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ERGODIC THEORY WITH RECURRENT WEIGHTS[1]

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It has generally been possible to prove ergodic theorems on continuous flows as corollaries of their discrete analogs, the idea being that a continuous average can be approximated by a discrete average. This method is no longer available, however, if the continuous average uses nonuniform weighting. Here the weighted average comes from the Lebesgue-Stieltjes measure generated by a solution v of the renewal equation $v = 1 + v * w$, where $v * w$ is the convolution of v with a probability distribution on the positive reals. The formulation has the advantage of including the classical discrete and continuous averages as special cases. Also included are the recurrent averages for the discrete case introduced by Baxter [1], [2], [5].

1. Statement of results. Let w be a probability measure on $[0, \infty)$ with $w\{0\} < 1$ and let v be the unique nondecreasing right continuous solution [7] to the renewal equation

$$(1.1) \quad v(\alpha) = 1 + \int_{[0,\alpha]} v(\alpha - \beta) w(d\beta), \alpha \geqq 0, \qquad v(\alpha) = 0, \alpha < 0.$$

Suppose that there is a Lebesgue measurable set V such that the sum of two members of V is in V and such that $\int_{V'} dv = 0$, where V' is the complement of V. Since w is absolutely continuous with respect to the measure generated by v, we have $w(V') = 0$.

EXAMPLES 1.1.

(a) $w(d\beta) = e^{-\beta} d\beta$, $v(\alpha) = 1 + \alpha$ for $\alpha \geqq 0$, $V = [0, \infty)$;

(b) $w\{1\} = 1$, $v(\alpha) = [\alpha] + 1$ for $\alpha \geqq 0$, $V = \{0, 1, \cdots\}$;

(c) $w\{1\} = w_1$, $w\{2\} = w_2, \cdots$, where $w_1 + w_2 + \cdots = 1$, $v(0) = 1$, $v(n) = 1 + w_1 v(n-1) + \cdots + w_n v(0)$, $V = \{0, 1, \cdots\}$.

We shall be concerned with a σ-finite measure space (S, Σ, μ) and a semigroup of positive linear operators $\{T^\alpha, \alpha \varepsilon V\}$, $T^0 = I$, $T^\alpha T^\beta = T^{\alpha+\beta}$, on the equivalence classes of real integrable functions $L_1(S, \Sigma, \mu)$ such that $\int T^\alpha |f| \, d\mu \leqq \int |f| \, d\mu$, $f \varepsilon L_1$. We require that T^α be strongly measurable [6]; that is, for each $f \varepsilon L_1$ there exist a real valued $g(\alpha, s)$ measurable with respect to the product of Lebesgue measure on V with (S, Σ, μ), such that $g(\alpha, \cdot)$ is in the equivalence class $T^\alpha f$ for each $\alpha \varepsilon V$. The integral $z_\beta(s; f) = \int_{[0,\beta]} (T^\alpha f)(s) \, dv(\alpha)$ is given meaning by reference to $\int_{[0,\beta]} g(\alpha, s) \, dv(\alpha)$. (Here and elsewhere integration over $[0, \beta]$ means integration over $[0, \beta] \cap V$).

Then we can state

THEOREM 1.2. *If* $f \varepsilon L_1$, $p \varepsilon L_1$, $p \geqq 0$, *then*

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$$\lim_{\beta\to\infty} z_\beta(s; f)/z_\beta(s; p) = Q(s)$$

exists and is finite for almost all s in the set $\{s: \sup_\alpha z_\alpha(s; p) > 0\}$.

A weaker version of this theorm (with different proof) appeared in the author's Ph.D. thesis at the University of Minnesota.

To discuss the limit $Q(s)$ we need a new positive linear operator $\bar{T}$ defined by

$$\bar{T}f = \int_{[0,\infty)} T^\beta f\, dw(\alpha); \tag{1.2}$$

note that $\|\bar{T}\|_1 \leqq 1$. The space S splits [9], p. 196, into two parts C, conservative and D, dissipative, such that for any $p > 0$, $p \in L_1$,

$$C = \{s: \textstyle\sum_{k=0}^{\infty} (\bar{T}^k p)(s) = \infty\}, \qquad D = \{s: \textstyle\sum_{k=0}^{\infty} (\bar{T}^k p)(s) < \infty\}. \tag{1.3}$$

The subsets C_1 of C for which the indicator function I_{C_1} is invariant on C under the adjoint $\bar{N}$ of $\bar{T}$ form a σ-algebra $\mathcal{C}$. Now consider the operator B_A defined for $A \in \Sigma$ by

$$B_A f = J_A f + J_A \bar{T} J_{A'} f + J_A \bar{T} J_{A'} \bar{T} J_{A'} f + \cdots$$

where J_A is the operator that multiplies by I_A. If f is a probability density and $\bar{T}$ is a Markov operator then $B_A f$ is the density for first entrance to the set A.

THEOREM 1.3. *The limit $Q(s)$ in Theorem 1.2 is given by*

$$Q(s) = E(B_c f \mid \mathcal{C})/E(B_c p \mid \mathcal{C}) \quad on \quad C \cap \{s: \sup_\alpha z_\alpha(s; p) > 0\}.$$

This generalizes a result of Chacon [4], [8]. (see Example 1.1(b)).

THEOREM 1.4. *If* $\bar{z}_k(s; f) = \sum_{m=0}^{k} (\bar{T}^m f)(s)$, *and if* $f, p \in L_1(S)$, $p \geqq 0$,

$$\lim_{k\to\infty} \bar{z}_k(s; f)/\bar{z}_k(s; p) = \lim_{\alpha\to\infty} z_\alpha(s; f)/z_\alpha(s; p)$$

for almost all s in $\{s: \sup_k \bar{z}_k(s; p) > 0\} = \{s: \sup_\alpha z_\alpha(s; p) > 0\}$.

This result follows from Theorem 3.4 and Theorem 1.3 and Chacon-Ornstein theory [4], [8].

Now define H_A to be the adjoint of B_A. If $\bar{T}$ is a Markov operator then $(H_A h)(s)$ is the expected value of h when entering A, having started at s. Letting $e_A = H_A I$, we have (see Brunel [3] and Meyer [8])

LEMMA 1.5. *If $B \in \Sigma$ is a subset of S on which* $\sup_\alpha z_\alpha(s; f) = \infty$, $f \in L_1(s)$, *then* $\int e_B f\, d\mu \geqq 0$.

Section 2 is devoted to proving the lemma, which is used in Section 3 to prove Theorems 1.2 and 1.3.

2. Proof of Lemma 1.5. If B is a measurable set of the type specified in the lemma, then $z_\alpha(s) = z_\alpha(s; f) \to \infty$ for $s \in B$. For each $t > 0$, let $M(t, s) = \sup_{\beta\in[0,t]} z_\beta(s)$, and define

$$\begin{aligned} r_t(s) &= \inf\{\alpha \in [0, t]: z_\alpha(s) > M(t, s) - t^{-1}\}, \quad s \in B, \quad M - t^{-1} > 0, \\ &= -1, \text{ otherwise} \end{aligned} \tag{2.1}$$

so that z_r is nearly the supremum M (z is right continuous). The following are easily checked:

PROPOSITION 2.1. (i) *$M(t, s)$ and $r_t(s)$ are μ-measurable for fixed t.*

(ii) $r_t(s) \uparrow \infty$ *as* $t \uparrow \infty$, *for each* $s \varepsilon B$.

(iii) $z_{r_t(s)}(s) > z_\alpha(s)$ *if* $0 \leqq \alpha < r_t(s)$.

Then designate, for $0 \leqq \alpha \leqq t < \infty$,

$$B(\alpha, t) = \{s : \alpha \leqq r_t(s)\} \tag{2.2}$$

with the following properties:

PROPOSITION 2.2. (i) $I_{B(\alpha,t)}(s)$ *is for fixed t measurable with respect to the product of* (S, Σ, μ) *with Lebesgue measure on* $[0, t]$.

(ii) $I_{B(\alpha,t)}(s) \uparrow I_B(s)$ *as* $t \uparrow \infty$, *for each* s.

Lemma 1.5 will be proved by first proving that $\int b(0, t) f \, d\mu \geqq 0$, where $b(0, t)$ is a bounded function such that $b(0, t) \to e_B$ as $t \to \infty$. Define, for $0 \leqq \alpha \leqq t$,

$$b(\alpha, t) = \sum_{k=0}^{\infty} (J_{B'(\alpha,t)} \bar{N}_r)^k I_{B(\alpha,t)} \quad \text{where } \bar{N}_t \text{ is defined by} \tag{2.3}$$

$$(\bar{N}_t h)(\alpha) = \int_{[0,t-\alpha]} N^\beta h(\alpha + \beta) w(d\beta). \tag{2.4}$$

That $b(\alpha, t)$ is bounded by 1 and nonincreasing in α may be ascertained by induction on the partial sums $b^n(\alpha, t)$ of (2.3), since $b^{n+1}(\alpha, t) = I_{B(\alpha,t)} + J_{B'(\alpha,t)} \bar{N}_t b^n(\alpha, t)$. It is apparent, taking limits, that

$$b(\alpha, t) = I_{B(\alpha,t)} + J_{B'(\alpha,t)} \bar{N}_t b(\alpha, t). \tag{2.5}$$

PROPOSITION 2.3. $\lim_{t\to\infty} b(\alpha, t) = e_B$.

PROOF. If $h(\alpha + \beta, s) \uparrow g(s) \varepsilon L_\infty(s)$, as $\alpha \uparrow \infty$, then as $t \uparrow \infty$, by (2.4)

$$(\bar{N}_t h)(\alpha, s) \uparrow \int_{[0,\infty)} N^\beta g w(d\beta) = (\bar{N} g)(s).$$

This fact shows by induction on n the truth of the following limit:

$$\sum_{k=0}^{n} (J_{B'(\alpha,t)} \bar{N}_t)^k I_{B(\alpha,t)} \uparrow \sum_{k=0}^{n} (J_{B'} \bar{N})^k I_B, \quad \text{as} \quad t \uparrow \infty, \tag{2.6}$$

since the formula's truth for $n = 0$ is given by Proposition 2.2(ii). Thus, denoting by $b^n(\alpha, t)$ the left side of 2.6,

$$\lim_{t\to\infty} b(\alpha, t) = \lim_{t\to\infty} \lim_{n\to\infty} b^n(\alpha, t) = \lim_{n\to\infty} \lim_{t\to\infty} b^n(\alpha, t)$$
$$= \sum_{k=0}^{\infty} (J_B \bar{N})^k I_B = H_B I = e_B$$

(see definitions following Theorem 1.4). The interchange of limits is permissible because $b^n(\alpha, t)$ is nondecreasing in n and in t.

PROPOSITION 2.4. *If* $B \subseteq C$, *the conservative part of* S, *then*

(i) $\bar{N} e_B = e_B$,

(ii) $H_C e_B = e_B$,

(iii) $e_B = I_{\bar{B}}$, *where* $\bar{B}$ *is of minimum measure among members of* $\mathcal{C}$ *that contain* B, if C is all of S.

These properties are proved by Meyer [8].

Now define

$$\Gamma(\alpha, t, s) = I_{B(\alpha,t)}(s)[1 - \int_{[0,t-\alpha]} N^\beta b(\alpha + \beta, t) w(d\beta)], \quad 0 \leqq \alpha \leqq t. \tag{2.7}$$

It follows from (2.5) that

$$\Gamma(\alpha, t, s) = b(\alpha, t, s) - \int_{[0,t-\alpha]} N^{\beta} b(\alpha + \beta, t) w(d\beta), \quad 0 \leqq \alpha \leqq t. \tag{2.8}$$

LEMMA 2.5. *The two measures on Lebesgue measurable subsets A of* $[0, t]$ *given respectively by* $\int_A d(v - I_{[0,\infty)})$ *and*

$$\int \int_{\alpha+\beta\varepsilon A} dv(\alpha) w(d\beta) = \int_{\alpha\varepsilon A} w(d\alpha) \int_{\alpha+\beta\varepsilon A} dv(\beta)$$

are equal.

PROOF. It is sufficient to establish the equality for intervals, for which the equality follows from the renewal equation (1.1) which defines v:

$$\int_{[0,\gamma]} d(v - I_{[0,\infty)}) = v(\gamma) - 1 = \int_{[0,\gamma]} w(d\alpha) v(\gamma - \alpha)$$
$$= \int_{[0,\gamma]} w(d\alpha) \int_{\alpha+\beta\varepsilon[0,\gamma]} dv(\beta).$$

In the following, dependence of Γ on s is suppressed.

PROPOSITION 2.6. (i) $\int_{[0,t]} N^{\alpha}\Gamma(\alpha, t)\, dv(\alpha) = b(0, t)$,

(ii) $\Gamma(\alpha, t)$ *is nondecreasing on* $[0, r_t]$, *for fixed* t,

(iii) $\Gamma(\alpha, t) = 0$, $\alpha \varepsilon (r_t, t]$.

PROOF. Property (i) follows from (2.8), using Lemma 2.5:

$$\int_{[0,t]} N^{\alpha}\Gamma(\alpha, t)\, dv(\alpha)$$
$$= \int_{[0,t]} N^{\alpha} b(\alpha, t)\, dv(\alpha) - \int_{[0,t]} N^{\alpha}\, dv(\alpha) \int_{[0,t-\alpha]} N^{\beta} b(\alpha + \beta, t) w(d\beta)$$
$$= \int_{[0,t]} N^{\alpha} b(\alpha, t)\, dv(\alpha) - \int_{\alpha+\beta\varepsilon[0,t]} N^{\alpha+\beta} b(\alpha + \beta, t)\, dv(\alpha) w(d\beta)$$
$$= \int_{[0,t]} N^{\alpha} b(\alpha, t)\, dv(\alpha) - [\int_{[0,t]} N^{\alpha} b(\alpha, t)\, dv(\alpha) - b(0, t)]$$
$$= b(0, t)$$

Property (ii) follows from (2.7) and (2.2) and the fact that $b(\alpha, t)$ is nonincreasing in α.

Property (iii) follows from (2.7) and (2.2).

Finally Lemma 1.5 can be proved. Since e_B is the limit of $b(0, t)$, by Proposition 2.3, it will be sufficient to show that $\int b(0, t) f\, d\mu \geqq 0$ for each t. First by Proposition 2.6,

$$\int b(0, t) f\, d\mu$$
$$= \int d\mu f \int_{[0,t]} N^{\alpha}\Gamma(\alpha, t)\, dv(\alpha) = \int_{[0,t]} \int d\mu f N^{\alpha}\Gamma(\alpha, t)\, dv(\alpha) \tag{2.9}$$
$$= \int_{[0,t]} \int d\mu \Gamma(\alpha, t) T^{\alpha} f\, dv(\alpha) = \int d\mu \int_{[0,r_t]} \Gamma(\alpha, t) T^{\alpha} f\, dv(\alpha).$$

In order to show that the inner integral is nonnegative, we approximate Γ by a continuous nondecreasing G, for which integration by parts is valid [6], p. 154. That is, if G is continuous, nondecreasing, nonnegative, we have

$$\int_{[0,r_t]} G\, dz$$
$$= G(r_t) z_{r_t+} - G(0) z_{0-} - \int_{[0,r_t]} z\, dG = z_{r_t} \int_{[0,r_t]} dG + z_{r_t} G(0) - \int_{[0,r_t]} z\, dG$$
$$= z_{r_t} G(0) + \int_{[0,r_t]} (z_{r_t} - z_{\alpha})\, dG(\alpha) \geqq 0$$

since z_{r_t} is nonnegative and $z_{r_t} \geqq z_\alpha$ for $\alpha < r_t$, by Proposition 2.1 (iii). Now $\Gamma(\alpha, s)$, $0 \leqq \alpha \leqq r_t$, is the pointwise limit of such G's, so we find

$$0 \leqq \int_{[0,r_t]} \Gamma\, dz = \int_{[0,r_t]} \Gamma(\alpha, s) T^\alpha f\, dv(\alpha).$$

3. The limit of the ratio. The first theorem of this section ties together the averages of the semigroup of operators T^α and the operator $\bar{T}$, facilitating use of the established theory of the latter. Then the fact that, if $p \geqq 0$, $\{\sum \bar{T}^k p > 0\} = \{\int T^\alpha p dv(\alpha) > 0\}$, combined with Meyer's technique [8] and Lemma 1.5, yields proofs of Theorem 1.2 and Theorem 1.3.

LEMMA 3.1. *If* $z_\alpha = z_\alpha(s; f)$ *then*

$$z_\alpha = f + \int_{[0,\alpha]} T^\beta z_{\alpha-\beta}\, dw(\beta). \tag{3.1}$$

PROOF. We have

$$\int_{[0,\alpha]} T^\beta z_{\alpha-\beta}\, dw(\beta)$$
$$= \int_{[0,\alpha]} T^\beta\, dw(\beta) \int_{[0,\alpha-\beta]} T^\gamma f\, dv(\gamma) = \int_{[0,\alpha]} T^\beta f\, d[v(\beta) - I_{[0,\infty)}(\beta)] = z_\alpha - f,$$

using Lemma 2.5.

LEMMA 3.2. *The equation*

$$g(\alpha) = f(\alpha) + \int_{[0,\alpha]} T^\beta g(\alpha - \beta)\, dw(\beta), \tag{3.2}$$

$g: [0, \infty) \to L_1(S, \Sigma, \mu)$, *has only one solution that is bounded on each finite interval in* L_1 *norm.*

PROOF. The previous lemma provides existence. If $g_1(\alpha)$ and $g_2(\alpha)$ are bounded and satisfy (3.2), then their difference $g(\alpha)$ satisfies (3.2) with $f = 0$. Let $w(\beta_0) < 1$ and let $M_0 = \sup_{0 \leqq \alpha \leqq \beta_0} \|g(\alpha)\|$. Then if $0 \leqq \alpha \leqq \beta_0$,

$$\|g(\alpha)\|_1 = \|\int_{[0,\alpha]} T^\beta g(\alpha - \beta)\, dw(\beta)\|_1 \leqq \int_{[0,\alpha]} \|g(\alpha - \beta)\|_1\, dw(\beta) \leqq w(\beta_0) M_0.$$

Thus $M_0 \leqq w(\beta_0) M_0$ and we see that M_0 must be zero. In this way we can show that $g(\alpha) = 0$, $\beta_0 \leqq \alpha \leqq 2\beta_0$, etc.

LEMMA 3.3. *If* $f \geqq 0$ *and we define* $z_\alpha^0 = f$,

$$z_\alpha^k = f + \int_{[0,\alpha]} T^\beta z_\alpha^{k-1}\, dw(\beta), \qquad k = 1, 2, \cdots, \tag{3.3}$$

then z_α^k *is nondecreasing in* α *and in* k *and* $z_\alpha^k \uparrow z_\alpha$ *as* $k \uparrow \infty$.

PROOF. An application of (3.3) to an induction argument shows that z_α^k is nondecreasing in α and in k. Another induction argument using (3.1) and (3.3) shows that $z_\alpha^k \leqq z_\alpha$. Then we let k go to ∞ in 3.3 and get

$$\lim_k z_\alpha^k = f + \int_{[0,\alpha]} T^\beta (\lim_k z_\alpha^k)\, dw(\beta). \tag{3.4}$$

Thus, by Lemma 3.2, $\lim_k z_\alpha^k = z_\alpha$.

THEOREM 3.4. *If* $f \geqq 0$, *then* $\sup_k \bar{z}_k = \sup_\alpha z_\alpha$; *that is,* $\sum_{k=0}^\infty \bar{T}^k f = \int_{[0,\infty)} T^\alpha f\, dv(\alpha)$, *with* ∞ *permitted as a value for the equality.*

PROOF. We proceed by induction to show that $z_\alpha^k \uparrow \bar{z}_k$ as $\alpha \uparrow \infty$, for all k. We have $z_\alpha^0 = f = \bar{z}_0$; and if $\lim_\alpha z_\alpha^{k-1} = \bar{z}'_{k-1}$, then

$$\lim_\alpha z_\alpha^k = \lim_\alpha (f + \int_{[0,\alpha]} T^\beta z_{\alpha-\beta}^{k-1} w(d\beta)) = f + \int_{[0,\infty)} T^\beta \bar{z}_{k-1} w(d\beta)$$
$$= f + \bar{T}\bar{z}_{k-1} = \bar{z}_k.$$

Combining this with Lemma 3.3,

$$\sup_\alpha z_\alpha = \sup_\alpha \sup_k z_\alpha^k = \sup_k \sup_\alpha z_\alpha^k = \sup_k \bar{z}_k .$$

LEMMA 3.5. *If* $p \geqq 0$ *and* $p \in L_1(S)$, $f \in L_1(S)$, *then* $\limsup_{\alpha\to\infty} z_\alpha(s; f)/z_\alpha(s; p) < \infty$ *for almost all* s *such that* $\sup_\alpha z_\alpha(s; p) > 0$.

PROOF. According to Theorem 3.4,

$$\{s: \sup_\alpha z_\alpha(s; p) > 0\} = \{s: \sup_n \bar{z}_n(s; p) > 0\} = \bigcup_n \{s: \bar{T}^n p > 0\}. \tag{3.5}$$

Now if $a > 0$

$$\begin{aligned}\{s: \limsup_\alpha z_\alpha(f)/z_\alpha(p) = \infty, \bar{T}^n p > 0\} &\subseteq \{s: \limsup_\alpha z_\alpha(f)/z_\alpha(p) > a, \bar{T}^n p > 0, \sup_\alpha z_\alpha(f) = \infty\}\\ &\subseteq \{s: \limsup_\alpha z_\alpha(f - ap) = \infty, \bar{T}^n p > 0\}.\end{aligned}$$

Calling the first set A, the last B, using Proposition 2.4(i) and Lemma 1.5,

$$0 \leqq \int (f - ap) e_B \, d\mu = \int (f - ap) \bar{N}^n e_B \, d\mu = \int (\bar{T}^n f - a\bar{T}^n p) e_B \, d\mu.$$

Thus

$$\int |f| \, d\mu \geqq \int \bar{T}^n f e_B \, d\mu \geqq \int a\bar{T}^n p e_B \, d\mu \geqq \int_B a\bar{T}^n p \, d\mu \geqq \int_A a\bar{T}^n p \, d\mu,$$

which can hold for all a only if $\mu(B) = 0$.

PROOF OF THEOREM 1.2. Without loss of generality assume $f \geqq 0$. By Lemma 3.5, where $z_\alpha(p)$ is bounded so is $z_\alpha(f)$ and both numerator and denominator have limits. Otherwise, if $a < b$, on

$$B = \{\liminf z_\alpha(f)/z_\alpha(p) < a < b < \limsup z_\alpha(f)/z_\alpha(p), \quad \bar{T}^n p > 0, \sup_\alpha z_\alpha(p) = \infty\},$$

we have $\limsup z_\alpha(f - bp) = \infty$ and $\limsup z_\alpha(ap - f) = \infty$. Then by Lemma 1.5 and Proposition 2.4 (i),

$$\int e_B \bar{T}^k (f - bp) \, d\mu = \int e_B (f - bp) \, d\mu \geqq 0,$$
$$\int e_B \bar{T}^k (ap - f) \, d\mu = \int e_B (ap - f) \, d\mu \geqq 0,$$

so that

$$\int e_B \bar{T}^k (ap - bp) \geqq 0,$$

which implies $\mu(B) = 0$. We again use (3.5).

THEOREM 3.6. *If* $f \geqq 0$ *and* $h = B_c f$ *then* $D_\alpha = z_\alpha(h)/z_\alpha(f) \to 1$ *on* $C \cap \{\sup_\alpha z_\alpha(f) > 0\}$.

PROOF (half). To see that the limit is at least one, let $b < 1$ and set

$$B = C \cap \{\bar{T}^k f > 0\} \cap \{\lim_\alpha D_\alpha < b\}.$$

We have $\bar{z}_n(f) \to \infty$ on B so $z_\alpha(f) \to \infty$ on B so $z_\alpha(bf - h) \to \infty$ on B. Then

$$0 \leqq \int e_B(bf - h)\, d\mu = \int e_B(bf - B_cf) d\mu = \int (be_B - H_ce_B) f\, d\mu = \int (be_B - e_B) f\, d\mu$$
$$= (b - 1) \int (\bar{N}^k e_B) f\, d\mu = (b - 1) \int e_B \bar{T}^k f\, d\mu,$$

using $H_ce_B = e_B$ and $\bar{N}e_B = e_B$ (Proposition 2.4). Thus $\mu(B) = 0$. And, using (3.5), we have $D_\alpha \geqq 1$ on $C \cap \{\sup_\alpha z_\alpha(f) > 0\}$.

COROLLARY 3.7. *If $f \geqq 0$, $p \geqq 0$ are μ-integrable, then*

$$\lim_{\alpha\to\infty} z_\alpha(B_cf)/z_\alpha(B_cp) = \lim_{\alpha\to\infty} z_\alpha(f)/z_\alpha(p)$$

on $\{\sup_\alpha z_\alpha(p) > 0\} \cap C$.

Note that B_cf has its support on C, and $B_cf = f$ if f has its support on C. The corollary combined with the following lemma will enable us to restrict T^β to the conservative part C in the discussion of the limiting behavior on C. If f has its support on C then so does $T^\alpha f$.

LEMMA 3.8. *If $f \geqq 0$ and $f = 0$ on D, then $T^\beta f = T^\beta f I_c$ for v-almost all β.*

PROOF. If $f = 0$ on D, then $\sum \bar{T}^k f = 0$ on D because

$$\int I_D \bar{T} f\, d\mu = \int \bar{N} I_D f\, d\mu \leqq \int I_D f\, d\mu.$$

Thus $\int T^\beta f\, dv(\beta) = 0$ on D by Theorem 3.4, and $T^\beta f = 0$ on D for v-almost all β

THEOREM 3.9. *If $\mu(D) = 0$ ($S = C$), and if f, p are nonnegative integrable, then on* $\{\sup z_\alpha(p) > 0\} = \{\sup z_\alpha(p) = \infty\}$, $\mu -$ a.e.,

$$\lim_{\alpha\to\infty} z_\alpha(f)/z_\alpha(p) = E(f \mid \mathcal{C})/E(p \mid \mathcal{C}) \tag{3.6}$$

PROOF. The set equality follows from Lemma 3.5, which implies that $0 < \sup z_\alpha < \infty$ cannot occur on C. To see that $E(p \mid \mathcal{C}) > 0$ on $\{\sup z_\alpha(p) > 0\}$, let $F = \{\bar{T}^k p > 0, E(p \mid \mathcal{C}) = 0\}$ and note that $E(p \mid \mathcal{C}) = 0$ on $\bar{F}$ (see Proposition 2.4 (iii)). Then

$$0 = \int I_{\bar{F}} E(p \mid \mathcal{C})\, d\mu = \int I_{\bar{F}} p\, d\mu = \int \bar{N}^k I_{\bar{F}}\, p d\mu = \int I_{\bar{F}} \bar{T}^k p\, d\mu \geqq \int I_F \bar{T}^k p\, d\mu$$

so $\mu(F) = 0$. Thus, by Theorem 3.4, $E(p \mid \mathcal{C}) > 0$ on $\{\sup z_\alpha(p) > 0\}$. Let the limit in (3.6) be Q and the ratio be R. Let $b > 0$ and

$$B = \{R < b < Q, \sup z_\alpha(p) = \infty\},$$

noting that $R < b$ on $\bar{B}$, (see Proposition 2.4(iii)) since R is $\mathcal{C}$-measurable. We have on B, $z_\alpha(f - bp) \to \infty$ so $\int e_B(f - bp)\, d\mu \geqq 0$, by Lemma 1.5. But $e_B = I_{\bar{B}}$, by Proposition 2.4(iii), and thus

$$\int_{\bar{B}} E(f \mid \mathcal{C})\, d\mu = \int_{\bar{B}} f\, d\mu \geqq b \int_{\bar{B}} p\, d\mu = b \int_{\bar{B}} E(p \mid \mathcal{C})\, d\mu,$$

contradicting $R < b$ on $\bar{B}$, unless $\mu(\bar{B}) = 0$. Similarly, if

$$A = \{R > b > Q, \sup z_\alpha(p) = \infty\},$$

we find $\mu(\bar{A}) = 0$. Thus $R = Q$ on $\{\sup z_\alpha(p) = \infty\}$.

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REFERENCES

[1] BAXTER, G. (1964). An ergodic theorem with weighted averages. *J. Math. Mech.* **13** 481–488.

[2] BAXTER, G. (1965). A general ergodic theorem with weighted averages. *J. Math. Mech.* **14** 277–288.

[3] BRUNEL, A. (1963). Sur une lemme ergodique voisin du lemme du E. Hopf, et sur une de ses applications. *C. R. Acad. Sci. Paris* **256** 5481–5484.

[4] CHACON, R. V. (1962). Identification of the limit of operator averages. *J. Math. Mech.* **11** 961–968.

[5] CHACON, R. V. (1964). Ordinary means imply recurrent means. *Bull. Amer. Math. Soc.* **70** 796–797.

[6] DUNFORD, N. and SCHWARTZ, J. T. (1958). *Linear Operators, Part I.* Interscience, New York.

[7] FELLER, W. (1941). On the integral equation of renewal theory. *Ann. Math. Statist.* **12** 243–267.

[8] MEYER, P. A. (1965). Le Theoreme ergodique de Chacon-Ornstein. *Seminaire Bourbaki* **17.3** 10–18.

[9] NEVEU, J. (1965). *Mathematical Foundations of the Calculus of Probability.* Holden-Day, San Francisco.