The Annals of Mathematical Statistics
1968, Vol. 39, No. 5, 1700–1710

HOW TO MINIMIZE OR MAXIMIZE THE PROBABILITIES OF EXTINCTION IN A GALTON-WATSON PROCESS AND IN SOME RELATED MULTIPLICATIVE POPULATION PROCESSES[1]

BY LEO A. GOODMAN

University of Chicago

1. Introduction. Let us consider a given population of N individuals who form, say, the 0th generation, who produce during their lifetime new individuals who form the 1st generation, who in turn produce during their lifetime new individuals who form the 2nd generation, who in turn produce during their lifetime new individuals who form the 3rd generation, etc. Let q_{ij} be the probability that an individual in the jth generation ($j = 0, 1, 2, \cdots$) will produce during his lifetime i new individuals ($i = 0, 1, 2, \cdots$) in the $(j + 1)$th generation. We assume that within each generation, given the past, individuals reproduce independently of one another, and that $\sum_{i=0}^{\infty} q_{ij} = 1$ for $j = 0, 1, 2, \cdots$. Let μ_j denote the mean number of new individuals produced during his lifetime by an individual in the jth generation; i.e., $\mu_j = \sum_{i=0}^{\infty} iq_{ij}$, for $j = 0, 1, 2, \cdots$. We call μ_j the Malthusian rate for the jth generation (see, e.g., Karlin (1966), p. 364). Let $\mathbf{m} = \{m_0, m_1, m_2, \cdots\}$ and $\mathbf{M} = \{M_0, M_1, M_2, \cdots\}$ denote two sequences of numbers which are such that

$$(1) \qquad m_j \leqq \mu_j \leqq M_j, \quad \text{for} \quad j = 0, 1, 2, \cdots.$$

For any given $\mathbf{m}$ and $\mathbf{M}$, we shall consider all possible values of q_{ij} which are such that Condition (1) is satisfied, and we shall answer the following four questions herein: (A) How can we minimize the probability that the jth generation ($j = 0, 1, 2, \cdots$) will become extinct? (B) How can we maximize the probability that the jth generation ($j = 0, 1, 2, \cdots$) will become extinct? (C) How can we minimize the probability of eventual extinction? (D) How can we maximize the probability of eventual extinction?

In a recent article, Freedman and Purves (1967) answered question (A) for the special case where it is assumed that

$$(2) \qquad q_{1j} = 0, \quad \text{for} \quad j = 0, 1, 2, \cdots,$$

and that Condition (1) is satisfied with $m_j = 0$ and $M_j = M < 2$ for $j = 0, 1, 2, \cdots$. The answer to question (A) under the special restriction (2) differs from the corresponding answer obtained herein when (2) is not assumed. In

Received 2 December 1967.
[1] This research was supported in part by Research Grant No. NSF GS 1905 from the Division of the Social Sciences of the National Science Foundation, and in part by the Army Research Office, Office of Naval Research, and Air Force Office of Scientific Research by Contract No. Nonr 2121(23), NR 342-043.

order to show the relationship between these two answers, we shall also answer herein a more general question which is presented below.

For each value of j ($j = 0, 1, 2, \cdots$), let H_j denote either the empty set or a given non-empty set of nonnegative integers. (In the following sections, we shall specify some further restrictions on the possible sets H_j that will be considered.) For a given $\mathbf{m}$ and $\mathbf{M}$, we shall consider all possible values of q_{ij} which are such that Condition (1) is satisfied and

$$(3) \qquad q_{hj} = 0 \quad \text{for} \quad h \varepsilon H_j, \quad \text{and} \quad j = 0, 1, 2, \cdots .$$

For these possible values of q_{ij}, we shall answer questions (A), (B), (C), and (D) given in the first paragraph above. By taking H_j as the empty set (for $j = 0, 1, 2, \cdots$), we note that the questions described in the first paragraph form a special case of those introduced in the present paragraph. By taking H_j as the set consisting of the integer 1 (for $j = 0, 1, 2, \cdots$), and setting $m_j = 0$ and $M_j = M < 2$ (for $j = 0, 1, 2, \cdots$), we also note that the question considered by Freedman and Purves (1967) is a special case of question (A) introduced in the present paragraph.

The process described in the first paragraph herein is a generalization of the Galton-Watson process. For the Galton-Watson process, it is usually assumed (see, e.g., Harris (1963), Chapter I) that $q_{ij} = q_{i\cdot}$ for all values of $j = 0, 1, 2, \cdots$. In other words, it is usually assumed that the q_{ij} do not depend upon the value of j; i.e., that the q_{ij} are not generation-dependent. In the present article, we shall consider both the case where the q_{ij} are generation-dependent (as described in the first paragraph above) and the case where they are not (as in the usual Galton-Watson process).

For a population of individuals whose growth can be described by a Galton-Watson process, the Malthusian rate μ (i.e., the mean number of new individuals produced by an individual during his lifetime) can be readily calculated from the appropriate demographic life-table and the age-specific birth-rates (see, e.g., Goodman (1967), (1968b)). When the demographic data are subject to error, a lower bound m and an upper bound M could also be calculated for the Malthusian rate μ (where $m \leqq \mu \leqq M$). For a given population for which the Malthusian rate μ, or the corresponding lower and upper bounds m and M, have been calculated, it would then seem natural to ask the following questions which we consider herein: (a)–(b) What are the smallest and largest possible values of the probability that the jth generation ($j = 0, 1, 2, \cdots$) will become extinct? (c)–(d) What are the smallest and largest possible values of the probability of eventual extinction?

For the study of the growth of populations of individuals, various kinds of multiplicative population processes may be of interest. Goodman (1967) showed how to calculate the probabilities of eventual extinction for three different kinds of multiplicative population processes which take explicitly into account the fact that the chances of death and the chances of giving birth, for a given individual in the population at time t, may depend upon his age at that time or upon the "phase" (or "stage") of life in which he is at that time. These three kinds of population processes were (i) a general age-dependent birth-and-death process in which the time-scale and the age-scale are discrete, (ii) a general multiphase birth-and-death process in which the time-scale is continuous and the phases (or stages) through which an individual proceeds during his lifetime are discrete, and (iii) a general age-dependent birth-and-death process in which the time-scale and the age-scale are continuous. Goodman (1967) showed, among other things, that the probability of eventual extinction satisfied an equation which was of the same form when considering the eventual extinction of the line of descent for an individual in the youngest age-interval in the population process (i), for an individual in the earliest phase of life in the population process (ii), for an individual who is newborn in the population process (iii), and for an individual in the usual Galton-Watson process. In other words, these particular probabilities of eventual extinction for the population processes (i), (ii) and (iii) satisfy an equation of the same form as the usual equation for the probability of eventual extinction in the Galton-Watson process. That is, in each of these cases the probability p of eventual extinction is the smallest root of the equation

$$p = \sum_{i=0}^{\infty} q_{i\cdot} p^{i}, \tag{4}$$

where $q_{i\cdot}$ is the probability that an individual will produce during his entire lifetime i new individuals. (In order to determine the smallest root of equation (4) for the population process (i), (ii), or (iii), it is actually not necessary to calculate first the values of the $q_{i\cdot}$ from the corresponding parameters of the population process, since simpler methods for determining the root are available without calculating the $q_{i\cdot}$; see Goodman (1967).)

Since the particular probabilities of eventual extinction which were considered in the preceding paragraph for the population processes (i), (ii), and (iii) satisfy equation (4) (which is the usual equation for the probability of eventual extinction in a Galton-Watson process), all the results presented herein pertaining to the minimization and the maximization of the probabilities of eventual extinction in a Galton-Watson process can be applied directly to the minimization and the maximization of the corresponding probabilities for the population processes (i), (ii), and (iii). A similar remark applies also to the population processes introduced in Goodman (1968a) for the study of the population growth of the two sexes both in the case where females are "marriage-dominant" and in the case where males are "marriage-dominant." For further discussion of the relationship between these two-sex population processes and the more usual one-sex population processes, see Goodman (1968a).

Before closing this section, we note that some of the questions considered herein and the answers obtained can be interpreted in a way that pertains to the determination of optimal strategies for certain kinds of gambling systems. Molenaar and van der Velde (1967) and Freedman (1967) showed that in order for a gambler to maximize the probability that he will survive for a fixed number of bets in a gambling house that offers certain kinds of fair and unfair bets, the optimal strategy would be to always make a certain kind of "timid" fair bet. On the other hand, Molenaar and van der Velde (1967) also noted that in a gambling house in which only certain kinds of unfair bets are offered (i.e., in a more realistic gambling house), "timid" betting need not be optimal and "bold" betting can be attractive. In contrast to these results, we provide the following interpretation for some of the results presented herein.

Let us suppose that a gambler wishes to make a fixed number k of consecutive visits to a gambling house in which each gamble on the $(j + 1)$th day ($j = 0, 1, \cdots, k - 1$) costs one dollar and in return the gambler receives i dollars ($i = 0, 1, 2, \cdots$) with probability q_{ij}. (Note that $\sum_{i=0}^{\infty} q_{ij} = 1$, and that the mean $\mu_j = \sum_{i=0}^{\infty} iq_{ij}$ is equal to one in a fair gamble.) The gambler plans to start with a fixed number N of dollars with which he will pay the cost of N gambles on the first day, and the returns he receives from his gambles on the jth day ($j = 1, 2, \cdots, k - 1$) will be used to pay the costs of the gambles on the $(j + 1)$th day. On the $(j + 1)$th day the gambling house offers a range of possible values of the q_{ij} which are such that Condition (1) is satisfied for some specified constants m_j and M_j, and Condition (3) is satisfied for some specified set H_j. (When $M_j < 1$, only unfair gambles are available; and when $M_j = 1$ both fair and unfair gambles are available if $m_j < 1$). From some of the results which we shall present herein, we find that, in order for the gambler to minimize the probability that he will go broke (i.e., maximize the probability that he will survive) during the k visits, the optimal strategy would be to always make a certain kind of "timid" unfair bet in the case where the gambling house offers only unfair bets, and to make a certain kind of "timid" fair bet in the case where the gambling house offers both fair and unfair bets. Furthermore, in this context, the worst possible strategy for the gambler would be to make a certain kind of "bold" bet; i.e., it would maximize the probability that the gambler would go broke.

2. Two lemmas. Our main results depend upon the following two lemmas. The first lemma is a generalization of the corresponding result given by Freedman and Purves (1967).

LEMMA 1. *Let g and M denote given constants, with $0 < g < 1$ and $M \geqq 0$. Let $\bar{H}$ denote a given set of nonnegative integers, which includes at least one integer $\geqq M$ and at least one integer $\leqq M$. Let M^* denote the smallest integer in $\bar{H}$ such that $M^* \geqq M$, and let M' denote the largest integer in $\bar{H}$ such that $M' \leqq M$. Let $D = M^* - M'$. Let Y be a random variable that takes on the values M' and M^* with probabilities $(M^* - M)/D$ and $(M - M')/D$, respectively. (When $M = M^* = M'$, then Y takes on the value M with probability one.) Let X be any random variable that takes on values in $\bar{H}$ and that has a mean value $\mu \leqq M$. Then*

$$E\{g^X\} \geqq E\{g^Y\}. \tag{5}$$

PROOF. In the case where $D = 0$, we make use of Jensen's inequality (see, e.g., Zygmund (1959), p. 21) and the fact that g^{μ} is a decreasing function of μ. Thus, $E\{g^X\} \geqq g^{\mu} \geqq g^M = E\{g^Y\}$. Now let us consider the case where $D > 0$.

Let $f(x) = [(M^* - x)g^{M'} + (x - M')g^{M^*}]/D$. Then $f(x) = g^x$ for $x = M^*$ and $x = M'$; and $f(x) < g^x$ for all other values of $x \varepsilon \bar{H}$. Note that $f(x)$ is a decreasing linear function of x. Thus,

$$E\{g^X\} \geqq E\{f(X)\} = f(E\{X\}) = f(\mu) \geqq f(M) = E\{g^Y\}. \qquad \square$$

LEMMA 2. *Let g and m denote given constants, with $0 < g < 1$ and $m \geqq 0$. Let T denote a given integer which is $\geqq m$. Let $\bar{H}^T$ denote a given subset of the integers $0, 1, 2, \cdots, T$, which includes at least one integer $\geqq m$ and at least one integer $\leqq m$. Let m^* denote the largest integer in $\bar{H}^T$, and let m' denote the smallest integer in $\bar{H}^T$. Let $d = m^* - m'$. Let Z denote a random variable that takes on the values m' and m^* with probabilities $(m^* - m)/d$ and $(m - m')/d$, respectively. (When $m = m^* = m'$, then Z takes on the value m with probability one.) Let X be any random variable that takes on values in $\bar{H}^T$ and that has a mean value $\mu \geqq m$. Then*

$$E\{g^X\} \leqq E\{g^Z\}. \tag{6}$$

PROOF. The case where $d = 0$ is trivial. Now let us consider the case where $d > 0$. Let

$$b(x) = [(m^* - x)g^{m'} + (x - m')g^{m^*}]/d.$$

Then $b(x) = g^x$ for $x = m^*$ and $x = m'$; and $b(x) > g^x$ for all other values of $x \varepsilon \bar{H}^T$. Note that $b(x)$ is a decreasing linear function of x. Thus,

$$E\{g^X\} \leqq E\{b(X)\} = b(E\{X\}) = b(\mu) \leqq b(m) = E\{g^Z\}. \qquad \square$$

Note that the inequalities (5) and (6) remain true in a trivial way when $g = 1$ or $g = 0$. (We can also obtain the inequalities (5) and (6) for $g = 0$ or $g = 1$ by continuity by taking $g \to 0$ or $g \to 1$.)

The set $\bar{H}$ defined in Lemma 1 can be interpreted as the set of possible numbers of new individuals produced by a given individual; and the set $\bar{H}^T$ defined in Lemma 2 can be interpreted similarly. The corresponding set $\bar{H}_j$ (for $j = 0, 1, 2, \cdots$), which we shall introduce in Section 3 (The minimization problem), can be interpreted as the set of possible numbers of new individuals in the $(j + 1)$th generation produced by a given individual in the jth generation; and the set $\bar{H}_j^T$ (for $j = 0, 1, 2, \cdots$), which we shall introduce in Section 4 (The maximization problem), can be interpreted similarly. The set H_j, which was referred to in the introduction herein, restricts by Condition (3) the possible numbers of new individuals in the $(j + 1)$th generation produced by a given individual in the jth generation. For the minimization problem, the set H_j to be considered is the set of nonnegative integers that are included in the complement of $\bar{H}_j$. For the maximization problem, the set H_j to be considered is the set of nonnegative integers that are included in the complement of $\bar{H}_j^T$.

We shall now make use of the above lemmas to prove our main results concerning the minimization and maximization of the probability that the kth generation $(k = 0, 1, \cdots)$ will become extinct.

3. The minimization problem. Let $\mathbf{M} = \{M_0, M_1, M_2, \cdots\}$ denote a sequence of numbers, with $M_j \geqq 0$ for $j = 0, 1, 2, \cdots$. For each value of j, let $\bar{H}_j$ denote a given set of nonnegative integers which includes at least one integer $\geqq M_j$ and at least one integer $\leqq M_j$. Let W_j be an integer-valued random variable for $j = 0, 1, 2, \cdots$. For a given nonnegative integer N and a given sequence $\mathbf{M}$, the sequence of random variables $W_0, W_1, W_2, \cdots$ will be called a $(N, \mathbf{0}, \mathbf{M})$-process if $W_0 = N$ and for all $j \geqq 0$, given $W_0, W_1, \cdots, W_j$, the conditional distribution of W_{j+1} is the sum of W_j independent random variables, each taking on values in $\bar{H}_j$ and having a mean $\mu_j \leqq M_j$.

Let M_j^* denote the smallest integer in $\bar{H}_j$ such that $M_j^* \geqq M_j$, and let M_j' denote the largest integer in $\bar{H}_j$ such that $M_j' \leqq M_j$. Let $D_j = M_j^* - M_j'$. Let us now consider the particular $(N, \mathbf{0}, \mathbf{M})$-process $(\bar{W}_0, \bar{W}_1, \bar{W}_2, \cdots)$ in which, given $\bar{W}_0, \bar{W}_1, \cdots, \bar{W}_j$, the conditional distribution of $\bar{W}_{j+1}$ is the sum of $\bar{W}_j$ independent random variables each taking on the values M_j' and M_j^* with probability $(M_j^* - M_j)/D_j$ and $(M_j - M_j')/D_j$, respectively. For this particular process, we find that

$$P\{\bar{W}_j = 0\} = [g(\mathbf{M}, j)]^N, \tag{7}$$

where

$$g(\mathbf{M}, 0) = 0,$$

and

$$g(\mathbf{M}, j + 1) = \{[g(\mathbf{M}^+, j)]^{M_0'}(M_0^* - M_0) + [g(\mathbf{M}^+, j)]^{M_0^*}(M_0 - M_0')\}/D_0,$$

with $\mathbf{M}^+ = \{M_1, M_2, M_3, \cdots\}$. (Note that $g(\mathbf{M}, j + 1)$ is the probability $P\{\bar{W}_{j+1} = 0\}$ for a $(1, \mathbf{0}, \mathbf{M})$-process $(\bar{W}_0, \bar{W}_1, \bar{W}_2, \cdots)$, and $g(\mathbf{M}^+, j)$ is the conditional probability that $\bar{W}_{j+1} = 0$ given that $\bar{W}_1 = 1$.)

Let K_j denote the set of integers $0, 1, \cdots, j - 1$, for $j \geqq 1$. For $j = 1, 2, 3, \cdots$, we note that $g(\mathbf{M}, j) = 0$ if $M_k' > 0$ for all $k \in K_j$; that $g(\mathbf{M}, j) = 1$ if $M_k = 0$ for at least one value of $k \in K_j$; and that $0 < g(\mathbf{M}, j) < 1$ otherwise.

We now prove the following theorem which is a direct extension of the Freedman-Purves (1967) result.

THEOREM 1. *For any $(N, \mathbf{0}, \mathbf{M})$-process $(W_0, W_1, W_2, \cdots)$, and for any nonnegative integer j, the probability that $W_j = 0$ is at least $[g(\mathbf{M}, j)]^N$.*

PROOF. We proceed by induction on j. For $j = 0$ the theorem is trivial. Suppose the theorem holds true for some value of $j > 0$ for all $(N, \mathbf{0}, \mathbf{M})$-processes. Considering the $(N, \mathbf{0}, \mathbf{M})$-process $(W_0, W_1, W_2 \cdots)$, we see that, given W_1, the process $(W_1, W_2, \cdots)$ is a $(W_1, \mathbf{0}, \mathbf{M}^+)$-process, where $\mathbf{M}^+ = \{M_1, M_2, \cdots\}$. Thus

$$P\{W_{j+1} = 0 \mid W_1\} \geqq [g(\mathbf{M}^+, j)]^{W_1},$$

and therefore

$$P\{W_{j+1} = 0\} \geqq E\{[g(\mathbf{M}^+, j)]^{W_1}\}.$$

Now W_1 is the sum of N independent random variables X_i (for $i = 1, 2, \cdots, N$), each taking on values in $\bar{H}_0$ and having mean $\mu_0 \leqq M_0$. From Lemma 1 we see that

$$E\{[g(\mathbf{M}^+, j)]^{X_i}\} \geqq E\{[g(\mathbf{M}^+, j)]^Y\},$$

where Y takes on the values M_0' and M_0^* with probabilities $(M_0^* - M_0)/D_0$ and $(M_0 - M_0')/D_0$, respectively, where $D_0 = M_0^* - M_0'$. Since $E\{g(\mathbf{M}^+, j)^Y\} = g(\mathbf{M}, j + 1)$, we obtain

$$P\{W_{j+1} = 0\} \geqq [g(\mathbf{M}, j + 1)]^N. \qquad \square$$

Since $[g(\mathbf{M}, j)]^N$ is the probability of extinction $P\{\bar{W}_j = 0\}$ for the particular $(N, \mathbf{0}, \mathbf{M})$-process $(\bar{W}_0, \bar{W}_1, \bar{W}_2, \cdots)$ defined earlier herein, Theorem 1 states that the corresponding probability of extinction for any other $(N, \mathbf{0}, \mathbf{M})$-process is at least $P\{\bar{W}_j = 0\}$. Recall that for the particular $(N, \mathbf{0}, \mathbf{M})$-process $(\bar{W}_0, \bar{W}_1, \cdots)$, each individual in the jth generation produces a random number Y of new individuals in the $(j + 1)$th generation, where Y takes on the values M_j' and M_j^* with probability $(M_j^* - M_j)/D_j$ and $(M_j - M_j')/D_j$, respectively.

In the special case where $\bar{H}_j = \bar{H}$ (for $j = 0, 1, 2, \cdots$) is the set of nonnegative integers excluding the integer 1, and $M_j = M$ (for $j = 0, 1, 2, \cdots$) is a constant such that $0 \leqq M \leqq 2$, then $M' = 0$, $M^* = 2$, and Y is the random variable taking on the values 0 and 2 with probabilities $1 - (M/2)$ and $(M/2)$, respectively. This agrees with the Freedman-Purves (1967) result.

In the special case where $\bar{H}_j = \bar{H}$ (for $j = 0, 1, 2, \cdots$) is the set of nonnegative integers, then M_j' is the largest integer $\leqq M_j$, and M_j^* is the smallest integer $\geqq M_j$, for $j = 0, 1, 2, \cdots$. Thus, in this special case, $D_j = 1$ (for $j = 0, 1, 2, \cdots$), unless $M_j = M_j^* = M_j'$ (i.e., unless M_j is a nonnegative integer). In particular, when $M_j = M$ (for $j = 0, 1, 2, \cdots$) is a constant such that $0 \leqq M \leqq 1$, then $M' = 0$ and $M^* = 1$ (unless $M' = M^*$), and Y is the random variable taking on the values 0 and 1 with probabilities $1 - M$ and M, respectively. Similarly, when M is such that $1 \leqq M \leqq 2$, then $M' = 1$ and $M^* = 2$ (unless $M' = M^*$), and Y takes on the values 1 and 2 with probabilities $2 - M$ and $M - 1$, respectively. More generally, when the $\bar{H}_j$ and the M_j are such that $D_j = 1$, then Y takes on the values M_j' and M_j^* (for $M_j^* = M_j' + 1$) with probabilities $M_j^* - M_j$ and $M_j - M_j'$, respectively.

In the special case where $\bar{H}_j = \bar{H}$ and $M_j = M$ (for $j = 0, 1, 2, \cdots$), the $(N, \mathbf{0}, \mathbf{M})$-process $(\bar{W}_0, \bar{W}_1, \cdots)$ is a Galton-Watson process. From Theorem 1 we see that the corresponding probability of extinction for any other Galton-Watson process, which has N individuals in the 0th generation and has a Malthusian rate $\leqq M$, will be at least $P\{\bar{W}_j = 0\}$.

It should be noted that, in general, there may be other $(N, \mathbf{0}, \mathbf{M})$-processes $(W_0, W_1, \cdots)$, in addition to the particular $(N, \mathbf{0}, \mathbf{M})$-process $(\bar{W}_0, \bar{W}_1, \cdots)$, for which $P\{W_j = 0\} = [g(\mathbf{M}, j)]^N$. For example, in the particular case where $\bar{H}_j = H$ (for $j = 0, 1, 2, \cdots$) is the set of nonnegative integers, and $M_j = M$ (for $j = 0, 1, 2, \cdots$) is a constant such that $1 \leqq M \leqq 2$, then $P\{W_j = 0\} = P\{\bar{W}_j = 0\} = 0$ for any $(N, \mathbf{0}, \mathbf{M})$-process $(W_0, W_1, \cdots)$ in which each individual in the jth generation ($j = 0, 1, 2, \cdots$) produces at least one new individual in the $(j + 1)$th generation (with probability 1). More generally, when the $\bar{H}_j$ and the M_j are such that $M_j' \geqq 1$ for $j \geqq 0$, then $P\{W_j = 0\} = P\{\bar{W}_j = 0\} = 0$ for any $(N, \mathbf{0}, \mathbf{M})$-process $(W_0, W_1, \cdots)$ in which each individual in the jth generation ($j = 0, 1, 2, \cdots$) produces at least one new individual in the $(j + 1)$th generation (with probability 1). On the other hand, by a slightly more detailed analysis following along the lines of the proof of Theorem 1, we find that when the $\bar{H}_j$ and the M_j are such that $M_j' = 0$ and $M_j > 0$ for all $j \geqq 0$, then the probability of extinction $P\{W_k = 0\}$, for any $(N, \mathbf{0}, \mathbf{M})$-process $(W_0, W_1, \cdots)$, is greater than $P\{\bar{W}_k = 0\}$ for $k \geqq 1$, unless the first $k + 1$ terms $W_0, W_1, \cdots, W_k$ are distributed like $\bar{W}_0, \bar{W}_1, \cdots, \bar{W}_k$.

Let $P_k = P\{W_k = 0\}$ for any $(N, \mathbf{0}, \mathbf{M})$-process $(W_0, W_1, \cdots)$, and let $\bar{P}_k = P\{\bar{W}_k = 0\}$. Since $P_0, P_1, P_2, \cdots$ and $\bar{P}_0, \bar{P}_1, \bar{P}_2, \cdots$ are both nondecreasing sequences which are bounded from above, each sequence will have a limit. Let $P = \lim P_k$ and $\bar{P} = \lim \bar{P}_k$. By an argument similar to that presented by Harris (1963), p. 7, we see that

$$P = P\{W_k \to 0\} = P\{W_k = 0 \quad \text{for some} \quad k\},$$

and a similar result applies to $\bar{P}$. Thus, P and $\bar{P}$ are the probabilities of eventual extinction for the processes $(W_0, W_1, \cdots)$ and $(\bar{W}_0, \bar{W}_1, \cdots)$, respectively. By applying Theorem 1, we see that $P \geqq \bar{P}$. In other words, the probability of eventual extinction for any $(N, \mathbf{0}, \mathbf{M})$-process is at least $\bar{P}$, which is the probability of eventual extinction for the particular $(N, \mathbf{0}, \mathbf{M})$-process $(\bar{W}_0, \bar{W}_1, \cdots)$ defined earlier herein.

Before closing this section, let us again consider the special case where $\bar{H}_j = \bar{H}$ and $M_j = M$ (for $j \geqq 0$). We noted earlier that the $(N, \mathbf{0}, \mathbf{M})$-process $(\bar{W}_0, \bar{W}_1, \cdots)$ is a Galton-Watson process in this special case. Therefore, the eventual-extinction probability $\bar{P}$ can be calculated by the usual equation (4) for the Galton-Watson process (see, e.g., Harris (1963), p. 7). The results in the preceding paragraph indicate that the probability of eventual extinction for any Galton-Watson process, which has N individuals in the 0th generation and has a Malthusian rate $\leqq M$, will be at least $\bar{P}$. (A more direct proof of this result can be obtained by applying Lemma 1 to the usual equation (4) for calculating the eventual-extinction probability for the Galton-Watson process.) If $M' \geqq 1$, the eventual-extinction probability can be reduced to zero as noted earlier herein; if $M' = 0$ and $M < 1$, the eventual-extinction probability will be one for any Galton-Watson process; if $M' = 0$ and $M \geqq 1$, the eventual-extinction probability is minimized only for the $(N, \mathbf{0}, \mathbf{M})$-process $(\bar{W}_0, \bar{W}_1, \cdots)$.

From the remarks in the introduction herein concerning the population processes (i), (ii), and (iii) which were considered in Goodman (1967), and the two-sex population processes which were considered in Goodman (1968a) for the case where females or males are "marriage-dominant", we see that the results presented in the preceding paragraph can be applied directly to each of these processes as well as to the Galton-Watson process.

4. The maximization problem. Let $\mathbf{m} = \{m_0, m_1, m_2, \cdots\}$ denote a sequence of numbers, with $m_j \geqq 0$ for $j = 0, 1, 2, \cdots$. For each value of j, let T_j denote a given nonnegative integer ($T_j \geqq m_j$), and let $\bar{H}_j^T$ denote a subset of the integers $0, 1, \cdots, T_j$ which includes at least one integer $\geqq m_j$ and at least one integer $\leqq m_j$. Let V_j be an integer-valued random variable, for $j = 0, 1, 2, \cdots$. For a given nonnegative integer N and a given sequence $\mathbf{m}$, the sequence of random variables $V_0, V_1, V_2, \cdots$ will be called a $(N, \mathbf{m}, \infty)$-process if $V_0 = N$ and for all $j \geqq 0$, given $V_0, V_1, \cdots, V_j$, the conditional distribution of V_{j+1} is the sum of V_j independent random variables, each taking on values in $\bar{H}_j^T$ and having a mean $\mu_j \geqq m_j$.

Let m_j^* denote the largest integer in $\bar{H}_j^T$, and let m_j' denote the smallest integer in $\bar{H}_j^T$. Let $d_j = m_j^* - m_j'$. Let us now consider the particular $(N, \mathbf{m}, \infty)$ process $(\bar{V}_0, \bar{V}_1, \bar{V}_2, \cdots)$ in which, given $\bar{V}_0, \bar{V}_1, \cdots, \bar{V}_j$, the conditional distribution of $\bar{V}_{j+1}$ is the sum of $\bar{V}_j$ independent random variables each taking on the values m_j' and m_j^* with probability $(m_j^* - m_j)/d_j$ and $(m_j - m_j')/d_j$, respectively. For this particular process, we find that

$$P\{\bar{V}_j = 0\} = [\gamma(\mathbf{m}, j)]^N,$$

where

$$\gamma(\mathbf{m}, 0) = 0, \tag{8}$$

and

$$\gamma(\mathbf{m}, j + 1) = \{[\gamma(\mathbf{m}^+, j)]^{m_0'}(m_0^* - m_0) + [\gamma(\mathbf{m}^+, j)]^{m_0^*}(m_0 - m_0')\}/d_0,$$

with $\mathbf{m}^+ = \{m_1, m_2, m_3, \cdots\}$. (Note that $\gamma(\mathbf{m}, j + 1)$ is the probability $P(\bar{V}_{j+1} = 0)$ for a $(1, \mathbf{m}, \infty)$-process $(\bar{V}_0, \bar{V}_1, \bar{V}_2, \cdots)$, and $\gamma(\mathbf{m}^+, j)$ is the conditional probability that $\bar{V}_{j+1} = 0$ given that $\bar{V}_1 = 1$.)

Let K_j denote the set of integers $0, 1, \cdots, j - 1$, for $j \geqq 1$. For $j = 1, 2, 3, \cdots$, we note that $\gamma(\mathbf{m}, j) = 0$ if $m_k' > 0$ for all $k \,\varepsilon\, K_j$; that $\gamma(\mathbf{m}, j) = 1$ if $m_k = 0$ for at least one $k \,\varepsilon\, K_j$; and that $0 < \gamma(\mathbf{m}, j) < 1$ otherwise.

We now introduce the following:

THEOREM 2. *For any $(N, \mathbf{m}, \infty)$-process $(V_0, V_1, V_2, \cdots)$, and for any nonnegative integer j, the probability that $V_j = 0$ is at most $[\gamma(\mathbf{m}, j)]^N$.*

PROOF. The method of proving Theorem 1 can be directly applied here, except that Lemma 2 (rather than Lemma 1) would be used in the proof. The details are left to the interested reader.

Since $\gamma(\mathbf{m}, j)^N$ is the probability of extinction for the $(N, \mathbf{m}, \infty)$-process $(\bar{V}_0, \bar{V}_1, \bar{V}_2, \cdots)$ defined earlier herein, Theorem 2 states that the probability of extinction for any other $(N, \mathbf{m}, \infty)$-process is at most $P\{\bar{V}_j = 0\}$. Recall that for the particular $(N, \mathbf{m}, \infty)$-process $(\bar{V}_0, \bar{V}_1, \bar{V}_2, \cdots)$, each individual in the jth generation produces a random number Z of new individuals in the $(j + 1)$th generation, where Z takes on the values m_j' and m_j^* with probability $(m_j^* - m_j)/d_j$ and $(m_j - m_j')/d_j$, respectively.

In the special case where $\bar{H}_j^T = H^T$ and $m_j = m$ (for $j = 0, 1, 2, \cdots$), the

$(N, \mathbf{m}, \boldsymbol{\infty})$-process $(\bar{V}_0, \bar{V}_1, \cdots)$ is a Galton-Watson process. From Theorem 2 we see that the corresponding probability of extinction for any other Galton-Watson process, which has N individuals in the 0th generation and has a Malthusian rate $\geqq m$, will be at most $P\{\bar{V}_j = 0\}$.

It should be noted that, in general, there may be other $(N, \mathbf{m}, \boldsymbol{\infty})$-processes $(V_0, V_1, \cdots)$, in addition to the particular $(N, \mathbf{m}, \boldsymbol{\infty})$-process $(\bar{V}_0, \bar{V}_1, \cdots)$, for which $P(V_j = 0) = [\gamma(\mathbf{m}, j)]^N$. For example, when the $\bar{H}_j^T$ and the m_j are such that $m_j' \geqq 1$ for $j \geqq 0$, then $P(V_j = 0) = P(\bar{V}_j = 0) = 0$ for any $(N, \mathbf{m}, \boldsymbol{\infty})$-process $(V_0, V_1, \cdots)$. On the other hand, we find that when the $\bar{H}_j^T$ and the m_j are such that $m_j' = 0$ and $m_j > 0$ for all $j \geqq 0$, then the probability of extinction $P(V_k = 0)$, for any $(N, \mathbf{m}, \boldsymbol{\infty})$-process $(V_0, V_1, \cdots)$, is less than $P(\bar{V}_k = 0)$ for $k \geqq 1$, unless the first $k + 1$ terms $V_0, V_1, \cdots, V_k$ are distributed like $\bar{V}_0, \bar{V}_1, \cdots, \bar{V}_k$.

Let $p_k = P\{V_k = 0\}$ for any $(N, \mathbf{m}, \boldsymbol{\infty})$-process $(V_0, V_1, \cdots)$, and let $\bar{p}_k = P\{\bar{V}_k = 0\}$. As in the preceding section, we find that the two sequences $p_0, p_1, p_2, \cdots$ and $\bar{p}_0, \bar{p}_1, \bar{p}_2, \cdots$ have limits, say, p and $\bar{p}$, respectively; and that these limits are the probabilities of eventual extinction for the processes $(V_0, V_1, \cdots)$ and $(\bar{V}_0, \bar{V}_1, \cdots)$, respectively. By applying Theorem 2, we see that $p \leqq \bar{p}$. In other words, the probability of eventual extinction for any $(N, \mathbf{m}, \boldsymbol{\infty})$-process is at most $\bar{p}$, which is the probability of eventual extinction for the particular $(N, \mathbf{m}, \boldsymbol{\infty})$-process $(\bar{V}_0, \bar{V}_1 \cdots)$.

Let us again consider the special case where $\bar{H}_j^T = \bar{H}^T$ and $m_j = m$ (for $j \geqq 0$). We noted earlier that the $(N, \mathbf{m}, \boldsymbol{\infty})$-process $(\bar{V}_0, \bar{V}_1, \cdots)$ is a Galton-Watson process in this special case. Therefore, the eventual-extinction probability $\bar{p}$ can be calculated by the usual equation (4) for the Galton-Watson process. The eventual-extinction probability for any Galton-Watson process, which has N individuals in the 0th generation and has a Malthusian rate $\geqq m$, will be at most $\bar{p}$. (This result can be obtained from the results given in the preceding paragraph, or more directly by applying Lemma 2 to the usual equation (4) for calculating the eventual-extinction probability for the Galton-Watson process.) If $m' \geqq 1$, the eventual-extinction probability will be zero; if $m' = 0$ and $m \leqq 1$, the eventual-extinction probability will be one for any Galton-Watson process having a Malthusian rate $\leqq 1$ (except for the process in which each individual in the jth generation produces during his lifetime one new individual who is in the $(j + 1)$th generation; i.e., in which $q_{1j} = 1$ for $j = 0, 1, 2, \cdots$); if $m' = 0$ and $m > 1$, the eventual-extinction probability will be maximized only for the $(N, \mathbf{m}, \boldsymbol{\infty})$-process $(\bar{V}_0, \bar{V}_1, \cdots)$.

As earlier herein, we note that the results presented in the preceding paragraph can be applied directly to the population processes (i), (ii), and (iii), and to the two-sex population processes, to which we referred in the introduction herein, as well as to the Galton-Watson process.

The results presented in this section pertain to certain kinds of population processes for which the Malthusian rate $\mu_j \geqq m_j$, for $j = 0, 1, \cdots$. These results will continue to hold for the subset of these population processes for which

$M_j \geqq \mu_j \geqq m_j$ (for $j = 0, 1, \cdots$), where $\mathbf{M}$ is any sequence for which $M_j \geqq m_j$ ($j = 0, 1, \cdots$). Similarly, the results presented in the preceding section, which pertain to certain kinds of population processes for which the Malthusian rate $\mu_j \leqq M_j$ (for $j \geqq 0$), will continue to hold for the subset of these population processes for which $m_j \leqq \mu_j \leqq M_j$ (for $j \geqq 0$), where $\mathbf{m}$ is any sequence for which $m_j \leqq M_j$ (for $j \geqq 0$).

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