussian inducing transformation had been applied to the observations. More specifically, if a random variable x is observed then it is assumed that $\xi = f^{-1}(x)$ has a Gaussian distribution with mean μ and variance σ^2. They find the minimum variance unbiased estimator (MVUE), $\hat{\theta}$, of $\hat{\theta} = E(x)$ in terms of the MVUE's of μ and σ^2. The heart of their solution is in taking a Taylor series expansion of $f(\xi)$ around the origin and showing that the resulting infinite power series behaves like a finite one in the sense that the operations of taking expectations and summing can be reversed as required provided only that $f(\xi)$ is an entire function of second order or less. This paper exploits the Taylor series expansion of $f(\xi)$ around the mean to find the MVUE of $\phi^2 = E(x - \theta)^2$. It was motivated in part by the fact that recent research in cloud seeding has shown that the variance may be a more important parameter than the mean [8], [15], and also by a general concern with the class of recursive transformations defined on p. 651 of [12]. In addition the MVUE of Var $(\hat{\theta})$ for particular transformations is derived.

The problems under consideration here have been considered by Finney [6], Sichel [14], and Meulenberg [11] for the logarithmic transformation i.e., $f(\xi) = e^{m\xi}$. Their results provide a check on mine.

Schmetterer [13] has interpreted the results of Neyman and Scott in terms of the solution $h(\hat{\mu}, \hat{\sigma}^2)$ of the integral equation

$$E[h(\hat{\mu}, \hat{\sigma}^2)] = E(x)$$

where $\hat{\mu}$, $\hat{\sigma}^2$ are the MVUE's for μ and σ^2. Kolmogorov [10] has also considered the problem of finding unbiased estimators in terms of the solutions of integral equations but he relies heavily upon the results of Blackwell [2] in using the sufficient statistics for (μ, σ^2) to turn unbiased but inefficient estimators into the MVUE.

The problem discussed here can be formulated as an integral equation: viz., find $h^*(\hat{\mu}, \hat{\sigma}^2)$ such that

$$E[h^*(\hat{\mu}, \hat{\sigma}_\rho{}^2)] = E[x - E(x)]^2$$

but the present author has not attempted to solve this problem in this way. Rather he has approached the problem by a straightforward application of the method of Neyman and Scott.

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2. Procedure. The first part of the paper shows that provided (i) θ exists and (ii) f is an entire function of second order, i.e., $f \varepsilon \mathfrak{F}$, where $\mathfrak{F}$ is the class of entire functions of second order or less, it is permissible to write

$$\theta = E[f(\xi)] = \sum_{n=0}^{\infty} (n!)^{-1} f_\mu^{(n)} E(\xi - \mu)^n = \sum_{n=0}^{\infty} f_\mu^{(2n)} (n!)^{-1} (\tfrac{1}{2}\sigma^2)^n$$

where $f_\mu^{(n)}$ stands for the nth derivative of f evaluated at μ. Further by expanding $f_\mu^{(2n)}$ as a Taylor series itself one obtains

$$\theta = \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} f_0^{(2m+k)} (k!\, m!\, 2^m)^{-1} \mu^k \sigma^{2m}.$$

It is then shown that if one substitutes the MVUE of $\mu^k \sigma^{2m}$ in this formula an unbiased estimate of θ can be obtained.

The second part of the paper shows that given (i), (ii) and (iii) ϕ^2 exists, one can legitimately write

$$\phi^2 = E\{f(\xi) - E[f(\xi)]\}^2$$
$$= \sum_{n=1}^{\infty} \sum_{k=1}^{n} f_\mu^{(k)} f_\mu^{(n-k+1)} (k!(n - k + 1)!)^{-1} [\mu_{n+1} - \mu_k \cdot \mu_{n-k+1}]$$

where $\mu_r = E(\xi - \mu)^r$. In an exactly analogous method to that above one can obtain the MVUE of ϕ^2. In the last part these results are used to find the MVUE of var $(\hat{\theta})$.

3. Preliminary results. The following results will be required in the sequel and it is convenient to prove them at the outset.

(i) If $g(x)$ is an entire function of order ρ $(0 \leqq \rho \leqq \infty)$ and type γ $(0 \leqq \gamma \leqq \infty)$ then $g_x^{(1)}$ and $g_x^{(2)}$ are entire functions of the same order and type. In particular $|g_\mu^{(1)}| < \infty$, $|g_\mu^{(2)}| < \infty$ for all finite μ. This is Theorem 2.4.1. of Boas [3].

(ii) A necessary and sufficient condition for the two series

$$\text{I}: \sum_{n=0}^{\infty} f_\mu^{(2n)} Z^n (n!)^{-1} \qquad \text{and} \qquad \text{II}: \sum_{n=0}^{\infty} f_\mu^{(2n+1)} Z^n (n!)^{-1}$$

to converge for all finite μ and Z is that $f \varepsilon \mathfrak{F}$. That f must be in $\mathfrak{F}$ is apparent by letting $\mu = 0$ and for this case Neyman and Scott have proved the converse. In order to prove the converse for $\mu \neq 0$ it is necessary only to expand $f_\mu^{(2n)}$ and $f_\mu^{(2n+1)}$ as a Taylor series in μ and to show that the resulting infinite double sum is absolutely convergent. This is easily done by showing that II for instance is absolutely convergent or divergent with the series

$$|\mu^{-1}| \sum_{p=0}^{\infty} (|Z| + \mu^2)^p |f_0^{(2p+1)}| (p!)^{-1} + \sum_{p=0}^{\infty} (|Z| + \mu^2)^p |f_0^{(2p)}| (p!)^{-1}$$

and this is absolutely convergent iff $f \varepsilon \mathfrak{F}$, [12].

(iii) Let

$$\gamma(r, s) = E[|(\xi - \mu)^r - E(\xi - \mu)^r|\, |(\xi - \mu)^s - E(\xi - \mu)^s|]; \quad r, s = 1, 2, \cdots,$$

then $\gamma(r, s) \leqq \beta_{r+s} + 3\beta_r \cdot \beta_s$ where $\beta_r = E\,|\xi - \mu|^r$. This is easily seen by expanding the right hand side of $\gamma(r, s)$.

(iv)

$$\beta_{2m-1} \leqq \sigma(\tfrac{1}{2}\sigma^2)^{m-1}(2m-1)!((m-1)!)^{-1}, \quad m = 1, 2, \cdots.$$

This is easily proved using the Cauchy-Schwarz inequality $\beta_{2m-1} \leqq (\mu_{2m-2}\cdot\mu_{2m})^{\frac{1}{2}}$ and then substituting for μ_{2m} and μ_{2m-2}.

(v) $\binom{2n+1}{2k} \leqq 2^{2n+1}\binom{n+1}{k}$ for $n = 1, 2, \cdots$ and $k = 1, 2, \cdots n$. Now

$$\begin{aligned}
&\Gamma(2n+2)[\Gamma(2k+1)\Gamma(2n-2k+2)]^{-1}\\
&= \Gamma(n+1)\Gamma(n+\tfrac{3}{2})\Gamma(\tfrac{1}{2})[\Gamma(k+\tfrac{1}{2})\Gamma(k+1)\Gamma(n-k+1)\Gamma(n-k+\tfrac{3}{2})]^{-1}\\
&= \binom{n+1}{k}\cdot(n+2)^{-1}\Gamma(n+\tfrac{3}{2})\Gamma(\tfrac{1}{2})[\Gamma(n-k+\tfrac{3}{2})\Gamma(k+\tfrac{1}{2})]^{-1}\\
&\leqq \tfrac{1}{2}\binom{n+1}{k}B(n+\tfrac{3}{2}, \tfrac{1}{2})[B(n-k+\tfrac{3}{2}, k+\tfrac{1}{2})]^{-1}\\
&\leqq \binom{n+1}{k}2^{2n+1}B(n+\tfrac{3}{2}, n+\tfrac{3}{2})[B(n-k+\tfrac{3}{2}, k+\tfrac{1}{2})]^{-1}\\
&\leqq \binom{n+1}{k}2^{2n+1}.
\end{aligned}$$

Since the beta function is a decreasing function of its arguments and where we have made extensive use of the properties of the beta and gamma functions, [17], [18]. In an exactly similar way one can show that

$$\binom{2n}{2k} \leqq 2^{2n}\binom{n}{k};\ \binom{2n}{2k-1} \leqq 2^{2n}\binom{n}{k-1} \text{ and } \binom{2n+1}{2k+1} \leqq 2^{2n+2}\binom{n}{k}$$

(vi) MVUE of $\sigma^{2m}\mu^k$. In their paper Washio, Morimoto and Ikeda [16] found the MVUE under general assumptions, of any analytic function of two unknown parameters of the exponential family. In particular they discuss the problem of unbiasedly estimating $\mu^\alpha\sigma^\beta$ (α a non-negative integer and $-\beta \leqq \alpha$) from a sample of n observations from $N(\mu, \sigma^2)$. They give several formulae based on the statistics $\hat{\mu}$ and S which, because of their sufficient and complete properties, ensure minimum variance estimators, where $\hat{\mu}$ is distributed according to $N(\mu, \lambda^2\sigma^2)$ and S/σ^2 according to $\chi^2(\nu)$ and independently of one another. λ^2 is the constant defining the variance of $\hat{\mu}$ in terms of the variance of the observations and ν is the degree of freedom of the residual sum of squares, S. Let $\boldsymbol{\theta} \varepsilon \Theta$ be a parameter in the general parametric space Θ and let $\mathbf{t}$ be the MVUE of $\boldsymbol{\theta}$, then the linear operator E^{-1} is defined by $\mathbf{t} = E^{-1}(\boldsymbol{\theta})$.

It is readily seen from Corollary 2.2 of [16] that $E^{-1}(\tau_1^{-\alpha}\tau_2^{-k})$, where $\tau_1 = \frac{1}{2}\sigma^{-2}$, $\tau_2 = \mu\sigma^{-2}\lambda^{-2}$, $u_2 = -\hat{\mu}$ is given by

$$S^{-(\nu/2-1)}\cdot(\Gamma(\alpha))^{-1}\cdot(\partial^k/\partial u_2^k)\{\textstyle\int_0^S x^{\alpha-1}(S-x)^{\nu/2-1}\,dx\}$$

where $k = 0, 1, 2, \cdots, \alpha \geqq -k$, and, since S is the residual sum of squares, we have $\partial S/\partial\hat{\mu} = -2\hat{\mu}/\lambda^2$. Thus, for $m \geqq -(k/2)$,

$$\begin{aligned}
&E^{-1}(\sigma^{2m}\mu^k)\\
&\quad= (-1)^k\lambda^{2k}B(m+k, \nu/2)[2^{m+k}S^{\nu/2-1}\Gamma(m+k)]^{-1}(\partial^k/\partial\hat{\mu}^k)\{S^{m+k+\nu/2-1}\}.
\end{aligned}$$

This partial derivative can be evaluated using the results of Faà di Bruno [3] or Götting [6].

From their results it is clear that

$$\partial^k(S^p)/\partial\hat{\mu}^k = k! \sum p!\, S^p[(p + N - k)!\, n_1!\, n_2!]^{-1}(-2\hat{\mu}/S\lambda^2)^{n_1}(-1/S\lambda^2)^{n_2}, \quad p = 1, 2, \cdots,$$

where the summation is taken over all partitions such that $N = 0, 1, \cdots k - 1$, $n_1 + n_2 = k - N$, $1n_1 + 2n_2 = k$, and n_1, n_2 non-negative integers, so that

$$\partial^k(S^p)/\partial\hat{\mu}^k = \sum_{N=0}^{[k/2]} p!\, k!\, S^{p-k+N}(2\hat{\mu})^{k-2N}[N!(p - k + N)!(k - 2N)!(-\lambda^2)^{k-N}]^{-1}.$$

Whence

$$E^{-1}(\sigma^{2m}\mu^k) = \Gamma(\nu/2)k!(S/2)^m \sum_{i=0}^{[k/2]} (-\tfrac{1}{4}S\lambda^2)^i\hat{\mu}^{k-2i}[i!(k - 2i)!\,\Gamma(m + \nu/2 + i)]^{-1}.$$

Alternatively if one knows this result one can check directly that its expectation is $\sigma^{2m}\mu^k$.

(vii) Upper bounds for $E\,|E^{-1}(\mu^k\sigma^{2m})|$, $m, k = 0, 1, \cdots$. Consider first the case when k is even. Then

$$E\,|E^{-1}(\mu^{2k}\sigma^{2m})| \leqq 2k!\,\sigma^{2m}(k!)^{-1}(\tfrac{1}{2}\sigma^2\lambda^2)^k + 2k!\,\sigma^{2m}\sum_{i=0}^{k-1}(\tfrac{1}{2}\sigma^2\lambda^2)^i E\,|\hat{\mu}|^{2k-2i}[(2k - 2i)!\,i!]^{-1}.$$

But

$$E\,|\hat{\mu}|^{2k-2i} \leqq \sum_{j=0}^{k-i}\binom{2k-2i}{2j}\beta_{2j}'\,|\mu|^{2k-2i-2j} + \sum_{j=1}^{k-i}\binom{2k-2i}{2j-1}\beta_{2j-1}'\,|\mu|^{2k-2i-2j+1}$$

where $\beta_r' = E\,|\hat{\mu} - \mu|^r$, $r = 0, 1, \cdots$, and

$$\sum_{j=0}^{k-i}\binom{2k-2i}{2j}\beta_{2j}'\,|\mu|^{2k-2i-2j} \leqq \sum_{j=0}^{k-i}(2k - 2i)![j!(2k - 2i - 2j)!]^{-1}(\tfrac{1}{2}\sigma^2\lambda^2)^j\mu^{2k-2i-2j} \leqq (2k - 2i)![(k - i)!]^{-1}[\tfrac{1}{2}\sigma^2\lambda^2 + \mu^2]^{k-i}.$$

For $\mu \neq 0$,

$$\sum_{j=1}^{k-i}\binom{2k-2i}{2j-1}\beta_{2j-1}'\mu^{2k-2i-2j+1} \leqq \sigma\lambda\,|\mu|^{-1}\sum_{j=1}^{k-i}(2k - 2i)!\,|\mu|^{2k-2i-2j+2}[(j - 1)!(2k - 2i - 2j + 1)!]^{-1}(\tfrac{1}{2}\sigma^2\lambda^2)^{j-1} \leqq \sigma\lambda\,|\mu|^{-1}(2k - 2i)![(k - i)!]^{-1}[\tfrac{1}{2}\sigma^2\lambda^2 + \mu^2]^{k-i}$$

where we have used the results of Section 3(iv). Thus if $\mu \neq 0$, $E\,|\hat{\mu}|^{2k-2i} \leqq [1 + \sigma\lambda/|\mu|](2k - 2i)![(k - i)!]^{-1}(\tfrac{1}{2}\sigma^2\lambda^2 + \mu^2)^{k-i}$ and if $\mu = 0$, $E\,|\hat{\mu}|^{2k-2i} = (2k - 2i)![(k - i)!]^{-1}(\tfrac{1}{2}\sigma^2\lambda^2)^{k-i}$ i.e. $E\,|\hat{\mu}|^{2k-2i} \leqq (1 + \rho_\mu)(2k - 2i)![(k - i)!]^{-1}\cdot(\tfrac{1}{2}\sigma^2\lambda^2 + \mu^2)^{k-i}$ where $\rho_\mu = 0$ if $\mu = 0$ and $\rho_\mu = \sigma\lambda|\mu|^{-1}$ if $\mu \neq 0$. So finally,

$$E\,|E^{-1}(\mu^{2k}\sigma^{2m})| \leqq 2k!(k!)^{-1}\sigma^{2m}(\tfrac{1}{2}\sigma^2\lambda^2)^k + (1 + \rho_\mu)2k!(k!)^{-1}\sigma^{2m}\sum_{i=0}^{k-1}\binom{k}{i}(\tfrac{1}{2}\sigma^2\lambda^2 + \mu^2)^{k-i} \leqq (1 + \rho_\mu)2k!(k!)^{-1}\sigma^{2m}(\sigma^2\lambda^2 + \mu^2)^k.$$

In an exactly analogous manner it can be shown that

$$E\,|E^{-1}(\mu^{2k+1}\sigma^{2m})| \leqq (2k+1)!(k!)^{-1}\sigma^{2m}[|\mu| + \sigma\lambda][\sigma^2\lambda^2 + \mu^2]^k.$$

4. Estimation of the mean. To justify writing

$$\theta = \sum_{n=0}^{\infty} f_\mu^{(2n)}(n!)^{-1}(\tfrac{1}{2}\sigma^2)^n = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} f_0^{(2n+k)}\mu^k\sigma^{2n}[n!\,k!\,2^n]^{-1}$$

and

$$\hat{\theta} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} f_0^{(2n+k)}[n!\,k!\,2^n]^{-1}E^{-1}(\mu^k\sigma^{2n})$$

it is necessary to use the following easily established results.

(i) The power series expansion of $f(\xi)$, i.e., $\sum_{n=0}^{\infty} f_\mu^{(n)}(\xi - \mu)^n(n!)^{-1}$ is absolutely convergent for all μ, σ^2 and $f \varepsilon \mathfrak{F}$.

(ii) The power series expansion of $E[f(\xi)]$, i.e., $\sum_{n=0}^{\infty} f_\mu^{(n)}E(\xi - \mu)^n(n!)^{-1}$ is absolutely convergent for all μ, σ^2 iff $f \in \mathfrak{F}$.

(iii) The expectation of the Taylor series expansion of $f(\xi)$ can be taken term by term, i.e., the series $\sum_{n=0}^{\infty} f_\mu^{(n)}(n!)^{-1}E\,|\xi - \mu|^n$ is absolutely convergent for all μ, σ^2 and $f \varepsilon \mathfrak{F}$.

To prove this it is sufficient to show that the two series

$$\sum_{n=0}^{\infty} f_\mu^{(2n)}(2n!)^{-1}E(\xi - \mu)^{2n} \quad \text{and} \quad \sum_{n=0}^{\infty} f_\mu^{(2n+1)}((2n+1)!)^{-1}E\,|\xi - \mu|^{2n+1}$$

are absolutely convergent for all μ, σ^2 and $f \varepsilon \mathfrak{F}$. The first part is immediate from Section 3(ii). For the second series, the result follows by using the results of Sections 3(ii) and 3 (iv).

(iv) The power series representation of $\hat{\theta}$ is absolutely convergent for all $\hat{\mu}$, S and $f \varepsilon \mathfrak{F}$.

To prove this we note that by simple algebra

$$|E^{-1}(\mu^{2k}\sigma^{2n})|(2k!\,n!\,2^n)^{-1} \leqq S^n(\lambda^2 S + 4\hat{\mu}^2)^k(n!\,k!\,2^{2n+2k})^{-1}$$

and

$$|E^{-1}(\mu^{2k+1}\sigma^{2n})|((2k+1)!\,n!\,2^n)^{-1} \leqq |\hat{\mu}|\,S^n(\lambda^2 S + 4\hat{\mu}^2)^k(n!\,k!\,2^{2n+2k})^{-1}$$

$$k = 0, 1 \cdots; \quad n = 0, 1, \cdots;$$

so that $\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} |f_0^{(2n+k)}E^{-1}(\mu^k\sigma^{2n})|(k!\,n!\,2^n)^{-1}$ is finite for all $\hat{\mu}$, S, $f \varepsilon \mathfrak{F}$ iff the same is true of both the series

$$\sum_{p=0}^{\infty}\sum_{n+k=p} |f_0^{(2n+2k)}|\,S^n(\lambda^2 S + 4\hat{\mu}^2)^k(k!\,n!\,2^{2n+2k})^{-1}$$

$$\text{and} \quad |\hat{\mu}|\sum_{p=0}^{\infty}\sum_{n+k=p} |f_0^{(2n+2k+1)}|\,S^n(\lambda^2 S + 4\hat{\mu}^2)^k(n!\,k!\,2^{2n+2k})^{-1}.$$

The finiteness of both these series for all $\hat{\mu}$, S and $f \varepsilon \mathfrak{F}$ follows almost immediately from the definition of $\mathfrak{F}$.

(v) $E\,|\hat{\theta}| < \infty$ for all μ, σ^2, $f \varepsilon \mathfrak{F}$.

Using the results of Section 3(vii) it is easily shown that

$$E\,|\hat{\theta}| \leqq (1 + \rho_\mu)\sum_{m=0}^{\infty}\sum_{k=0}^{\infty} |f_0^{(2m+2k)}|(k!\,m!)^{-1}(\tfrac{1}{2}\sigma^2)^m(\sigma^2\lambda^2 + \mu^2)^k$$
$$+ (|\mu| + \sigma\lambda)\sum_{m=0}^{\infty}\sum_{k=0}^{\infty} |f_0^{(2m+2k+1)}|(k!\,m!)^{-1}(\tfrac{1}{2}\sigma^2)^m(\sigma^2\lambda^2 + \mu^2)^k$$

so that $E|\hat{\theta}| < \infty$ iff, for all μ, σ^2 and $f \varepsilon \mathfrak{F}$,

$$\sum_{p=0}^{\infty} \sum_{m+k=p} |f_0^{(2m+2k)}|(k!\,m!)^{-1}(\tfrac{1}{2}\sigma^2)^m(\sigma^2\lambda^2 + \mu^2)^k < \infty$$

and

$$\sum_{p=0}^{\infty} \sum_{m+k=p} |f_0^{(2m+2k+1)}|(k!\,m!)^{-1}(\tfrac{1}{2}\sigma^2)^m(\sigma^2\lambda^2 + \mu^2)^k < \infty.$$

This follows almost immediately as before. Thus the expectation of $\hat{\theta}$ exists and it is clear that $E(\hat{\theta}) = \theta$.

5. Estimation of the variance.

THEOREM 1. (A)

$$\phi^2 = \sum_{n=1}^{\infty} \sum_{k=1}^{n} f_\mu^{(k)} f_\mu^{(n-k+1)}(k!(n-k+1)!)^{-1}[\mu_{n+1} - \mu_k \cdot \mu_{n-k+1}]$$

and

$$\hat{\phi}^2 = \sum_{n=1}^{\infty} \sum_{k=1}^{n} \sum_{j=0}^{\infty} \sum_{i=0}^{j} f_0^{(k+i)} f_0^{(n-k+1+j-i)}(i!\,k!(n-k+1)!(j-i)!)^{-1} \cdot E^{-1}[\mu^j(\mu_{n+1} - \mu_k \cdot \mu_{n-k+1})]$$

are absolutely convergent for μ, σ^2 iff $f \varepsilon \mathfrak{F}$.

(B) $\hat{\phi}^2$ *is the* MVUE *of* ϕ^2.

To obtain the expression for $\phi^2 = \text{var}(x)$ we exploit the method of Neyman and Scott and the traditional large-sample procedure for finding an approximate answer to the problem, [9]. Now,

$$\begin{aligned}\phi^2 &= E\{[x - E(x)]^2\} \\ &= E\{[f(\xi) - E\{f(\xi)\}]^2\} \\ &= E\{[\textstyle\sum_{n=1}^{\infty} (n!)^{-1} f_\mu^{(n)}\{(\xi - \mu)^n - E(\xi - \mu)^n\}]^2\} \\ &= E[\textstyle\sum_{n=1}^{\infty} \sum_{k=1}^{n} f_\mu^{(k)} f_\mu^{(n-k+1)}(n!\,(n-k+1)!)^{-1}\{(\xi - \mu)^k - E(\xi - \mu)^k\} \\ &\qquad \cdot \{(\xi - \mu)^{n-k+1} - E(\xi - \mu)^{n-k+1}\}].\end{aligned}$$

These manipulations are justified as it has been shown that $f(\xi)$ and $E\{f(\xi)\}$ can both be expanded as absolutely convergent Taylor series around μ for all μ, ξ, σ^2 and $f \varepsilon \mathfrak{F}$. Thus the Taylor series expansion of $f(\xi) - E\{f(\xi)\}$ is absolutely convergent and so therefore is the expansion of $[f(\xi) - E\{f(\xi)\}]^2$. The next step is to take, and this will be justified later, the expectation inside the double summation giving

$$\begin{aligned}\phi^2 &= \textstyle\sum_{n=1}^{\infty} \sum_{k=1}^{n} f_\mu^{(k)} f_\mu^{(n-k+1)}(k!(n-k+1)!)^{-1} \operatorname{cov}\{(\xi - \mu)^k, \\ &\qquad (\xi - \mu)^{n-k+1}\} \\ &= \textstyle\sum_{n=1}^{\infty} \sum_{k=1}^{n} f_\mu^{(k)} f_\mu^{(n-k+1)}(k!(n-k+1)!)^{-1}[\mu_{n+1} - \mu_k\mu_{n-k+1}]\,.\end{aligned} \tag{1}$$

To demonstrate that this double series is absolutely convergent for all μ, σ^2 iff $f \varepsilon \mathfrak{F}$ assume the convergence of right hand side of (1), then

$$\infty > \sum_{n=1}^{\infty} \sum_{k=1}^{n} |f_\mu^{(k)} f_\mu^{(n-k+1)}| \cdot |\mu_{n+1} - \mu_k\mu_{n-k+1}|(k!(n-k+1)!)^{-1}$$

$$\geqq \sum_{n=1}^{\infty} |f_\mu^{(1)} f_\mu^{(n)}| (n!)^{-1} |\mu_{n+1}|$$
$$\geqq \sigma^2 |f_\mu^{(1)}| \sum_{n=0}^{\infty} |f_\mu^{(2n+1)}| (n!)^{-1} (\tfrac{1}{2}\sigma^2)^n$$

so that $f \varepsilon \mathfrak{F}$.

To show the converse we note that to show that expectations can be taken term by term it is necessary to show that if $f \varepsilon \mathfrak{F}$ then the double series

$$\sum_{n=1}^{\infty} \sum_{k=1}^{n} |f_\mu^{(k)} f_\mu^{(n-k+1)}| (k!(n-k+1)!)^{-1} \cdot E\{|(\xi-\mu)^k - E(\xi-\mu)^k| \cdot |(\xi-\mu)^{n-k+1} - E(\xi-\mu)^{n-k+1}|\}$$

is convergent for all μ, σ^2 and this will be proved in Theorem 2. But

$$|\text{cov}\{(\xi-\mu)^k, (\xi-\mu)^{n-k+1}\}| \leqq \beta_{n+1} + \beta_k \beta_{n-k+1} \leqq \gamma(k, n-k+1)$$

where we have used the result of Section 3(iii). It is clear therefore that if the series

$$\sum_{n=1}^{\infty} \sum_{k=1}^{n} f_\mu^{(k)} f_\mu^{(n-k+1)} (k!(n-k+1)!)^{-1} \gamma(k, n-k+1)$$

is absolutely convergent for all μ, σ^2 and $f \varepsilon \mathfrak{F}$ then so is the series

$$\sum_{n=1}^{\infty} \sum_{k=1}^{n} f_\mu^{(k)} f_\mu^{(n-k+1)} (k!(n-k+1)!)^{-1} (\mu_{n+1} - \mu_k \mu_{n-k+1}).$$

THEOREM 2. *The series*

$$\sum_{n=1}^{\infty} \sum_{k=1}^{n} f_\mu^{(k)} f_\mu^{(n-k+1)} (k!(n-k+1)!)^{-1} \gamma(k, n-k+1)$$

is absolutely convergent for all μ, σ^2 *and* $f \varepsilon \mathfrak{F}$.

By rearranging the absolute values of the terms of this series according to whether n and k are odd or even, one finds that the series is bounded above by the sum of four double series all of which can be shown individually to be finite for μ, σ^2 and $f \varepsilon \mathfrak{F}$.

Consider for example the double series (k odd, m odd)

$$\sum_{m=1}^{\infty} \sum_{k=1}^{m} |f_\mu^{(2k-1)} f_\mu^{(2m-2k+2)}| ((2k-1)!(2m-2k+2)!)^{-1} \gamma(2k-1, 2m-2k+2).$$

Using the results of Section 3(iii), (iv), (v) fairly straightforward algebra shows that this latter double series is bounded above by

$$7\sigma^2 \sum_{m=1}^{\infty} \sum_{k=1}^{m} |f_\mu^{(2k-1)} f_\mu^{(2m-2k+2)}| (2\sigma^2)^m ((k-1)!(m-k+1)!)^{-1}$$
$$= 7\sigma^2 \left(\sum_{k=0}^{\infty} |f_\mu^{(2k+1)}| (2\sigma^2)^k (k!)^{-1}\right) \left(\sum_{k=1}^{\infty} |f_\mu^{(2k)}| (2\sigma^2)^k (k!)^{-1}\right).$$

But both these series are convergent for all μ, σ^2 and $f \varepsilon \mathfrak{F}$. The proofs for the other three constituent series are similar and are omitted. Together they prove this theorem and hence also Theorem 1A.

It remains to show that (i) the series expansion for $\hat{\phi}^2$ is absolutely convergent for all $\hat{\mu}$, S and $f \varepsilon \mathfrak{F}$ and (ii) $E|\hat{\phi}^2| < \infty$ for all μ, σ^2 and $f \varepsilon \mathfrak{F}$. Now

$$\phi^2 = \sum_{n=1}^{\infty} \sum_{k=1}^{n} f_\mu^{(k)} f_\mu^{(n-k+1)} (k!(n-k+1)!)^{-1} [\mu_{n+1} - \mu_k \cdot \mu_{n-k+1}]$$
$$= \sum_{n=1}^{\infty} \sum_{k=1}^{n} \sum_{j=0}^{\infty} \sum_{i=0}^{j} f_0^{(k+i)} f_0^{(n-k+1+j-i)} (i!\,k!(n-k+1)!(j-i)!)^{-1} \cdot \mu^j [\mu_{n+1} - \mu_k \cdot \mu_{n-k+1}]$$

and so to prove (i) one has to show the convergence for all $\hat{\mu}$, S, $f \varepsilon \mathfrak{F}$ of

$$A = \sum_{n=1}^{\infty} \sum_{k=1}^{n} \sum_{j=0}^{\infty} \sum_{i=0}^{j} |f_0^{(k+i)} f_0^{(n-k+1+j-i)} E^{-1}(\mu^j \mu_{n+1})| \cdot (i!k!(n-k+1)!(j-i)!)^{-1}$$

and

$$B = \sum_{n=1}^{\infty} \sum_{k=1}^{n} \sum_{j=0}^{\infty} \sum_{i=0}^{j} |f_0^{(k+i)} f_0^{(n-k+1+j-i)} E^{-1}(\mu^j \mu_k \mu_{n-k+1})| \cdot (i!k!(n-k+1)!(j-i)!)^{-1}$$

Now

$$\begin{aligned} A &\leqq \sum_{m=1}^{\infty} \sum_{k=1}^{2m-1} \sum_{j=0}^{\infty} \sum_{i=0}^{j} |f_0^{(k+i)} f_0^{(2m-k+j-i)} E^{-1}(\mu^j \mu_{2m})| \cdot (i!k!(2m-k)!(j-i)!)^{-1} \\ &\leqq \sum_{m=1}^{\infty} \sum_{k=1}^{2m-1} \sum_{j=0}^{\infty} \sum_{i=0}^{2j} |f_0^{(k+i)} f_0^{(2m-k+2j-i)} E^{-1}(\mu^{2j} \mu_{2m})| \cdot (i!k!(2m-k)!(2j-i)!)^{-1} \\ &\quad + \sum_{m=1}^{\infty} \sum_{k=1}^{2m-1} \sum_{j=0}^{\infty} \sum_{i=0}^{2j+1} |f_0^{(k+i)} f_0^{(2m-k+2j+1-i)} E^{-1}(\mu^{2j+1} \mu_{2m})| \cdot (i!k!(2m-k)!(2j+1-i)!)^{-1} \end{aligned}$$

as the odd central moments of a Gaussian distribution are zero. It is not difficult to show that

$$\begin{aligned} |E^{-1}(\mu_{2m} \mu^{2j})| &\leqq 2m!(m!)^{-1} 2j!(j!)^{-1} \sum_{i=0}^{j} \binom{j}{i} (\tfrac{1}{4} S \lambda^2)^i \hat{\mu}^{2j-2i} \\ &\leqq 2m!(m!)^{-1} 2j!(j!)^{-1} Y^m Z^j \end{aligned}$$

where $Y = \frac{1}{4}S$ and $Z = \frac{1}{4}S\lambda^2 + \hat{\mu}^2$. Similarly,

$$|E^{-1}(\mu_{2m} \mu^{2j+1})| \leqq 2m!(m!)^{-1}(2j+1)!(j!)^{-1} |\hat{\mu}| Y^m Z^j.$$

If $X = \max(Y, Z)$ then,

$$\begin{aligned} A &\leqq \sum_{m=1}^{\infty} \sum_{k=1}^{2m-1} \sum_{j=0}^{\infty} \sum_{i=0}^{2j} (2j)!(2m)! X^{m+j} \cdot (i!k!m!j!(2m-k)!(2j-i)!)^{-1} |f_0^{(k+i)} f_0^{(2m-k+2j-i)}| \\ &\quad + \sum_{m=1}^{\infty} \sum_{k=1}^{2m-1} \sum_{j=0}^{\infty} \sum_{i=0}^{2j+1} |\hat{\mu}| X^{m+j} (2j+1)!(2m)! |f_0^{(k+i)} f_0^{(2m-k+2j+1-i)}| \cdot (i!k!m!j!(2m-k)!(2j+1-i)!)^{-1}. \end{aligned}$$

To show that these series converge one must split each one into four parts according as i or k is odd or even.

Consider, for example, the first series when i is even and k is odd, i.e. the series,

$$A_{11} = \sum_{m=1}^{\infty} \sum_{k=1}^{m} \sum_{J=0}^{\infty} \sum_{i=0}^{J} \binom{2J}{2i} \binom{2m}{2k} X^{m+J} |f_0^{(2k+2i-1)} f_0^{(2m-2k+1+2J-2i)}| (J!m!)^{-1}$$

But from Section 3(v), $\binom{2J}{2i} \leqq 2^{2J} \binom{J}{i}$ and $\binom{2m}{2k-1} \leqq 2^{2m} \binom{m}{k-1}$ so that

$$\begin{aligned} \binom{2J}{2i} \binom{2m}{2k} (J!m!)^{-1} &\leqq 2^{2m+2J}((i+k-1)!(m-k+1+J-i)!)^{-1} \binom{i+k-1}{i} \binom{m-k+1+J-i}{J-i} \\ &\leqq 8^{m+j}((i+k-1)!(m-k+1+J-i)!)^{-1}. \end{aligned}$$

Thus $A_{11} \leqq \sum_{m=1}^{\infty} \sum_{k=1}^{m} \sum_{J=0}^{\infty} \sum_{i=0}^{J} (8X)^{m+J} |f_0^{(2k+2i-1)} f_0^{(2m-2k+2J-2i+1)}|$
$$\cdot((i + k - 1)!(m - k + 1 + J - i)!)^{-1}$$
$$\leqq 8X[\sum_{m=0}^{\infty} \sum_{J=0}^{\infty} |f_0^{(2m+2J+1)}|((m + J)!)^{-1}(8X)^{m+J}]^2.$$

Now this is finite for all $\hat{\mu}$, S, $f \varepsilon \mathfrak{F}$ iff

$$\sum_{m=0}^{\infty} \sum_{J=0}^{\infty} |f_0^{(2m+2J+1)}|((m + J)!)^{-1}(8X)^{m+J} < \infty;$$

i.e., iff

$$\sum_{p=0}^{\infty} \sum_{m+J=p} |f_0^{(2m+2J+1)}|((m + J)!)^{-1}(8X)^{m+J} < \infty;$$

i.e., iff

$$\sum_{p=0}^{\infty} (p + 1)(8X)^p(p!)^{-1}|f_0^{(2p+1)}| < \infty.$$

This can be shown to be convergent for all X and thus all $\hat{\mu}$, S by the Cauchy root test using the further known result that if $f \varepsilon \mathfrak{F}$, $\lim_{p\to\infty} |(p!)^{-1} f_0^{(2p+1)}|^{1/p} = 0$, [12]. Thus $A_{11} < \infty$ for all $\hat{\mu}$, S and $f \varepsilon \mathfrak{F}$. In an exactly similar way the other seven component series of A can be shown to be convergent for all $\hat{\mu}$, S and $f \varepsilon \mathfrak{F}$.

In considering B we note that unless k is even and n odd $\mu_k \cdot \mu_{n-k+1} \equiv 0$ so that

$$B \leqq \sum_{m=1}^{\infty} \sum_{k=1}^{m} \sum_{j=0}^{\infty} \sum_{i=0}^{j} |f_0^{(2k+i)} f_0^{(2n-2k+j-i)}|(i!2k!(2n - 2k)!(j - i)!)^{-1}$$
$$\cdot |E^{-1}(\mu^j \mu_{2k} \cdot \mu_{2n-2k})|.$$

It is not difficult to show that

$$|E^{-1}(\mu^{2J} \mu_{2k} \cdot \mu_{2n-2k})| \leqq 2k!(k!)^{-1}(2n - 2k)!((n - k)!)^{-1} 2J!(J!)^{-1} Y^n Z^J$$

and

$|E^{-1}(\mu^{2J+1} \mu_{2k} \cdot \mu_{2n-2k})|$
$$\leqq |\hat{\mu}| 2k!(k!)^{-1}(2n - 2k)!((n - k)!)^{-1}(2J + 1)!(J!)^{-1} Y^n Z^J.$$

We again split B into four parts according as i and j are odd or even. For example, for i and j even,

$$B_{11} = \sum_{n=1}^{\infty} \sum_{k=1}^{n} \sum_{J=0}^{\infty} \sum_{i=0}^{J} \binom{2J}{2i} |f_0^{(2k-2i)} f_0^{(2n-2k+2J-2i)}| Y^n Z^J (k!(n - k)!J!)^{-1}.$$

It can be shown that

$$\binom{2J}{2i}(k!(n - k)!J!)^{-1} \leqq 2^{2J}((i + k)!(J - i + n - k)!)^{-1} \binom{i+k}{i} \binom{J-i+n-k}{n-k}$$
$$\leqq 2^{2J} \cdot 2^{J+n}((i + k)!(J - i + n - k)!)^{-1}$$

so that

$$B_{11} \leqq \sum_{n=1}^{\infty} \sum_{k=1}^{n} \sum_{J=0}^{\infty} \sum_{i=0}^{J} |f_0^{(2k+2i)} f_0^{(2n-2k+2J-2i)}| X_1^{(n+J)}$$
$$\cdot((i + k)!(n - k + J - i)!)^{-1}$$
$$\leqq [\sum_{n=1}^{\infty} \sum_{J=0}^{\infty} |f_0^{(2n+2J)}| X_1^{n+J}((n + J)!)^{-1}]$$

$$\cdot\left[\sum_{n=1}^{\infty}\sum_{J=0}^{\infty}|f_0^{(2n+2J-2)}|X_1^{n+J-1}((n+J-1)!)^{-1}\right]$$

where $X_1 = \max(8Z, 2Y)$. B_{11} can be shown to be $< \infty$ by the same methods used in dealing with A_{11}.

In an exactly analogous way the other parts of B can be shown to be convergent for all $\hat{\mu}$, $S, f \varepsilon \mathfrak{F}$. Together they prove that B itself is convergent and added to A, it means that the series expansion for $\hat{\phi}^2$ is absolutely convergent for all $\hat{\mu}$, S, $f \varepsilon \mathfrak{F}$.

We are thus left to prove $E|\hat{\phi}^2| < \infty$ for all μ, σ^2 and $f \varepsilon \mathfrak{F}$. Now

$$E|\hat{\phi}^2| \leqq \sum_{n=1}^{\infty}\sum_{k=1}^{n}\sum_{j=0}^{\infty}\sum_{i=0}^{j}|f_0^{(k+i)}f_0^{(n-k+1+j-i)}|(k!(n-k+1)!(j-i)!i!)^{-1}$$
$$\cdot\{E|E^{-1}(\mu^j\mu_{n+1})| + E|E^{-1}(\mu^j\mu_k\mu_{n-k+1})|\}$$

and the proof follows that of showing $\hat{\phi}^2 < \infty$. This expression can be simplified by noting that $\mu_k = 0$ if k odd and $\mu_k \cdot \mu_{n-k+1} \neq 0$ only if k is even and n odd. Moreover by recalling that

$$E|E^{-1}[\mu^{2J}(\tfrac{1}{2}\sigma^2)^n]| \leqq (1 + \rho_\mu)2J!(J!)^{-1}(\tfrac{1}{2}\sigma^2)^n(\sigma^2\lambda^2 + \mu^2)^J,$$
$$E|E^{-1}[\mu^{2J+1}(\tfrac{1}{2}\sigma^2)^n]| \leqq (|\mu| + 2^{-\frac{1}{2}}\sigma\lambda)(2J+1)!(J!)^{-1}(\tfrac{1}{2}\sigma^2)^n(\sigma^2\lambda^2 + \mu^2)^J$$

and by letting $Z = \sigma^2\lambda^2 + \mu^2$, $Y = \frac{1}{2}\sigma^2$ it is clear that the resulting series are those considered in showing $\hat{\phi}^2 < \infty$. These have been shown to be absolutely convergent for all Z, Y and $f \varepsilon \mathfrak{F}$ and hence $E|\hat{\phi}^2| < \infty$ for all μ, σ^2 and $f \varepsilon \mathfrak{F}$.

6. Particular expressions for the variance. Since $\mu_{n+1} - \mu_{k+1}\cdot\mu_{n-k+1} = 0$ for n even and all k,

$$\phi^2 = \sum_{n=1}^{\infty}\sum_{k=1}^{2n-1} f_\mu^{(k)} f_\mu^{(2n-k)}(k!(2n-k)!)^{-1}[\mu_{2n} - \mu_k\cdot\mu_{2n-k}].$$

(i) *Square-root transformation.* Here $x = f(\xi) = \xi^2$ so that $f \varepsilon \mathfrak{F}$ and $\phi^2 = 4\mu^2\sigma^2 + 2\sigma^4$.

(ii) *Cube-root transformation.* In this case $x = f(\xi) = \xi^3$ so that $f \varepsilon \mathfrak{F}$ and $\phi^2 = 15\sigma^6 + 9\mu^4\sigma^2 + 36\sigma^4\mu^2$.

(iii) *Recursive class of transformation.* This class of transformations was introduced by Neyman and Scott [12] and is defined by the differential equation

$$f^{(2n)}(x) = AB^{n-1} + B^n f(x), \quad n = 1, 2, \cdots, \quad \text{so} \quad f^{(2n+1)}(x) = B^n f^{(1)}(x)$$

Note that for $B = 0$, this is the square root transformation which we have just considered. In what follows then, we will assume $B \neq 0$. Neyman and Scott show that several well known transformations are members of this class e.g. $\log x$ and $\sin^{-1}(x^{\frac{1}{2}})$. Now

$$\phi^2 = \sum_{n=2}^{\infty}\sum_{k=1}^{n-1} f_\mu^{(2k)} f_\mu^{(2n-2k)}(2k!(2n-2k)!)^{-1}(\mu_{2n} - \mu_{2n-2k}\cdot\mu_k)$$
$$+ \sum_{n=1}^{\infty}\sum_{k=1}^{n} f_\mu^{(2k-1)} f_\mu^{(2n-2k+1)}((2k-1)!(2n-2k+1)!)^{-1}$$
$$\cdot(\mu_{2n} - \mu_{2n-2k+1}\cdot\mu_{2k-1})$$
$$= \sum_{n=2}^{\infty}(\tfrac{1}{2}\sigma^2)^n\sum_{k=1}^{n-1} f_\mu^{(2k)} f_\mu^{(2n-2k)}(2k!(2n-2k)!)^{-1}$$

$$\cdot[(2n)!(n!)^{-1} - (2n-2k)!(2k)!((n-k)!k!)^{-1}]$$
$$+ \sum_{n=1}^{\infty} (\tfrac{1}{2}\sigma^2)^n \sum_{k=1}^{n} f_\mu^{(2k-1)} f_\mu^{(2n-2k+1)}((2k-1)!(2n-2k+1)!)^{-1}$$
$$\cdot 2n!(n!)^{-1}$$
$$\equiv \sum_{n=2}^{\infty} (\tfrac{1}{2}\sigma^2)^n (U_n - V_n) + \sum_{n=1}^{\infty} (\tfrac{1}{2}\sigma^2)^n W_n.$$

But for a recursive transformation, for $n = 1, 2, \cdots$ and $k = 1, 2, \cdots n$,

$$f_\mu^{(2k)} f_\mu^{(2n-2k)} = A^2 B^{n-2} + 2AB^{n-1} f_\mu + B^n f_\mu^2$$

and

$$f_\mu^{(2k-1)} f_\mu^{(2n-2k+1)} = B^{n-1} f_\mu^{(1)2}$$

so that

$$U_n = (A^2 B^{n-2} + 2AB^{n-1} f_\mu + B^n f_\mu^2)(n!)^{-1} \sum_{k=1}^{n-1} \binom{2n}{2k}$$
$$= B^n (AB^{-1} + f_\mu)^2 (2^{2n-1} - 2)(n!)^{-1},$$
$$V_n = B^n (AB^{-1} + f_\mu)^2 (2^n - 2)(n!)^{-1} \quad \text{and}$$
$$W_n = B^{n-1} 2^{2n-1} (n!)^{-1} f_\mu^{(1)2}.$$

Thus

$$\phi^2 = (e^{2\sigma^2 B} - 1)\tfrac{1}{2}[(AB^{-1} + f_\mu)^2 + (f_\mu^{(1)})^2/B] - (e^{\sigma^2 B} - 1)(AB^{-1} + f_\mu)^2.$$

For example

(a) $f(\xi) = \sin^2 \xi \cdot (A = 2, B = -4)$.
Thus $\phi^2 = \frac{1}{8}(e^{-8\sigma^2} - 1) \cos 4\mu - \frac{1}{4}(e^{-4\sigma^2} - 1) \cos^2 2\mu$

(b) $f(\xi) = e^{m\xi}$. Here $A = 0$, $B = m^2$, and thus

$$\phi^2 = e^{2m\mu + m^2\sigma^2}(e^{m^2\sigma^2} - 1).$$

(c) $f(\xi) = \sinh^2 \xi$. Here $A = 2$, $B = 4$, and

$$\phi^2 = \tfrac{1}{8}(e^{8\sigma^2} - 1) \cosh 4\mu - \tfrac{1}{4}(e^{4\sigma^2} - 1) \cosh^2 2\mu.$$

7. Estimation of the variance.

(i) Square-root transformation:

$$\hat{\phi}^2 = E^{-1}(4\mu^2\sigma^2 + 2\sigma^4)$$
$$= 4\hat{\sigma}^2\hat{\mu}^2 + 2\nu(\nu + 2)^{-1}(1 - 2\lambda^2)\hat{\sigma}^4$$

where $\hat{\sigma}^2 = S\nu^{-1}$.

(ii) Cube-root transformation:

$$\hat{\phi}^2 = E^{-1}(15\sigma^6 + 9\mu^4\sigma^2 + 36\sigma^4\mu^2)$$
$$= 9\hat{\sigma}^2\hat{\mu}^4$$
$$+ 18\nu(\nu + 2)^{-1}(2 - 3\lambda^2)\hat{\mu}^2\hat{\sigma}^4 + 3\nu^2(9\lambda^4 - 12\lambda^2 + 5)[(\nu + 2)(\nu + 4)]^{-1}\hat{\sigma}^6.$$

(iii) Recursive transformations: Now ϕ^2 is a linear function of $e^{a\sigma^2/2}$, $f_\mu e^{b\sigma^2/2}$, f_μ, $f_\mu^2 e^{c\sigma^2/2}$, f_μ^2, $f_\mu^{(1)2}$ and $f_\mu^{(1)2} e^{d\sigma^2/2}$ for suitably chosen a, b, c, d so that we need to find unbiased estimators for each of these to obtain $\hat{\phi}^2$. Again it is only necessary to consider $B \neq 0$.

(1) $E^{-1}(e^{a\sigma^2/2})$: Neyman and Scott show that

$$E^{-1}(e^{a\sigma^2/2}) = \Phi(aS, \nu) = \sum_{n=0}^{\infty} (n!)^{-1}\Gamma(\tfrac{1}{2}\nu)[\Gamma(\tfrac{1}{2}\nu + n)]^{-1}(\tfrac{1}{4}aS)^n.$$

This we will write as $\Phi(a)$ for there will be no confusion in what follows.

(2) $E^{-1}(f_\mu^{b\sigma^2/2})$. Consider

$$f(\hat{\mu})\Phi(b - B\lambda^2) - AB^{-1}[\Phi(b) - \Phi(b - B\lambda^2)].$$

This has an expected value, using equation (48) of Neyman and Scott's paper, of

$$[f_\mu e^{B\lambda^2\sigma^2/2} + AB^{-1}e^{B\lambda^2\sigma^2/2} - AB^{-1}]e^{(b-B\lambda^2)\sigma^2/2} - AB^{-1}[e^{b\sigma^2/2} - e^{(b-B\lambda^2)\sigma^2/2}] = f_\mu e^{b\sigma^2/2}.$$

Thus

$$E^{-1}(f_\mu e^{b\sigma^2/2}) = f_{\hat{\mu}}\Phi(b - B\lambda^2) - AB^{-1}[\Phi(b) - \Phi(b - B\lambda^2)] \quad \text{for all} \quad b.$$

(3) $E^{-1}(f_\mu^2)$: Consider expanding $g(\mu) = f_\mu^2$ as a Taylor series around the origin—this is clearly permissible since $f \varepsilon \mathfrak{F}$. Then

$$g(\mu) = g(0) + \sum_{n=0}^{\infty} \mu^{2n+1}((2n + 1)!)^{-1}g^{(2n+1)}(0) + \sum_{n=1}^{\infty} \mu^{2n}((2n)!)^{-1}g^{(2n)}(0)$$

and by Leibniz's theorem

$$g^{(2n)}(x) = 2f_x f_x^{(2n)} + \sum_{k=0}^{n-1} \binom{2n}{2k+1} f_x^{(2n-2k-1)} f_x^{(2k+1)} + \sum_{k=1}^{n-1} \binom{2n}{2k} f_x^{(2k)} f_x^{(2n-2k)}.$$

By using the fact that we are considering a recursive transformation it is easily found that

$$g^{(2n)}(0) = 2^{2n-1}[B^{n-1}f_0^{(1)2} + A^2B^{n-2} + B^n f_0^2 + 2AB^{n-1}f_0] - 2A^2B^{n-2} - 2AB^{n-1}f_0$$

and

$$g^{(2n+1)}(0) = 2^{2n+1}[AB^{n-1}f_0^{(1)} + f_0 f_0^{(1)}B^n] - 2AB^{n-1}f_0^{(1)}.$$

If we assume $B > 0$, then by substituting these results and evaluating the summations over n, it is readily found that,

$$f_\mu^2 = f_0^2 - M + N + \tfrac{1}{2}KB^{-\frac{1}{2}}(e^{2B^{\frac{1}{2}}\mu} - e^{-2B^{\frac{1}{2}}\mu}) - \tfrac{1}{2}LB^{-\frac{1}{2}}(e^{B^{\frac{1}{2}}\mu} - e^{-B^{\frac{1}{2}}\mu})$$
$$+ \tfrac{1}{2}M(e^{2B^{\frac{1}{2}}\mu} + e^{-2B^{\frac{1}{2}}\mu}) - \tfrac{1}{2}N(e^{B^{\frac{1}{2}}\mu} + e^{-B^{\frac{1}{2}}\mu})$$

where

$$K = AB^{-1}f_0^{(1)} + f_0 f_0^{(1)}, \quad L = 2AB^{-1}f_0^{(1)}$$

$$M = \tfrac{1}{2}B^{-2}(Bf_0^{(1)2} + A^2 + B^2f_0^2 + 2ABf_0), \qquad N = 2A^2B^{-2} + AB^{-1}f_0.$$

So recalling that, [1], [6],

$$E[e^{\alpha\hat{\mu}}\Phi(\beta)] = \exp\{\alpha\mu + (\alpha^2\lambda^2 + \beta)\sigma^2/2\}$$

it is easy to see that

$$E^{-1}[f_\mu^2] = f_0^2 - M + N + \tfrac{1}{2}KB^{-\frac{1}{2}}(e^{2B^{\frac{1}{2}}\hat{\mu}} - e^{-2B^{\frac{1}{2}}\hat{\mu}})\Phi(-4B\lambda^2)$$

$$+ \tfrac{1}{2}M(e^{2B^{\frac{1}{2}}\hat{\mu}} + e^{-2B^{\frac{1}{2}}\hat{\mu}})\Phi(-4B\lambda^2) - \tfrac{1}{2}LB^{-\frac{1}{2}}(e^{B^{\frac{1}{2}}\hat{\mu}} - e^{-B^{\frac{1}{2}}\hat{\mu}})\Phi(-B\lambda^2)$$

$$- \tfrac{1}{2}N(e^{B^{\frac{1}{2}}\hat{\mu}} + e^{-B^{\frac{1}{2}}\hat{\mu}})\Phi(-B\lambda^2)$$

$$= f_{\hat{\mu}}^2 + [KB^{-\frac{1}{2}} \sinh (2B^{\frac{1}{2}}\hat{\mu}) + M \cosh (2B^{\frac{1}{2}}\hat{\mu})][\Phi(-4B\lambda^2) - 1]$$

$$- [LB^{-\frac{1}{2}} \sinh (B^{\frac{1}{2}}\hat{\mu}) + N \cosh (B^{\frac{1}{2}}\hat{\mu})][\Phi(-B\lambda^2) - 1].$$

If $B < 0$ it is clear what should happen—replace $B^{-\frac{1}{2}} \sinh (B^{\frac{1}{2}}\hat{\mu})$ by $(|B|)^{-\frac{1}{2}}$ sin $(+|B|^{\frac{1}{2}}\hat{\mu})$ and $\cosh (B^{\frac{1}{2}}\hat{\mu})$ by $\cos (+|B|^{\frac{1}{2}}\hat{\mu})$. This is evident from the definition of the hyperbolic and trigonometric functions in terms of the exponential function.

(4) $E^{-1}[f_\mu^2 e^{c\sigma^2/2}]$: It follows almost immediately that

$$E^{-1}[f_\mu^2 e^{c\sigma^2/2}]$$

$$= [f_0^2 - M + N]\Phi(c) + [KB^{-\frac{1}{2}} \sinh (2B^{\frac{1}{2}}\hat{\mu}) + M \cosh (2B^{\frac{1}{2}}\hat{\mu})]\Phi(c - 4B\lambda^2)$$

$$+ [LB^{-\frac{1}{2}} \sinh (B^{\frac{1}{2}}\hat{\mu}) + N \cosh (B^{\frac{1}{2}}\hat{\mu})]\Phi(c - B\lambda^2)$$

$$= f_{\hat{\mu}}^2\Phi(c) + [KB^{-\frac{1}{2}} \sinh (2B^{\frac{1}{2}}\hat{\mu}) + M \cosh (2B^{\frac{1}{2}}\hat{\mu})][\Phi(c - 4B\lambda^2) - \Phi(c)]$$

$$- [LB^{-\frac{1}{2}} \sinh (B^{\frac{1}{2}}\hat{\mu}) + N \cosh (B^{\frac{1}{2}}\hat{\mu})][\Phi(c - B\lambda^2) - \Phi(c)].$$

(5) $E^{-1}[f_\mu^{(1)^2}]$: By expanding $h(\mu) = f_\mu^{(1)^2}$ as a power series in μ and by using the properties of the derivatives of f it can be readily shown, in a similar manner to finding f_μ^2, that

$$h(\mu) = f_0^{(1)^2} + \tfrac{1}{2}KBB^{-\frac{1}{2}}(e^{2B^{\frac{1}{2}}\mu} - e^{-2B^{\frac{1}{2}}\mu}) + \tfrac{1}{2}MB(e^{2B^{\frac{1}{2}}\mu} + e^{-2B^{\frac{1}{2}}\mu} - 2)$$

so that

$$E^{-1}[f_\mu^{(1)^2}] = f_{\hat{\mu}}^{(1)^2} + [KBB^{-\frac{1}{2}} \sinh (2B^{\frac{1}{2}}\hat{\mu}) + \tfrac{1}{2}MB \cosh (2B^{\frac{1}{2}}\hat{\mu})][\Phi(-4B\lambda^2) - 1].$$

(6) $E^{-1}[f_\mu^{(1)^2} e^{d\sigma^2/2})$: It follows immediately that

$$E^{-1}[f_\mu^{(1)^2} e^{d\sigma^2/2}] = f_{\hat{\mu}}^{(1)^2}\Phi(d) + [KBB^{-\frac{1}{2}} \sinh (2B^{\frac{1}{2}}\hat{\mu}) + MB \cosh (2B^{\frac{1}{2}}\hat{\mu})]$$

$$\cdot[\Phi(d - 4B\lambda^2) - \Phi(d)].$$

Thus on substituting these expressions for suitably chosen a, b, c, d in the expression for $E^{-1}(\phi^2) = \hat{\phi}^2$ one obtains, after some simplification,

$$\hat{\phi}^2 = \tfrac{1}{2}[f_{\hat{\mu}}^2 - A^2B^{-2}][\Phi(4B) - 2\Phi(2B) + 1] + \tfrac{1}{2}B^{-1}f_{\hat{\mu}}^{(1)2}[\Phi(4B) - 1]$$

$$+ AB^{-1}[f_{\hat{\mu}} + AB^{-1}][\Phi(4B - 4B\lambda^2) + \Phi(-B\lambda^2) - 2\Phi(B - 2B\lambda^2)]$$

$$- \tfrac{1}{2}\psi(LB^{-\frac{1}{2}}, N, B^{\frac{1}{2}}\hat{\mu})[\Phi(4B - B\lambda^2) - \Phi(4B) - \Phi(-B\lambda^2) + 2\Phi(2B)$$

$$- 2\Phi(2B - B\lambda^2) - 1] + \psi(KB^{-\frac{1}{2}}, M, 2B^{\frac{1}{2}}\hat{\mu})$$

$$\cdot[\Phi(4B - 4B\lambda^2) - \Phi(4B) + \Phi(2B) - \Phi(2B - 4B\lambda^2)]$$

where

$$\psi(P\alpha^{-\frac{1}{2}}, Q, \alpha^{\frac{1}{2}}x) = P|\alpha|^{-\frac{1}{2}} \sinh (|\alpha|^{\frac{1}{2}}x) + Q \cosh (|\alpha|^{\frac{1}{2}}x), \qquad \alpha > 0,$$
$$= P, \qquad \alpha = 0,$$
$$= P|\alpha|^{-\frac{1}{2}} \sin (|\alpha|^{\frac{1}{2}}x) + Q \cos (|\alpha|^{\frac{1}{2}}x), \qquad \alpha < 0.$$

For example

(a) $\xi = \log_{10} x$. Here $A = 0$, $B = m^2$, $m = \log_e 10$, $k = 1$, $L = 0$, $M = 1$, $N = 0$. Thus $\psi(LB^{-\frac{1}{2}}, N, B^{\frac{1}{2}}\hat{\mu}) = 0$, and $\psi(KB^{-\frac{1}{2}}, M, 2B^{\frac{1}{2}}\hat{\mu}) = \sinh (2m\hat{\mu}) + \cosh (2m\hat{\mu}) = e^{2m\hat{\mu}}$ whence

$$\hat{\phi}^2 = e^{2m\hat{\mu}}[\Phi(4m^2 - 4m^2\lambda^2) - \Phi(2m^2 - 4m^2\lambda^2)]$$

which agrees with Finney [6], Sichel [14], and Meulenberg [11].

(b) $\xi = \sin^{-1} x^{\frac{1}{2}}$. Here $A = 2$, $B = -4$, $K = L = 0$, $M = \frac{1}{8}$, $N = \frac{1}{2}$ so that

$$\psi(LB^{-\frac{1}{2}}, N, B^{\frac{1}{2}}\hat{\mu}) = \tfrac{1}{2} \cos 2\hat{\mu} \quad \text{and} \quad \psi(KB^{-\frac{1}{2}}, M, 2B^{\frac{1}{2}}\hat{\mu}) = \tfrac{1}{8} \cos 4\hat{\mu}.$$

Thus after some simplification

$$\hat{\phi}^2 = \tfrac{1}{8}[1 - \Phi(-8)] + \tfrac{1}{8} \cos 4\hat{\mu}[\Phi(-16 + 16\lambda^2) - \Phi(-8 + 16\lambda^2)]$$
$$+ \tfrac{1}{4} \cos 2\hat{\mu}[2\Phi(4\lambda^2) - 2\Phi(-4 + 8\lambda^2) + 2\Phi(-8 + 4\lambda^2) - \Phi(-16 + 4\lambda^2)$$
$$+ \Phi(-16 + 16\lambda^2)].$$

(c) $\xi = \sinh^{-1} x^{\frac{1}{2}}$. Here $A = 2$, $B = 4$, $K = L = 0$, $M = \frac{1}{8}$, $N = \frac{1}{2}$, so that

$$\psi(KB^{-\frac{1}{2}}, M, 2B^{\frac{1}{2}}\hat{\mu}) = \tfrac{1}{8} \cosh 4\hat{\mu},$$
$$\psi(LB^{-\frac{1}{2}}, N, B^{\frac{1}{2}}\hat{\mu}) = \tfrac{1}{2} \cosh 2\hat{\mu}.$$

Thus, after some simplification,

$$\hat{\phi}^2 = \tfrac{1}{8}[1 - \Phi(8)] + \tfrac{1}{8} \cosh (4\hat{\mu}[\Phi(16 - 16\lambda^2) - \Phi(8 - 16\lambda^2)]$$
$$+ \tfrac{1}{4} \cosh (2\hat{\mu})[2\Phi(-4\lambda^2) - 2\Phi(4 - 8\lambda^2) + 2\Phi(8 - 4\lambda^2) + \Phi(16 - 16\lambda^2)$$
$$- \Phi(16 - 4\lambda^2)].$$

8. Estimating var $(\hat{\theta})$**.** Neyman and Scott have obtained the MVUE of θ but, it is clearly desirable to have some idea of the variability of $\hat{\theta}$. One could presumably set an approximate 95 % confidence interval for θ as $\hat{\theta} \pm 2[(E^{-1}\{\text{var}(\hat{\theta})\})^{\frac{1}{2}}]$ but the whole problem requires further investigation. An alternative procedure is to compute in the usual manner a confidence interval of μ, say $P\{\hat{\mu}_L \leqq \mu \leqq \hat{\mu}_U\} = \alpha$ and then to use the formula for $\hat{\theta}$ in terms of $\hat{\mu}$, i.e. $\hat{\theta} = \hat{\theta}(\hat{\mu}, S)$, to provide approximate α upper and lower limits, that is to say use

$$\hat{\theta}_U = \hat{\theta}(\hat{\mu}_U, S) \quad \text{and} \quad \hat{\theta}_L = \hat{\theta}(\hat{\mu}_L, S)$$

as upper and lower points of an α confidence interval. The author has no idea of the properties of either of these two procedures but hopes to investigate them in the future. One would not expect them to give substantially different results.

Clearly since var $(\hat{\theta}) = E(\hat{\theta}^2) - \theta^2$ we have only to find $E^{-1}(\theta^2)$ to obtain $E^{-1}(\text{var}(\hat{\theta})) = \hat{\theta}^2 - E^{-1}(\theta^2)$.

(i) Square-root transformation: Since

$$E^{-1}(\theta^2) = \hat{\mu}^4 + (2 - 6\lambda^2)\hat{\sigma}^2\hat{\mu}^2 + \nu\hat{\sigma}^4(\nu + 2)^{-1}(1 - 2\lambda^2 + 3\lambda^4)$$

it is easily seen that

$$E^{-1}[\text{Var}(\hat{\theta})] = 4\lambda^2\hat{\sigma}^2\hat{\mu}^2 + \hat{\sigma}^4\{(1 - \lambda^2)^2 - \nu(\nu + 2)^{-1}(1 - 2\lambda^2 + 3\lambda^4)\}.$$

(ii) Cube-root transformation: By elementary calculations

$$E^{-1}[\text{var}(\hat{\theta})] = 9\lambda^2\hat{\mu}^4\hat{\sigma}^2 + 9[(1 - \lambda^2)^2 - \nu(\nu + 2)^{-1}(1 - 4\lambda^2 + 5\lambda^4)]\hat{\mu}^2\hat{\sigma}^4 - 3\lambda^2\nu^2\hat{\sigma}^6((\nu + 2)(\nu + 4))^{-1}(3 - 6\lambda^2 + 5\lambda^4).$$

(iii) Recursive class of transformations: Now $\theta = f(\mu)e^{B\sigma^2/2} - AB^{-1} + AB^{-1}e^{B\sigma^2/2}$ so that

$$E^{-1}(\theta^2) = E^{-1}[f_\mu^{\,2}e^{2B\sigma^2/2} + A^2B^{-2} + A^2B^{-2}e^{2B\sigma^2/2} - 2AB^{-1}f_\mu e^{B\sigma^2/2} - 2A^2B^{-2}e^{B\sigma^2/2} + 2AB^{-1}f_\mu e^{2B\sigma^2/2}]$$

and the right hand side is known. After some simplification,

$$\begin{aligned} E^{-1}[\text{Var}(\hat{\theta})] &= \hat{\theta}^2 - A^2B^{-2}[1 - \Phi(2B)] - 2AB^{-1}[f_{\hat{\mu}} + AB^{-1}][\Phi(2B - B\lambda^2) - \Phi(B - B\lambda^2)] \\ &\quad - \psi(KB^{-\frac{1}{2}}, M, 2B^{\frac{1}{2}}\hat{\mu})[\Phi(2B - 4B\lambda^2) - \Phi(2B)] \\ &\quad + \psi(LB^{-\frac{1}{2}}, N, B^{\frac{1}{2}}\hat{\mu})[\Phi(2B - B\lambda^2) - \Phi(2B)] \\ &\quad - f_{\hat{\mu}}^{\,2}\Phi(2B). \end{aligned}$$

For example

(a) $\xi = \log_{10} x$. It is found that

$$\begin{aligned} E^{-1}[\text{var}(\hat{\theta})] &= \hat{\theta}^2 - e^{2m\hat{\mu}}[\Phi(2m^2 - 4\lambda^2m^2) - \Phi(2m^2)] - e^{2m\hat{\mu}}\cdot\Phi(2m^2) \\ &= \hat{\theta}^2 - e^{2m\hat{\mu}}\Phi(2m^2 - 4\lambda^2m^2) \\ &= e^{2m\hat{\mu}}[\Phi^2(m^2 - m^2\lambda^2) - \Phi(2m^2 - 4\lambda^2m^2)] \end{aligned}$$

where $\hat{\theta} = \Phi(m^2 - m^2\lambda^2)e^{m\hat{\mu}}$.

(b) $\xi = \sin^{-1} x^{\frac{1}{2}}$. For this transformation

$$\hat{\theta} = \tfrac{1}{2} + (\sin^2\hat{\mu} - \tfrac{1}{2})\Phi(-4 + 4\lambda^2)$$

and it turns out that, after some simplification,

$$E^{-1}[\text{var}(\hat{\theta})] = \hat{\theta}^2 - \tfrac{1}{4} - \tfrac{1}{8}\Phi(-8) + \tfrac{1}{2}\cos 2\hat{\mu}\cdot\Phi(-4 + 4\lambda^2) - \tfrac{1}{8}\cos 4\hat{\mu}\cdot\Phi(-8 + 16\lambda^2).$$

(c) $\xi = \sinh^{-1} x^{\frac{1}{2}}$. For this transformation

$$\hat{\theta} = \Phi(4 - 4\lambda^2)[\sinh^2\hat{\mu} + \tfrac{1}{2}] - \tfrac{1}{2}$$

and it turns out, that, after some simplification,

$$E^{-1}[\text{var}(\hat{\theta})] = \hat{\theta}^2 - \tfrac{1}{4} - \tfrac{1}{8}\Phi(8) + \tfrac{1}{2}\cosh 2\hat{\mu}\cdot\Phi(4 - 4\lambda^2) - \tfrac{1}{8}\Phi(8 - 16\lambda^2)\cosh 4\hat{\mu}.$$

9. Estimating the variance of the difference between two treatment effects. In an analysis of variance situation one is often interested in the difference between two treatments effects and in our case it is clear that we are concerned with the original (or natural) units and not the transformed data. It is obvious that we can estimate this difference as $\hat{\theta}_1 - \hat{\theta}_2$ and in this section I give expressions for $\hat{V}(\hat{\theta}_1 - \hat{\theta}_2)$. To do this we suppose that $(\hat{\mu}_1, \hat{\mu}_2)$ has a bivariate normal distribution with mean (μ_1, μ_2) and variance-covariance matrix

$$\sigma^2\begin{bmatrix} \lambda_1^2 & \lambda_{12} \\ \lambda_{12} & \lambda_2^2 \end{bmatrix}.$$

(i) Square-root transformation: Using the known $\hat{\theta}_i$, the results of Cook [5], for the cumulants of the bivariate normal, and the known formula for Var $(\hat{\theta}_i)$ it is easily shown that

$$\begin{aligned}\text{Var}(\hat{\theta}_1 - \hat{\theta}_2) &= \sigma^4[(\lambda_1^2 + \lambda_2^2)^2 + (\nu + 2)\nu^{-1}(\lambda_1^2 - \lambda_2^2)^2 - 4\lambda_{12}^2] \\ &\quad + 4\sigma^2[\lambda_1^2\mu_1^2 + \lambda_2^2\mu_2^2 - 2\lambda_{12}\mu_1\mu_2]\end{aligned}$$

and

$$\begin{aligned}\hat{V}(\hat{\theta}_1 - \hat{\theta}_2) &= 4\hat{\sigma}^2(\lambda_1^2\hat{\mu}_1^2 + \lambda_2^2\,\hat{\mu}_2^2 - 2\lambda_{12}\hat{\mu}_1\hat{\mu}_2) + 4\nu\hat{\sigma}^4(\nu + 2)^{-1}(\lambda_{12}^2 - \lambda_1^2\lambda_2^2) \\ &\quad - 2((\nu - 1)(\nu + 2)^{-1})\hat{\sigma}^4(\lambda_1^2 - \lambda_2^2)^2\end{aligned}$$

or alternatively,

$$\begin{aligned}\hat{V}(\hat{\theta}_1 - \hat{\theta}_2) &= (\hat{\theta}_1 - \hat{\theta}_2)^2 - (\hat{\mu}_1^2 - \hat{\mu}_2^2)^2 \\ &\quad + 2\hat{\sigma}^2[\lambda_1^2(3\hat{\mu}_1^2 - \hat{\mu}_2^2) + \lambda_2^2(3\hat{\mu}_2^2 - \hat{\mu}_1^2) - 4\lambda_{12}\hat{\mu}_1\hat{\mu}_2] \\ &\quad + \nu(\nu + 2)^{-1}\hat{\sigma}^4[-3(\lambda_1^2 - \lambda_2^2)^2 + 4(\lambda_{12}^2 - \lambda_1^2\lambda_2^2)].\end{aligned}$$

(ii) Cube-root transformation: In a like manner, it is easily found that

$$\begin{aligned}\text{cov}(\hat{\theta}_1, \hat{\theta}_2) &= 9\sigma^2(1 - \lambda_1^2)(1 - \lambda_2^2)(\nu + 2)\nu^{-1}(\mu_1\mu_2 + \lambda_{12}\sigma^2) \\ &\quad + 9\lambda_{12}\sigma^2[\mu_1^2\mu_2^2 + \sigma^2\mu_1^2 + \sigma^2\mu_2^2] \\ &\quad + 9\sigma^4\mu_1\mu_2[2\lambda_{12}^2 + \lambda_1^2 + \lambda_2^2 - \lambda_1^2\lambda_2^2] \\ &\quad + 3\sigma^6\lambda_{12}[2\lambda_{12}^2 + 3\lambda_1^2 + 3\lambda_2^2 - 3\lambda_1^2\lambda_2^2]\end{aligned}$$

and that an unbiased estimator for this is,

$$\begin{aligned}\hat{C}(\hat{\theta}_1, \hat{\theta}_2) &= 9\hat{\sigma}^4\hat{\mu}_1\hat{\mu}_2[(1 - \lambda_1^2)(1 - \lambda_2^2) - \nu(\nu + 2)^{-1}(1 + 2\lambda_{12}^2 - \lambda_1^2 \\ &\quad - \lambda_2^2 + \lambda_1^2\lambda_2^2)] \\ &\quad + 9\nu(\nu + 2)^{-1}\hat{\sigma}^4\lambda_{12}[\hat{\mu}_1^2(1 - \lambda_2^2) + \hat{\mu}_2(1 - \lambda_1^2)] + 9\lambda_{12}\hat{\mu}_1\hat{\mu}_2\hat{\sigma}^2 \\ &\quad - \nu^2\hat{\sigma}^6[(\nu + 2)(\nu + 4)]^{-1}[6\lambda_{12}^3 + 9\lambda_1^2\lambda_2^2\lambda_{12} + 9\lambda_{12} - 9\lambda_2^2\lambda_{12} - 9\lambda_{12}\lambda_1^2].\end{aligned}$$

There appears to be no useful simplification of the expression

$$V(\hat{\theta}_1 - \hat{\theta}_2) = V(\hat{\theta}_1) + V(\hat{\theta}_2) - 2\,\mathrm{cov}\,(\hat{\theta}_1, \hat{\theta}_2)$$

and

$$\begin{aligned}\hat{V}(\hat{\theta}_1 - \hat{\theta}_2) = {} & (\hat{\theta}_1 - \hat{\theta}_2)^2 - (\hat{\mu}_1^{\,3} - \hat{\mu}_2^{\,3})^2 \\ & + \hat{\sigma}^6\nu^2((\nu + 2)(\nu + 4))^{-1}\{9(\lambda_1^{\,2} - \lambda_2^{\,2})^2 - 30(\lambda_1^{\,6} + \lambda_2^{\,6}) \\ & + 18\lambda_{12}(\lambda_1^{\,2} + \lambda_2^{\,2} + \lambda_1^{\,2}\lambda_2^{\,2} - 6\lambda_{12}(3 + 6\lambda_{12} - 4\lambda_{12}^2))\} \\ & - \hat{\sigma}^4\nu(\nu + 2)^{-1}[\{18\lambda_{12}(1 - \lambda_2^{\,2}) + 9(1 - 5\lambda_1^{\,2})(1 + \lambda_1^{\,2})\}\hat{\mu}_1^{\,2} \\ & + \{18\lambda_{12}(1 - \lambda_1^{\,2}) + 9(1 - 5\lambda_2^{\,2})(1 + \lambda_2^{\,2})\}\hat{\mu}_2^{\,2} \\ & - \{18(1 - \lambda_1^{\,2} - \lambda_2^{\,2} + 2\lambda_{12})\}\hat{\mu}_1\hat{\mu}_2] \\ & - \hat{\sigma}^2[18\lambda_{12}\hat{\mu}_1^{\,2}\hat{\mu}_2^{\,2} + (6 - 15\lambda_1^{\,2})\hat{\mu}_1^{\,4} + (6 - 15\lambda_2^{\,2})\hat{\mu}_2^{\,4} \\ & - 6(1 - \lambda_2^{\,2})\hat{\mu}_2\hat{\mu}_1^{\,3} - 6\hat{\mu}_1\hat{\mu}_2^{\,3}(1 - \lambda_1^{\,2})].\end{aligned}$$

(iii) Recursive transformations: Since we already have expressions for $E^{-1}(\theta^2)$ $(i = 1, 2)$ we concentrate on finding an estimate (again treating $B \neq 0$), for

$$\begin{aligned}\theta_1\theta_2 = {} & A^2B^{-2} - 2A^2B^{-2}e^{B\sigma^2/2} + A^2B^{-2}e^{B\sigma^2} + f(\mu_1)f(\mu_2)e^{B\sigma^2} \\ & - AB^{-1}e^{B\sigma^2/2}[f(\mu_1) + f(\mu_2)] + AB^{-1}e^{B\sigma^2}[f(\mu_1) + f(\mu_2)].\end{aligned}$$

The only expression for which we do not already have an estimate is $f(\mu_1)f(\mu_2)e^{B\sigma^2}$. To find this we will need to remember, since

$$E(e^{ax+by}) = \exp\{a\mu_x + b\mu_y + \tfrac{1}{2}a^2\sigma_x^{\,2} + ab\rho\sigma_x\sigma_y + \tfrac{1}{2}b^2\sigma_y^{\,2}\}$$

where x and y have the obvious bivariate normal distribution, that

$$E^{-1}(e^{a\mu_1+b\mu_2}e^{c\sigma^2/2}) = e^{a\hat{\mu}_1+b\hat{\mu}_2}\Phi(c - a^2\lambda_1^{\,2} - 2ab\lambda_{12} - b^2\lambda_2^{\,2}).$$

The method is to write $g(x, y) = f(x)f(y)$ as a bivariate power series in x and y,

$$g(x, y) = g(0, 0) + \textstyle\sum_{n=1}^{\infty} g_n(0, 0)/n!$$

where

$$g_n(0, 0) = (x(\partial/\partial x) + y(\partial/\partial y))^n g(x, y)|_{x=0,y=0}\,.$$

It turns out, using the specific knowledge of the derivatives of $f(x)$ that,

$$\begin{aligned}g_{2n}(0, 0) = {} & -(x^{2n} + y^{2n})B^n\tfrac{1}{2}N \\ & + (x + y)^{2n}B^n M. + (x - y)^{2n}B^n H. \quad (n = 1, 2, \cdots)\end{aligned}$$

where N and M are as before and $H = \frac{1}{2}B^{-2}[-Bf_0^{(1)2} + A^2 + 2ABf_0 + B^2f_0^{\,2}]$. Likewise

$$g_{2n+1}(0, 0) = KB^{-\frac{1}{2}}(x + y)^{2n+1}(B^{\frac{1}{2}})^{2n+1} - (x^{2n+1} + y^{2n+1})(B^{\frac{1}{2}})^{2n+1}\tfrac{1}{2}LB^{-\frac{1}{2}}.$$

It is easily verified that by summing over n, and simplifying that,

$$\begin{aligned} g(\mu_1, \mu_2) &= f(\mu_1)f(\mu_2) \\ &= (f_0^2 - M + N - H) + \tfrac{1}{2}KB^{-\frac{1}{2}}\{e^{(\mu_1+\mu_2)B^{\frac{1}{2}}} - e^{-(\mu_1+\mu_2)B^{\frac{1}{2}}}\} \\ &\quad + \tfrac{1}{2}M\{e^{(\mu_1+\mu_2)B^{\frac{1}{2}}} + e^{-(\mu_1+\mu_2)B^{\frac{1}{2}}}\} \\ &\quad + \tfrac{1}{2}H\{e^{(\mu_1-\mu_2)B^{\frac{1}{2}}} + e^{-(\mu_1-\mu_2)B^{\frac{1}{2}}}\} \\ &\quad - \tfrac{1}{4}LB^{-\frac{1}{2}}\{e^{\mu_1B^{\frac{1}{2}}} - e^{-\mu_1B^{\frac{1}{2}}} + e^{\mu_2B^{\frac{1}{2}}} - e^{-\mu_2B^{\frac{1}{2}}}\} \\ &\quad - \tfrac{1}{4}N\{e^{\mu_1B^{\frac{1}{2}}} + e^{-\mu_1B^{\frac{1}{2}}} + e^{\mu_2B^{\frac{1}{2}}} + e^{-\mu_2B^{\frac{1}{2}}}\}. \end{aligned}$$

It follows that

$$\begin{aligned} E^{-1}[f(\mu_1)f(\mu_2)e^{2B\sigma^2/2}] &= f(\hat{\mu}_1)f(\hat{\mu}_2)\Phi(2B) \\ &+ \psi(KB^{-\frac{1}{2}}, M, \{\hat{\mu}_1 + \hat{\mu}_2\}B^{\frac{1}{2}})[\Phi(2B - B\lambda_1^2 - 2B\lambda_{12} - B\lambda_2^2) - \Phi(2B)] \\ &+ H \cosh\{B^{\frac{1}{2}}(\hat{\mu}_1 - \hat{\mu}_2)\}[\Phi(2B - B\lambda_1^2 + 2\lambda_{12}B - B\lambda_2^2) - \Phi(2B)] \\ &- \tfrac{1}{2}\psi(LB^{-\frac{1}{2}}, N, B^{\frac{1}{2}}\hat{\mu}_1)[\Phi(2B - B\lambda_1^2) - \Phi(2B)] \\ &- \tfrac{1}{2}\psi(LB^{-\frac{1}{2}}, N, B^{\frac{1}{2}}\hat{\mu}_2)[\Phi(2B - B\lambda_2^2) - \Phi(2B)]. \end{aligned}$$

It is thus possible to find $E^{-1}(\theta_1\theta_2)$ and it is easy to find the expression for $\hat{V}(\hat{\theta}_1 - \hat{\theta}_2) = \hat{\theta}_1^2 + \hat{\theta}_2^2 - 2E^{-1}(\theta_1\theta_2)$ yielding

$$\begin{aligned} \hat{V}(\hat{\theta}_1 - \hat{\theta}_2) &= (\hat{\theta}_1 - \hat{\theta}_2)^2 - \Phi(2B)[f(\hat{\mu}_1) - f(\hat{\mu}_2)]^2 \\ &- \psi(KB^{-\frac{1}{2}}, M, 2B^{\frac{1}{2}}\hat{\mu}_1)[\Phi(2B - 4B\lambda_1^2) - \Phi(2B)] \\ &- \psi(KB^{-\frac{1}{2}}, M, 2B^{\frac{1}{2}}\hat{\mu}_2)[\Phi(2B - 4B\lambda_2^2) - \Phi(2B)] \\ &+ 2\psi(KB^{-\frac{1}{2}}, M, B^{\frac{1}{2}}\{\hat{\mu}_1 + \hat{\mu}_2\})[\Phi(2B - B\lambda_1^2 - 2B\lambda_{12} - B\lambda_2^2) \\ &- \Phi(2B)] \\ &+ 2H \cosh B^{\frac{1}{2}}(\hat{\mu}_1 - \hat{\mu}_2)[\Phi(2B - B\lambda_1^2 + 2B\lambda_{12} - B\lambda_2^2) - \Phi(2B)]. \end{aligned}$$

For example

(a) $\xi = \log_{10} x$; $H = 0$.

$$\begin{aligned} \hat{V}(\hat{\theta}_1 - \hat{\theta}_2) &= (\hat{\theta}_1 - \hat{\theta}_2)^2 - e^{2m\hat{\mu}_1}\Phi(2m^2(1 - 2\lambda_1^2)) \\ &- e^{2m\hat{\mu}_2}\Phi(2m^2(1 - 2\lambda_2^2)) \\ &+ 2e^{m(\hat{\mu}_1+\hat{\mu}_2)}\Phi(m^2(2 - \lambda_1^2 - 2\lambda_{12} - \lambda_2^2)). \end{aligned}$$

(b) $\xi = \sin^{-1} x^{\frac{1}{2}}$; $H = \frac{1}{8}$.

$$\begin{aligned} \hat{V}(\hat{\theta}_1 - \hat{\theta}_2) &= (\hat{\theta}_1 - \hat{\theta}_2)^2 - \Phi(-8)[\sin 2\hat{\mu}_1 - \sin 2\hat{\mu}_2]^2 \\ &- \tfrac{1}{8}\cos 4\hat{\mu}_1[\Phi(-8 + 16\lambda_1^2) - \Phi(-8)] - \tfrac{1}{8}\cos 4\hat{\mu}_2[\Phi(-8 + 16\lambda_2^2) \\ &- \Phi(-8)] \end{aligned}$$

$$+ \tfrac{1}{4}\cos(2\hat{\mu}_1 + 2\hat{\mu}_2)[\Phi(-8 + 4\lambda_1^2 + 8\lambda_{12} + 4\lambda_2^2) - \Phi(-8)]$$
$$+ \tfrac{1}{4}\cos(2\hat{\mu}_1 - 2\hat{\mu}_2)[\Phi(-8 + 4\lambda_1^2 - 8\lambda_{12} + 4\lambda_2^2) - \Phi(-8)].$$

(c) $\xi = \sinh^{-1} x^{\frac{1}{2}}$, $H = \frac{1}{8}$.

$$\hat{V}(\hat{\theta}_1 - \hat{\theta}_2) = (\hat{\theta}_1 - \hat{\theta}_2)^2 - \Phi(8)[\sinh 2\hat{\mu}_1 - \sinh 2\hat{\mu}_2]^2$$
$$- \tfrac{1}{8}\cosh 4\hat{\mu}_1[\Phi(8 - 16\lambda_1^2) - \Phi(8)] - \tfrac{1}{8}\cosh 4\hat{\mu}_2[\Phi(8 - 16\lambda_2^2) - \Phi(8)]$$
$$+ \tfrac{1}{4}\cosh(2\hat{\mu}_1 + 2\hat{\mu}_2)[\Phi(8 - 4\lambda_1^2 - 8\lambda_{12} - 4\lambda_2^2) - \Phi(8)]$$
$$+ \tfrac{1}{4}\cosh(2\hat{\mu}_1 - 2\hat{\mu}_2)[\Phi(8 - 4\lambda_1^2 + 8\lambda_{12} - 4\lambda_2^2) - \Phi(8)].$$

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