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# SOME MULTIVARIATE $t$-DISTRIBUTIONS[1]

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**1. Introduction.** It is our objective in this paper to consider various distributions which may be called "generalized $t$-distributions". Certain types of generalized $t$-distributions (and some of their applications) have been investigated in [1]–[8]. We have derived, under various hypotheses, many different frequency functions which could be called generalized $t$-distributions. However, if the distribution functions are complicated, for example, if they are represented as an infinite series of higher transcendental functions, then their practical usefulness is limited. With this in mind we have restricted ourselves to reporting only those situations where we have been able to obtain the density function in closed form.

Let $X, X_1, X_2, \cdots, X_p$ be Gaussian vectors of dimensions $p, n_1, n_2, \cdots, n_p$ respectively. Let $t_k = x_k/r_k$, $1 \leqq k \leqq p$, where $X = \{x_1, \cdots, x_p\}$ and $r_k = |X_k|$, $1 \leqq k \leqq p$, is the norm of $X_k$. Then $T = \{t_1, \cdots, t_p\}$ will be called a *generalized $p$-dimensional $t$ random vector*. Of course, in general, $X, X_1, X_2, \cdots, X_p$ will be correlated. Many "generalized $t$ variates" we have observed in the literature may be subsumed under the above definition.

**2. Multivariate $t$-distributions.** In the following theorems we shall use the standard notation $\Gamma(\cdot)$, $B(\cdot, \cdot)$, $K(\cdot)$, $E(\cdot)$, $D_{-\nu}(\cdot)$, ${}_1F_1(\cdot, \cdot; \cdot)$, ${}_2F_1(\cdot, \cdot, \cdot; \cdot)$ to denote, respectively, the gamma function, the beta function, the complete elliptic integral of the first kind, the complete elliptic integral of the second kind, the parabolic cylinder function, the confluent hypergeometric function, and the hypergeometric function. If $\Sigma$ is a square matrix, we shall write $|\Sigma|$ to indicate the determinant of $\Sigma$. Also, if $Y = \{y_1, \cdots, y_n\}$ and $Z = \{z_1, \cdots, z_n\}$ are $n$-dimensional random vectors with $\{y_k, z_k\}$, $1 \leqq k \leqq n$, independent and identically distributed normal $N(0, M)$, then we shall say $\{Y, Z\}$ is of class $N_n(M)$.

Our main technique is simple. We write the density function of our $t$ random vector as a multiple integral and apply our knowledge of special functions (see, for example, [9]) to reduce the integrals to closed form. We shall merely sketch the proofs. In particular, we shall have a number of occasions to consider various special cases of the integral:

$$J(a, b, c, n, m) \equiv \int_0^\infty \int_0^\infty x^n y^m e^{-\frac{1}{2}(ax^2+by^2)} e^{-cxy}\, dx\, dy. \tag{2.1}$$

The results we need are expressed in the following lemma.

LEMMA. *Let $a$ and $b$ be positive, $c$ a real number, and $n$ a nonnegative integer.*

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*Let $c^2 < ab$. Then*

$$(2.2)\quad J(a, b, c, n, n) = \Gamma(n+1)B(\tfrac{1}{2}, n+1)(c+(ab)^{\frac{1}{2}})^{-(n+1)} \cdot {}_2F_1\,(n+1, \tfrac{1}{2}, n+\tfrac{3}{2}; (c-(ab)^{\frac{1}{2}})/(c+(ab)^{\frac{1}{2}}))$$

$$(2.3)\quad J(a, b, c, n, n) = \tfrac{1}{2}(-1)^{n+1}(\partial^{n+1}/\partial c^{n+1})(\text{arc cos } c/(ab)^{\frac{1}{2}})^2,$$

$$(2.4)\quad J(a, b, 0, n, n) = 2^{n-1}(ab)^{-\frac{1}{2}(n+1)}\Gamma^2(\tfrac{1}{2}(n+1)).$$

PROOF. If we make the change of variable $x = u$, $y = t/u$, and let $m = n$ in (2.1), then the integral with respect to $u$ may be evaluated [10], page 114, leaving

$$(2.5)\quad J(a, b, c, n, n) = \int_0^\infty t^n e^{-ct} K_0(t(ab)^{\frac{1}{2}})\, dt$$

where $K_0(\cdot)$ is the modified Bessel function of the second kind and order zero. A reference to [9], page 50, reduces (2.5) to (2.2), and a reference to [10], page 115, reduces (2.5) to (2.3). We obtain (2.4) directly from (2.1) by letting $c = 0$ and $m = n$.

Our first theorem concerns bivariate $t$-distributions.

THEOREM 1. *Let $X = \{x_1, x_2\}$ be normal $N(0, \Sigma)$ and let $\{Y, Z\}$ be of class $N_n(M)$ where $X$ is independent of $\{Y, Z\}$. Let $U = \{u_1, u_2\}$ where $u_1 = x_1/|Y|$, and $u_2 = x_2/|Z|$.*

(i) *If $x_1$ and $x_2$ are independent, then the density function of $U$ is*

$$(2.6)\quad h(U) = [B^2(\tfrac{1}{2}n, \tfrac{1}{2})(\alpha\beta\,|\Sigma|)^{\frac{1}{2}}(\alpha\beta\,|M|)^{\frac{1}{2}n}]^{-1}\,{}_2F_1\,(\tfrac{1}{2}(n+1), \tfrac{1}{2}(n+1), \tfrac{1}{2}n; \lambda^2).$$

(ii) *If $Y$ and $Z$ are independent, then the density function of $U$ is*

$$(2.7)\quad h(U) = nB(n+1, \tfrac{1}{2})[B(\tfrac{1}{2}n, \tfrac{1}{2}n)\pi 2^{n-1}\,|\Sigma|^{\frac{1}{2}}\,|M|^{\frac{1}{2}n}]^{-1} \cdot(\gamma+(\alpha\beta)^{\frac{1}{2}})^{-(n+1)}\,{}_2F_1\,(n+1, \tfrac{1}{2}, \tfrac{1}{2}(2n+3); (\gamma-(\alpha\beta)^{\frac{1}{2}})/(\gamma+(\alpha\beta)^{\frac{1}{2}})).$$

(iii) *If in* (i) *we let $n = 2$, then*

$$(2.8)\quad h(U) = [2\pi\,|\Sigma|^{\frac{1}{2}}\,|M|(\alpha\beta)^{3/2}(1-\lambda^2)^2]^{-1}[2E(\lambda) - (1-\lambda^2)K(\lambda)].$$

(iv) *If in* (ii) *we let $n = 2$, then*

$$(2.9)\quad h(U) = -[4\pi\,|\Sigma|^{\frac{1}{2}}\,|M|]^{-1}(\partial^3/\partial\gamma^3)(\text{arc cos } \gamma/(\alpha\beta)^{\frac{1}{2}})^2.$$

*In the above formulas*

$$\alpha = \phi_{11}u_1^2 + w_{11}, \qquad \beta = \phi_{22}u_2^2 + w_{22},$$

$$\gamma = \phi_{12}u_1u_2, \qquad \lambda^2 = w_{12}^2/\alpha\beta$$

*where $M^{-1} = \|w_{ij}\|_{1\leq i,j\leq 2}$ and $\Sigma^{-1} = \|\phi_{ij}\|_{1\leq i,j\leq 2}$.*

PROOF. We may write

$$(2.10)\quad h(U) = \int_0^\infty \int_0^\infty rsg(r, s)f(ru_1, su_2)\, dr\, ds$$

where $f(\cdot, \cdot)$ is the Gaussian density function and $g(\cdot, \cdot)$ is the Rayleigh density function [10], page 34. If we expand the Bessel function that appears in $g(\cdot, \cdot)$, then (2.10) reduces to

$$(2.11)\quad h(U) = [\pi 2^{n-1}\Gamma(\tfrac{1}{2}n)|\Sigma|^{\frac{1}{2}}|M|^{\frac{1}{2}n}]^{-1} \cdot \textstyle\sum_{k=0}^{\infty}[k!\,\Gamma(\tfrac{1}{2}n+k)]^{-1}(\tfrac{1}{2}w_{12})^{2k}J(\alpha,\beta,\gamma,n+2k,n+2k).$$

Now (2.4) reduces (2.11) to (2.6), and (2.2) reduces (2.11) to (2.7). The identity

$${}_2F_1(\tfrac{3}{2},\tfrac{3}{2},1;\lambda^2) = 2(\pi(1-\lambda^2)^2)^{-1}[2E(\lambda)-(1-\lambda^2)K(\lambda)]$$

reduces (2.6) to (2.8) and (2.3) reduces (2.11) to (2.9).

For the density functions of (2.6) and (2.7) we may compute the moments $\mathcal{E}u_1^p u_2^q$ (where $p$ and $q$ are nonnegative integers less than $n$) in terms of the hypergeometric function.

THEOREM 2. *Let $X = \{x_1, \cdots, x_p\}$ be normal $N(A,\Sigma)$ and let $Y$ be an $n$-dimensional vector normal $N(B, mI)$ where $I$ is the $n \times n$ identity matrix. Let $X$ and $Y$ be independent. Let $T = \{t_1, \cdots, t_p\}$ where $t_k = x_k/|Y|$, $1 \leqq k \leqq p$.*

(i) *If $|Y|^2$ is central chi-square, then the density function of $T$ is*

$$(2.12)\quad h(T) = 2w^{\frac{1}{2}n}[\Gamma(n+p)/\Gamma(\tfrac{1}{2}n)][2^{\frac{1}{2}(n+p)}\pi^{\frac{1}{2}p}|\Sigma|^{\frac{1}{2}}(T'\Phi T+w)^{\frac{1}{2}(n+p)}]^{-1} \cdot e^{-\frac{1}{2}A'\Phi A}e^{(T'\Phi A)^2/4(T'\Phi T+w)}D_{-(n+p)}(-T'\Phi A/(T'\Phi T+w)^{\frac{1}{2}}).$$

(ii) *If $X$ has mean zero, then the density function of $T$ is*

$$(2.13)\quad h(T) = w^{\frac{1}{2}n}[\Gamma(\tfrac{1}{2}n+\tfrac{1}{2}p)/\Gamma(\tfrac{1}{2}n)][\pi^{\frac{1}{2}p}|\Sigma|^{\frac{1}{2}}(T'\Phi T+w)^{\frac{1}{2}(n+p)}]^{-1} \cdot e^{-\frac{1}{2}wb^2}\,{}_1F_1(\tfrac{1}{2}(n+p),\tfrac{1}{2}n;b^2w^2/2(T'\Phi T+w)).$$

(iii) *If $A$ and $B$ are both zero, then*

$$(2.14)\quad h(T) = w^{\frac{1}{2}n}[\Gamma(\tfrac{1}{2}n+\tfrac{1}{2}p)/\Gamma(\tfrac{1}{2}n)][\pi^{\frac{1}{2}p}|\Sigma|^{\frac{1}{2}}(T'\Phi T+w)^{\frac{1}{2}(n+p)}]^{-1}.$$

*In the above formulas $w = m^{-1}$, $\Phi = \Sigma^{-1}$, and $b = |B|$.*

PROOF. We may write

$$h(T) = \int_0^\infty r^p f(rT)g(r)\,dr$$

where $f(\cdot)$ is Gaussian and $g(\cdot)$ is noncentral chi. Expanding the Bessel function in $g(\cdot)$, as in Theorem 1, we are led to an infinite series of integrals of the form

$$\int_0^\infty r^{n+p+2k-1}e^{-\frac{1}{2}r^2(T'\Phi T+w)}e^{r(T'\Phi A)}\,dr.$$

But this integral is essentially a parabolic cylinder function [9], page 119. Thus $h(T)$ becomes an infinite series of parabolic cylinder functions. In (i) we have $B = 0$, which immediately reduces the infinite series to (2.12); and the fact that $A = 0$ and

$$D_{-2\nu}(0) = [\Gamma(\nu)/\Gamma(2\nu)]2^{\nu-1}$$

reduces the infinite series to (2.13). Equation (2.14) is a special case of (2.13) (with $B = 0$) or a special case of (2.12) (with $A = 0$).

**3. Ratios of Gaussian variates.** The results of the previous section suggest the following theorems, analogous to Theorems 1 and 2 respectively.

THEOREM 3. *Let* $X = \{x_1, x_2, x_3, x_4\}$ *be normal* $N(0, \Sigma)$. *Let*

$$z = x_1/x_2 \quad \text{and} \quad \zeta = x_3/x_4 .$$

*Then the joint frequency function* $p(z, \zeta)$ *of* $z$ *and* $\zeta$ *is*

$$p(z, \zeta) = [2\pi^2 |\Sigma|^{\frac{1}{2}}]^{-1}(\partial^2/\partial c^2)(\text{Arc} \sin c/(ab)^{\frac{1}{2}})^2 \tag{3.1}$$

*where*

$$a = \phi_{11}z^2 + 2\phi_{12}z + \phi_{22},$$

$$b = \phi_{33}\zeta^2 + 2\phi_{34}\zeta + \phi_{44},$$

$$c = \phi_{13}z\zeta + \phi_{14}z + \phi_{23}\zeta + \phi_{24},$$

*and* $\Sigma^{-1} = \|\phi_{ij}\|_{1 \leqq i,j \leqq 4}$.

(The fact that $c^2 < ab$ is easily verified.)

In the spirit of Theorem 2 we may compute a multivariate density function involving correlated Gaussian variates. The techniques used above readily yield the following theorem.

THEOREM 4. *Let* $X = \{x_1, \cdots, x_p\}$ *be a p-dimensional Gaussian vector and let* $\{X, y\}$ *be normal* $N(A, \Sigma)$. *Then the joint frequency function* $\psi(Z)$ *of* $Z = \{z_1, \cdots, z_p\}$ *where* $z_k = x_k/y$, $1 \leqq k \leqq p$, *is given by*

$$\begin{aligned} \psi(Z) = (2\pi)^{-\frac{1}{2}(p+1)}\Gamma(p+1)|\Phi|^{\frac{1}{2}} e^{-\frac{1}{2}A'\Phi A} e^{(A'\Phi V)^2/4V'\Phi V} \\ \cdot (V'\Phi V)^{-\frac{1}{2}(p+1)}[D_{-(p+1)}(-A'\Phi V/(V'\Phi V)^{\frac{1}{2}}) \\ + D_{-(p+1)}(A'\Phi V/(V'\Phi V)^{\frac{1}{2}})] \end{aligned} \tag{3.2}$$

*where* $V = \{Z, 1\}$ *and* $\Phi = \Sigma^{-1}$. *If* $A = 0$, *then*

$$\psi(Z) = \Gamma(\tfrac{1}{2}p + \tfrac{1}{2})|\Phi|^{\frac{1}{2}}(\pi V'\Phi V)^{-\frac{1}{2}(p+1)}. \tag{3.3}$$

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