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AN INEQUALITY FOR THE RATIO OF TWO QUADRATIC FORMS IN NORMAL VARIATES

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The distribution of ratios of quadratic forms has been investigated by many authors. Two simple inequalities for the ratio of quadratic forms in independent normal variates are presented.

THEOREM. *If* $\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_n > 0$, $d_1 \geqq d_2 \geqq \cdots \geqq d_n > 0$, *and* $Z_1, \cdots, Z_n$ *are positive random variables, then*

$$\text{(i)} \qquad P\{\textstyle\sum_1^n \lambda_i d_i Z_i / \sum_1^n d_i Z_i \leqq \nu\} \leqq P\{\sum_1^n d_i Z_i / \sum_1^n Z_i \leqq \nu\}.$$

If $X_1, \cdots, X_n$ *are independent* $N(0, 1)$ *random variates, and* $(d_1^*, \cdots, d_n^*)$ *is any rearrangement of* $d_1, \cdots, d_n$, *then*

$$\text{(ii)} \quad P\{\textstyle\sum_1^n \lambda_i d_i^* X_i^2 / \sum_1^n d_i^* X_i^2 \leqq \nu\} \leqq P\{\sum_1^n \lambda_i d_{n-i+1} X_i^2 / \sum d_{n-i+1} X_i^2 \leqq \nu\}.$$

PROOF. (i) The proof[1] follows from the fact that

$$\textstyle\sum_1^n \lambda_i d_i Z_i / \sum_1^n d_i Z_i \geqq \sum_1^n \lambda_i Z_i / \sum_1^n Z_i,$$

which follows as a special case of an inequality due to Tchebychef [1], p. 168. It is clear, of course, that, if the d's are increasingly ordered, the inequality in (i) would go the opposite way.

$$\text{(ii)} \qquad P\{\textstyle\sum_1^n \lambda_i d_i^* X_i^2 / \sum_1^n d_i^* X_i^2 \leqq \nu\}$$
$$= P\{\textstyle\sum_1^n (\lambda_i - \nu) d_i^* X_i^2 \leqq 0\} = (2\pi)^{-n/2} \int_E e^{-1/2x'x}\, dx,$$

where E is the set appearing on the left hand side of (ii). Now if $d_1^* \neq d_n$ there exists a k such that $d_k^* < d_1^*$. And,

$$P\{\textstyle\sum_1^n (\lambda_i - \nu) d_i^* X_i^2 \leqq 0\}$$
$$= P\{(\lambda_1 - \nu_1) d_1^* X_1^2 + (\lambda_k - \nu) d_k^* X_k^2 \leqq \chi_0\} / P(\chi \leqq \chi_0)$$

where $\chi = -\sum_{i \neq 1,k} (\lambda_i - \nu) d_i^* X_i^2$. But,[2]

$$\text{(1)} \quad P\{(\lambda_1 - \nu) d_1^* X_1^2 + (\lambda_k - \nu) d_k^* X_k^2 \leqq \chi_0\}$$
$$= (2\pi)^{-1} \int\int_{E_0} e^{-\frac{1}{2}(x_1^2 + x_2^2)}\, dx_1\, dx_2 \leqq (2\pi)^{-1} \int\int_{E_1} e^{-\frac{1}{2}(x_1^2+x_2^2)}\, dx_1\, dx_2$$

where $E_0 = \{x_1, x_2 : (\lambda_1 - \nu) d_1^* x_1^2 + (\lambda_k - \nu) d_k^* x_2^2 \leqq \chi_0\}$

and $E_1 = \{x_1, x_2 : (\lambda_1 - \nu) d_k^* x_1^2 + (\lambda_k - \nu) d_1^* x_2^2 \leqq \chi_0\}$.

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[1] The author is indebted to the referee who suggested this proof which is more direct and simpler than the original proof.

[2] If $(\lambda_1 - \nu) \geqq 0$ and $(\lambda_k - \nu) \leqq 0$ the fact that the inequality in (1) follows is trivial. If both $(\lambda_1 - \nu)$ and $(\lambda_k - \nu)$ are strictly positive (1) follows from the fact that the areas of the two sets E_0 and E_1 are equal.

This procedure can be repeated until the coefficient of $(\lambda_1 - \nu)X_1^2$ is smallest, the coefficient of $(\lambda_2 - \nu)X_2^2$ is second smallest and so on until the coefficient of $(\lambda_n - \nu)X_n^2$ is d_1. Therefore, we have

$$P\{\sum_1^n \lambda_i d_i^* X_i^2 / \sum_1^n d_i^* X_i^2 \leqq \nu\} \leqq P\{\sum_1^n \lambda_i d_{n-i+1} X_i^2 / \sum_1^n d_{n-i+1} X_i^2 \leqq \nu\}.$$

AN APPLICATION. Let it be required to test the null hypothesis, H_0 that Var $(X_i) = d$ and $EX_i = 0$ against the alternative hypothesis, H_1 that Var $(X_i) = d_i^*$ and $EX_i = 0$ based on a sample of size one. Then if one proposes a ratio of the type

$$\sum_{i=1}^n \lambda_i X_i^2 / \sum_{i=1}^n X_i^2 \tag{2}$$

as a test statistic for H_0 against H_1, with the critical region, ω, defined by

$$\omega = \{X : \sum_{i=1}^n \lambda_i X_i^2 / \sum_{i=1}^n X_i^2 \leqq \nu\} \tag{3}$$

then, a best test within the class (3) is obtained by matching λ_i with X_i whose variance is d_{n-i+1}. One advantage of a statistic of the type (2) is that its distribution is tabulated for selected values of n [3], [4] when $\lambda_i = 2 - 2\cos(n + 1 - i)\pi/n + 1$. The most powerful test, of course, is obtained by choosing $\lambda_i = 1/\text{Var}(X_i)$ [2], [5]. Inequality (i) says that any test of the form (3) is unbiased if the λ's are properly matched.

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