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SOME EXAMPLES OF MULTI-DIMENSIONAL INCOMPLETE BLOCK DESIGNS

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0. Introduction and summary. This paper presents some examples of "multi-dimensional incomplete block designs," a class of designs introduced as such by Potthoff [2] to provide analogues for the familiar two-factor incomplete block design. Much of the content of [2] was subsequently presented in [3], [4], [5], and [6]. Three-dimensional incomplete block ("3DIB") designs and four-dimensional incomplete block ("4DIB") designs for additive models are discussed in general terms in [3] and [4] respectively; our purpose in this paper will be to provide examples to which the tools developed in [3] and [4] can be applied. The original results of this paper have been presented in [1].

1. Preliminary results. In this section we indicate some of the results already obtained by Potthoff. First let us consider the 3DIB additive designs of [3]. Suppose, as in [3], that we begin with a situation in which we have three factors—factors 1, 2, and 3—at m, n, and p levels respectively, and h observations. The 3DIB additive model equation is given in [3], equations (3.1a) and (3.1b). It will be assumed that each level of factor 1 appears equally often, and thus h/m times; that each level of factor 2 appears h/n times; and that each level of factor 3 appears h/p times. Suppose that, as is indicated in [3], paragraph 3(g), we can find a vector Q_1 of the form [3], equation (3.3), whose expectation, given by [3], equation (3.4), depends only on the effects corresponding to the levels of factor 1. Suppose further that a "conditional inverse" of C_{11}, as defined in [3], equation (3.4), can be found, where we define C^* to be a conditional inverse of a square matrix C if and only if $C = CC^*C$. In this case we can conveniently find [3], Sections 4 and 6, Gauss-Markov estimators for contrasts in the effects for factor 1, formulas for the variances of these estimators, and [3], equation (1.10), a sum of squares to be attributed to variability in the effects of factor 1.

Next, we might interchange the role of factor 1 with that of either factor 2 or factor 3, and obtain the same results, if possible, for those factors. For the 3DIB designs which we present in Sections 2 and 3, these results can be obtained only for factors 1 and 2; we can thence find sums of squares to be attributed to error and to variability in factor 3, by application of the conditional error theorem quoted in [3], Section 2.

We now consider the problem of obtaining a relation of the form [3], equation

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(3.3), for factor 1. We can begin with the three equations:

$$(h/m)t_1 + H_{12}t_2 + H_{13}t_3 = E(Y_1), \tag{1.1}$$

$$H_{21}t_1 + (h/n)t_2 + H_{23}t_3 = E(Y_2), \tag{1.2}$$

$$H_{31}t_1 + H_{32}t_2 + (h/p)t_3 = E(Y_3). \tag{1.3}$$

Here the vectors t_u and Y_u, and the matrices H_{uv}, $1 \leqq u, v \leqq 3$ and $u \neq v$, are defined as in [3], Section 3. We note that, generally for $u \neq v$, the (k, L) entry of the "marginal matrix" H_{uv} is the number of times that level k of factor u and level L of factor v are observed in combination among the h design observations. A general method of obtaining [3], equation (3.3), is to multiply equation (1.2) on the left by an $m \times n$ matrix A_{12} and equation (1.3) on the left by an $m \times p$ matrix A_{13} and then to add the transformed equations to (1.1), where A_{12} and A_{13} are chosen so that $H_{12} + (h/n)A_{12} + A_{13}H_{32}$ is the $m \times n$ zero matrix and $H_{13} + A_{12}H_{23} + (h/p)A_{13}$ is the $m \times p$ zero matrix, so that an equation of form [3], equation (3.4), is obtained. We will implicitly follow this procedure in subsequent sections.

We now consider the 4DIB additive designs of [4]. Here we begin with the three factors of the 3DIB design and the h observations, with a fourth factor, "factor 4," at q levels. As for the 3DIB design, each level of factor 1 will appear h/m times, etc. We will begin with a system of equations, analogous to (1.1), (1.2), and (1.3) for the 3DIB design:

$$(h/m)t_1 + H_{12}t_2 + H_{13}t_3 + H_{14}t_4 = E(Y_1), \tag{1.4}$$

$$H_{21}t_1 + (h/n)t_2 + H_{23}t_3 + H_{24}t_4 = E(Y_2), \tag{1.5}$$

$$H_{31}t_1 + H_{32}t_2 + (h/p)t_3 + H_{34}t_4 = E(Y_3), \tag{1.6}$$

$$H_{41}t_1 + H_{42}t_2 + H_{43}t_3 + (h/q)t_4 = E(Y_4), \tag{1.7}$$

in the notation of [4]—analogous to that of [3]. From these equations we will obtain a vector of form [4], equation (2.3), whose expectation is given in [4], equation (2.4). The procedure for finding the 4DIB design vector is analogous to that for finding the vector [3], equation (3.3), for the 3DIB design; the results that can be obtained are analogous to the 3DIB results.

In subsequent sections, we shall let I denote the identity matrix, J a matrix (not necessarily square) whose entries are all 1's, and 0 a matrix whose entries are all 0; the number of rows and columns in these matrices will be clear in context. A scalar multiple of J will be referred to as flat. Notation introduced in this section will be used in other sections; notation introduced in other sections will be used only in the section where introduced except as is explicitly indicated.

2. Some 3DIB designs. In this section we construct 3DIB designs; by a slight variation, we may obtain some further designs for 4 and 5 factors as well.

Before the actual design construction, we present some preliminaries. Let n_1 and n_2 be integers greater than 2. Suppose that we can choose an integer d_1, with $1 < d_1 < n_1$, such that $c_1 = d_1(d_1 - 1)/(n_1 - 1)$ is an integer; similarly suppose that we can choose d_2, $1 < d_2 < n_2$, such that $c_2 = d_2(d_2 - 1)/(n_2 - 1)$

is an integer. Let G_1 be the additive group of integers $0, 1, \cdots, n_1 - 1$, based on addition mod n_1; suppose that we can choose a set D_1 of d_1 distinct integers r_i, $i = 0, \cdots, d_1 - 1$, belonging to G_1, so that the set of $d_1(d_1 - 1)$ nonzero differences $r_{i'} - r_i$, $i' \neq i$, $i, i' = 0, \cdots, d_1 - 1$, includes, under addition in G_1, exactly c_1 repetitions of each of the $n_1 - 1$ nonzero members of G_1. Suppose that a $\neq 0$ belongs to G_1; the set of d_1 elements r_i, $i = 0, \cdots, d_1 - 1$, and the set of d_1 elements $a + r_i$, $i = 0, \cdots, d_1 - 1$, based on addition in G_1, have exactly c_1 elements in common. To see this, we note that for $0 \leqq i, i' \leqq d_1 - 1$, the element $r_{i'}$ equals the element $a + r_i$ if and only if $r_{i'} - r_i = a$. As is indicated above, there are exactly c_1 combinations (i, i'), $i \neq i'$, $i, i' = 0, \cdots, d_1 - 1$, such that $r_{i'} - r_i = a$, so that the number of common elements is in fact c_1. We suppose that we can similarly choose a set D_2 of d_2 distinct integers s_j, $j = 0, \cdots, d_2 - 1$, such that, based on additiòn mod n_2 in the additive group G_2 of integers $0, 1, \cdots, n_2 - 1$, the set of differences $s_{j'} - s_j$, $j \neq j', j, j' = 0, \cdots, d_2 - 1$, includes exactly c_2 repetitions of each of the nonzero members of G_2. Then if $a \neq 0$ belongs to G_2, the set of elements s_j, $j = 0, \cdots, d_2 - 1$, and the set $a + s_j$, $j = 0, \cdots, d_2 - 1$, have exactly c_2 elements in common.

Under these circumstances, we construct 3DIB designs with $m = d_1 n_2$, $n = d_2 n_1$, $p = n_1 n_2$, and $h = mn$. For example, we might choose $n_1 = 4$, $n_2 = 3$, $d_1 = 3$, and $d_2 = 2$, so that $m = 9$, $n = 8$, $p = 12$, and $h = 72$. The difference set D_1 can consist of $r_0 = 0$, $r_1 = 1$, and $r_2 = 2$; D_2 can consist of $s_0 = 0$ and $s_1 = 1$. The procedure of construction is to set up what might be termed a $n_2 \times n_1$ array of $d_1 \times d_2$ matrices, to obtain an overall $d_1 n_2 \times d_2 n_1$ array. We let rows correspond to the levels of factor 1, columns to the levels of factor 2, and entries to the levels of factor 3. Entries are defined as follows: the $(y, z) d_1 \times d_2$ matrix will have (i, j) entry

$$n_2(r_i + z) + (s_j + y), \tag{2.1}$$

where $i = 0, \cdots, d_1 - 1$, $j = 0, \cdots, d_2 - 1$, $y = 0, \cdots, n_2 - 1$, and $z = 0, \cdots, n_1 - 1$. The quantities $r_i + z$ and $s_j + y$ are reduced mod n_1 and mod n_2 respectively to integers between 0 and $n_1 - 1$ inclusive and between 0 and $n_2 - 1$ inclusive respectively; all entries will thus be between 0 and $n_1 n_2 - 1$ inclusive. For example, for $n_1 = 4$, $n_2 = 3$, $d_1 = 3$, $d_2 = 2$, $r_2 = 2$, $s_1 = 1$, $y = 2$, and $z = 3$, we would obtain from (2.1) the quantity $3(2 + 3) + (1 + 2) = 3 \times 1 + 0 = 3$. The entire design for $n_1 = 4$, $n_2 = 3$, $d_1 = 3$, and $d_2 = 2$ can be represented

		0	1	0	1	0	1	0	1
	0	0	1	3	4	6	7	9	10
	1	3	4	6	7	9	10	0	1
	2	6	7	9	10	0	1	3	4
	0	1	2	4	5	7	8	10	11
(2.2)	1	4	5	7	8	10	11	1	2
	2	7	8	10	11	1	2	4	5
	0	2	0	5	3	8	6	11	9
	1	5	3	8	6	11	9	2	0
	2	8	6	11	9	2	0	5	3

In the general case, we let the levels of each factor be numbered in natural order; suppose that rows, corresponding to the levels of factor 1, are numbered from 0 to $d_1n_{22} - 1$, columns from 0 to $d_2n_1 - 1$, and entries from 0 to $n_1n_2 - 1$. For fixed i the n_i quantities $r_i + z \pmod{n_1}$, $z = 0, \cdots, n_1 - 1$, comprise the n_1 integers $0, \cdots, n_1 - 1$; thus the d_1n_1 quantities $r_i + z$, $z = 0, \cdots, n_1 - 1$, $i = 0, \cdots, d_1 - 1$, include d_1 occurrences of each of the n_1 integers $0, \cdots, n_1 - 1$. Likewise the d_2n_2 quantities $s_j + y \pmod{n_2}$ include d_2 occurrences of each of the n_2 integers $0, \cdots, n_2 - 1$. Thus the $d_1 d_2 n_1 n_2$ design entries given in (2.1) include exactly $d_1 d_2$ occurrences of each integer $an_2 + b$, for $a = 0, \cdots, n_1 - 1$ and $b = 0, \cdots, n_2 - 1$, and thus exactly $d_1 d_2$ occurrences of each integer $0, 1, \cdots, n_1n_2 - 1$. As a result, each level of factor 3 will appear exactly $d_1 d_2$ times (equally often) in our design. The marginal matrix H_{12} will be J (each row and each column coinciding exactly once). We now want to look at the matrix products $H_{13}H_{31}$, $H_{23}H_{32}$, and $H_{13}H_{32}$.

For $k, L = 0, \cdots, d_1n_2 - 1$, the (k, L) entry of $H_{13}H_{31}$ is the number of entries common to rows k and L of the overall design array. Consider the number common to rows $d_1y_0 + i_0$ and $d_1y_1 + i_1$, where $y_0, y_1 = 0, \cdots, n_2 - 1$ and $i_0, i_1 = 0, \cdots, d_1 - 1$. Using (2.1), it is easily shown that the d_2n_1 elements in each of these rows are, respectively, $n_2z + (s_j + y_0)$, $z = 0, \cdots, n_1 - 1$, $j = 0, \cdots, d_2 - 1$, and $n_2z + (s_j + y_1)$, $z = 0, \cdots, n_1 - 1$, $j = 0, \cdots, d_2 - 1$, where $s_j + y_0$ and $s_j + y_1$ are reduced mod n_2. For $y_0 = y_1$ the number of common elements is just d_2n_1. For $y_0 \neq y_1$, it is tedious but straightforward to show that the number of common elements is just n_1 multiplied by the number of repetitions (based on addition mod n_2) of $y_1 - y_0$ among the nonzero differences $s_j - s_{j'}$, $j \neq j'$, $j, j' = 0, \cdots, d_2 - 1$. But we already know that there are exactly c_2 such repetitions; hence the number of common elements is just c_2n_1. Using this information, we can write $H_{13}H_{31}$ as an $n_2 \times n_2$ array of $d_1 \times d_1$ matrices, where the $d_1 \times d_1$ matrix along the diagonal is d_2n_1J, and off the diagonal is c_2n_1J.

In order to find $H_{23}H_{32}$, we consider the number of elements common to columns $d_2z_0 + j_0$ and $d_2z_1 + j_1$, $z_0, z_1 = 0, \cdots, n_1 - 1$ and $j_0, j_1 = 0, \cdots, d_2 - 1$. Using (2.1), we obtain that the d_1n_2 elements in each of these columns are, respectively, $n_2(r_i + z_0) + y$, $y = 0, \cdots, n_2 - 1$, $i = 0, \cdots, d_1 - 1$, and $n_2(r_i + z_1) + y$, $y = 0, \cdots, n_2 - 1$, $i = 0, \cdots, d_1 - 1$, where $r_i + z_0$ and $r_i + z_1$ are reduced mod n_2. Arguing as for $H_{13}H_{31}$, we find that we can write $H_{23}H_{32}$ as an $n_1 \times n_1$ array of $d_2 \times d_2$ matrices where the $d_2 \times d_2$ matrix along the diagonal is d_1n_2J, and off the diagonal is c_1n_2J.

For $k = 0, \cdots, d_1n_2 - 1$ and $L = 0, \cdots, d_2n_1 - 1$, the (k, L) entry of $H_{13}H_{32}$ will be the number of elements that row k of the overall array and column L of the overall array have in common. Using (2.1), we have (as above) that row $d_1y_0 + i_0$ comprises the elements $n_2z + (s_j + y_0)$, $j = 0, \cdots, d_2 - 1$, $z = 0, \cdots, n_1 - 1$, and that column $d_2z_0 + j_0$ comprises the elements $n_2(r_i + z_0) + y$, $i = 0, \cdots\ d_1 - 1$, $y = 0, \cdots, n_2 - 1$. The common elements are the $d_1 d_2$

elements $n_2(r_i + z_0) + (s_j + y_0)$, $i = 0, \cdots, d_1 - 1, j = 0, \cdots, d_2 - 1$. Thus each row and each column of the overall array have $d_1 d_2$ elements in common, and $H_{13}H_{32} = d_1 d_2 J$.

We now obtain convenient results for factors 1 and 2. The matrix $H_{12} - (1/d_1 d_2)H_{13}H_{32}$ equals 0, so that we can easily form an equation of form [3], equation (3.4), where $Q_1 = Y_1 - (1/d_1 d_2)H_{13}Y_3$ and $C_{11} = d_2 n_1 I - (1/d_1 d_2)H_{13}H_{31}$. The matrix $C_{11} + (c_2 n_1/d_1 d_2)J$ is an $n_2 \times n_2$ array of $d_1 \times d_1$ matrices, with matrix D appearing along the diagonal and matrix 0 appearing off the diagonal, where the $d_1 \times d_1$ matrix D equals $(n_1/d_1)(d_1 d_2 I - (1 - ((d_2 - 1)/(n_2 - 1)))J)$. Finding D^{-1} is a simple matter; the $n_2 \times n_2$ array of $d_1 \times d_1$ matrices with D^{-1} appearing along the diagonal and 0 appearing off the diagonal is a conditional inverse of C_{11}. Interchanging factors 1 and 2 and the subscripts 1 and 2, we obtain a like result for factor 2. For the design given in (2.2), the 3×3 matrix D in terms of which the conditional inverse is defined will be $(\frac{4}{3})(6I - (\frac{1}{2})J)$; the inverse D^{-1}, $(\frac{1}{72})(9I + J)$. For factor 2, the 2×2 matrix corresponding to D is $(\frac{1}{2})(18I - J)$, inverse $(\frac{1}{144})(16I + J)$. Using the relationship $H_{12} - (1/d_1 d_2)H_{13}H_{32} = 0$, we can show that contrast estimators corresponding to factor 1 and the sum of squares attributed to variability in that factor are the same as what would be obtained if the effects of factor 2 were nil, and vice versa; and also that the estimators and sum of squares for factor 1 are distributed independently of the corresponding quantities for factor 2.

Some sets of parameter values for which 3DIB designs can be constructed according to this section are

n_1	d_1	n_2	d_2	m	n	p	h
4	3	3	2	9	8	12	72
4	3	4	3	12	12	16	144
4	3	5	4	15	16	20	240
3	2	5	4	10	12	15	120
3	2	6	5	12	15	18	180
3	2	7	3	14	9	21	126
4	3	7	3	21	12	28	252
3	2	3	2	6	6	9	36

We now replace the single factor at $d_1 n_2$ levels corresponding to rows by two factors, say $1A$ and $1B$, at d_1 and n_2 levels respectively: we let level i_a of factor $1A$ be associated with row i_a of each of the $d_1 \times d_2$ arrays of (2.2), $i_a = 0, \cdots, d_1 - 1$; for $y_b = 0, \cdots, n_2 - 1$, we let level y_b of factor $1B$ be associated with all of the rows in the $d_1 \times d_2$ arrays (y_b, z), $z = 0, \cdots, n_1 - 1$. We thus will have a 4DIB design with factors at d_1, n_2, $d_2 n_1$, and $n_1 n_2$ levels, and with the number of observations again $d_1 d_2 n_1 n_2$. We can let the factors of the 4DIB design now be numbered 1 (corresponding to factor $1A$), 2 (corresponding to factor $1B$), 3 corresponding to columns of the overall design, and 4 corresponding to entries. Arguing much as for the 3DIB designs, we can find the marginal matrices: we obtain

$$H_{12} = d_2 n_1 J, \quad H_{13} = n_2 J, \quad H_{14} = d_2 J, \quad H_{23} = d_1 J, \quad H_{24}H_{42} = d_1^2 n_1(c_2 J + (d_2 - c_2)I),$$

and $H_{24}H_{43} = d_1^2 d_2 J$. The matrix $H_{34}H_{43}$ will be the matrix $H_{23}H_{32}$ obtained for the 3DIB design. Using these results in conjunction with the equations (1.4)–(1.7), we may easily obtain results involving factors 1, 2, and 3; sums of squares attributed to error and to each of the four factors can then be easily calculated.

Let us now assume in conjunction with the 4DIB designs that our four-factor model is not purely additive, but a model with one interaction term, between factors 1 and 2. The problem of finding Gauss-Markov estimators of factor effects, variances of these estimators, and appropriate sums of squares is equivalent to the problem, already considered, of obtaining the same results for the 3DIB designs of this section, in conjunction with a purely additive model.

For $n_1 = 4$, $d_1 = 3$, $n_2 = 3$, and $d_2 = 2$, we can obtain a 4DIB design with $m = 3$, $n = 3$, $p = 8$, $q = 12$, and $h = 72$, from the 3DIB design given in (2.2). Alongside each of the 9 rows in (2.2), we place a pair whose entries correspond to levels of factors "$1A$" and "$1B$" respectively; the pairs are, from top to bottom, (0, 0), (1, 0), (2, 0), (0, 1), (1, 1), (2, 1), (0, 2), (1, 2), and (2, 2).

We may obtain a class of 5DIB designs by replacing the factor corresponding to columns in (2.2) by two factors, as well as the factor corresponding to rows. These 5DIB designs can be used in conjunction with a model allowing for two 2-factor interactions involving four distinct factors; mathematically we will be dealing with the equivalent of both the 3DIB and 4DIB designs of this section in conjunction with additive models, as well as with the 5DIB design itself in conjunction with additive models (easily handled because of the simplicity of the marginal matrices). In (2.2) we let the pairs corresponding to columns be, from left to right, (0, 0), (1, 0), (0, 1), (1, 1), (0, 2), (1, 2), (0, 3), and (1, 3).

3. Use of finite fields. In this section we obtain some 4DIB designs and 3DIB designs, based on the finite ("Galois") field of s elements for s a power of a prime. Such fields have been constructed from Galois polynomials and discussed in, for example, [7].

Let F denote the field of s letters, the symbols $0 = f_0, f_1, \cdots, f_{s-1}$ denote the elements of F; and F_0 denote the set of $s - 1$ nonzero elements of F—a group under field multiplication. If s is a prime, the field is just the integers $0, 1, \cdots, s - 1$ based on reduction mod s; in this case we may let $f_j = j$, $j = 0, \cdots, s - 1$. Suppose that for $1 < d < s - 1$ we can find an element c of F_0 which is of order d, that is, $c^d = 1$ but $c^i \neq 1$ for 1 for $i = 1, \cdots, d - 1$. Let T_0 denote the subgroup of d elements c^i, $i = 0, \cdots, d - 1$, generated by c. The group F_0 may be divided into $(s - 1)/d$ congruence classes, or cosets, each containing d elements, with any two elements of F_0 belonging to the same coset, if and only if, their quotient under field multiplication belongs to T_0. We let aT_0 denote the coset to which the element a belongs. Let $(s - 1)/d$ be denoted by r. Suppose that g is an element of F_0 such that the r elements g^y, $y = 0, \cdots, r - 1$, belong to r distinct cosets. (This must be the case if none of the $r - 1$ elements g^y, $y = 1, \cdots, r - 1$, belong to T_0; for if g^y and $g^{y'}$ belonged to the same coset for $0 \leqq y < y' \leqq r - 1$, we would have to conclude that $g^{y'-y}$ belonged to T_0, with $1 \leqq y' - y \leqq r - 1$.) For all values of s, d, and r that we have investigated, corresponding field elements c and g satisfying these requirements can be found. Some sets of values of s, d, r, c, and g are: (1) 5, 2, 2, 4, and 2; (2) 7, 2, 3, 6, and 3; (3) 7, 3, 2, 2, and 6; (4) 11, 5, 2, 4, and 2; (5) 11, 2, 5, 10, and 2; (6) 13, 4, 3, 5, and 2; (7) 13, 3, 4, 3, and 2; (8) 13, 2, 6, 12, and 2; (9) 19, 9, 2, 4, and 2; (10) 23, 11, 2, 3, and 5; (11) 31, 15, 2, 7, and 3; (12) 9, 4, 2, x, and $x + 1$ where the field is based on reduction of Galois polynomials mod $(x^2 + 1)$; and (13) 9, 2, 4, 2, and $x + 1$, where the field is the same as for (12).

We first present a class of 4DIB designs. We begin by forming r $d \times s$ subarrays which we arrange lengthwise to form an overall $d \times rs$ array. Numbering smaller arrays from 0 to $r - 1$, rows of each (smaller) array from 1 to d, and columns of each array from 0 to $s - 1$, we let the entry in row i and column j of array y be given by the pair $(g^y c^i + f_j, g^{y+1} c^i + f_j)$, in terms of field multiplication and addition. We let factor 1 be at s levels with level k of factor 1 corresponding to the occurrence of f_k as the 1st component of an entry, $k = 0, \cdots, s - 1$; likewise we let Factor 2 be at s levels with level k of factor 2 corresponding to the occurrence of f_k as the 2nd component of an entry. Factor 3 will be at rs levels, corresponding to the rs columns of the overall array; and factor 4 will be at d levels, corresponding to the d rows of the overall array. The number of design observations will be $s(s - 1)$. We claim that s distinct field elements appear as 1st entry component in each row i of each array y; these are the distinct field elements f_j, $j = 0, \cdots, s - 1$, plus $g^y c^i$ added to each of them. Hence in the overall design, each field element appears exactly r times in each row, so that $H_{14} = rJ$. Likewise $H_{24} = rJ$. Since each row and each column coincide exactly once, we have $H_{34} = J$.

We now investigate the matrix $H_{13}H_{31}$. Numbering its rows and columns from 0 to $s - 1$, we have that the (k, L) entry of this matrix is the number of times that field elements f_k and f_L appear in the same column, throughout the overall design. For $k = L$ this number equals $s - 1$; we claim that for $k \neq L$, it is $d - 1$. We have already shown that in array y of the overall design the field element x appears exactly once as 1st entry component in row i, $i = 1, \cdots, d$. Suppose that in row i' of array y x thus appears in column $j = j(x, y, i')$, so that $x = f_j + g^y c^{i'}$. The remaining 1st components appearing in this same column with x are the elements $f_j + g^y c^i$, $i \neq i'$, $i = 1, \cdots, d$, or $x + g^y(c^i - c^{i'})$, $i' \neq i$, $i = 1, \cdots, d$, so that if $a \neq 0$, $x + a$ will also appear as a 1st component, if and only if, $a = g^y(c^i - c^{i'})$ for some i. Letting y and i' vary, we obtain that the number of times that x and $x + a$ appear as 1st components in the same column, in the overall design, will be the number of times that $a \neq 0$ appears among the $(s - 1)(d - 1)$ nonzero quantities $g^y(c^i - c^{i'})$, $i \neq i'$, $i, i' = 1, \cdots,$

d, $y = 0, \cdots, r-1$, or

(3.1) $\quad g^y c^i(c^{i'} - 1), \quad i = 1, \cdots, d, i' = 1, \cdots, d-1, \qquad y = 0, \cdots, r-1.$

For fixed $i' = 1, \cdots, d-1$, the $s-1$ quantities $g^y c^i(c^{i'} - 1)$, $i = 1, \cdots, d$, $y = 0, \cdots, r-1$, comprise the $s-1$ nonzero elements of F, so that the $(d-1)(s-1)$ quantities (3.1) include exactly $d-1$ occurrences of each nonzero element of F. Thus x and $x + a$, where $a \neq 0$, appear at 1st components in the same column exactly $d-1$ times; and thus for $k \neq L$, the (k, L) entry of $H_{13}H_{31}$ is $d-1$. Hence $H_{13}H_{31} = (d-1)J + (s-d)I$. A similar argument shows that $H_{23}H_{32} = H_{13}H_{31}$.

We now show that $H_{12} = J - I$. For $i = 1, \cdots, d$ and $y = 0, \cdots, r-1$, we again know that x appears in row i and array y of the overall design in a single column $j = j(x, y, i)$. The 2nd component appearing in combination with 1st component x in row i of array y is $f_j + g^{y+1}c^i$, or $x + (g-1)g^y c^i$. Thus the number of times that 1st component x and 2nd component $x + a$ are observed together is the number of times that a appears among the $s-1$ quantities $(g-1)g^y c^i$, $y = 0, \cdots, r-1$, $i = 1, \cdots, d$. These quantities include one occurrence of each nonzero member of F, so that for $k, L = 0, \cdots, s-1$, the matrix H_{12} has (k, L) entry 0 for $k = L$ and 1 for $k \neq L$. Thus H_{12} equals $J - I$.

We now show that $H_{13}H_{32} = d(J - I)$. For $k, L = 0, \cdots, s-1$, the (k, L) entry of this matrix is equal to the number of times that level k of factor 1 (1st components) and level L of Factor 2 (2nd components) appear in the same column —this is to be distinguished from the two levels' being observed in combination together. Appearing in the same column with 1st component x in row i and column $j = j(x, y, i)$ of array y will be the 2nd components $x + g^{y+1}c^{i'} - g^y c^i$, $i, i' = 1, \cdots, d$, $y = 0, \cdots, r-1$, or $g^y c^i(gc^{i'} - 1)$, $i, i' = 1, \cdots, d$, $y = 0, \cdots, r-1$. For fixed i', the $s-1$ quantities $g^y c^i(gc^{i'} - 1)$, $i = 1, \cdots, d$, $y = 0, \cdots, r-1$, comprise the $s-1$ nonzero members of F; thus the $d(s-1)$ quantities $g^y c^i(gc^{i'} - 1)$, $i, i' = 1, \cdots, d$, $y = 0, \cdots, r-1$, include d occurrences of each nonzero member of F (and no occurrences of the zero element). Using this information, we may conclude that the (k, L) entry of $H_{13}H_{32}$ is 0 for $k = L$ and d for $k \neq L$, so that $H_{12} - (1/d)H_{13}H_{32} = 0$.

From the 4DIB equations (1.4)–(1.7) we obtain convenient results for factors 1, 2, and 4, using the matrix relationships thus found. For factor 1, we obtain an equation of form [4], equation (2.3), with $Q_1 = Y_1 - (1/d)H_{13}Y_3$ and $C_{11} = ((d-1)/d)(sI - J)$; for factor 2 we obtain the same result with factors 1 and 2 interchanged. A conditional inverse of C_{11} is just $(d/(d-1)s)I$. We will obtain the same results involving factor 4 as we would if the effects of factors 1, 2, and 3 were nil; the Gauss-Markov estimators of contrasts involving factor 1 are, as for the 3DIB designs of Sections 2, distributed independently of those involving factor 2, and are the same as those obtained if the effects of factors 2 and 4 are assumed to be nil.

The simplest example of a 4DIB design constructed and dealt with as above (appropriate as an example of construction, but practically not too useful because it allows only 1 degree of freedom for error estimation) is, for $s = 5$, $d = 2$, $r = 2$, $c = 4$, and $g = 2$, given by

(3.2)

	0	1	2	3	4	0	1	2	3	4
1	4, 3	0, 4	1, 0	2, 1	3, 2	3, 1	4, 2	0, 3	1, 4	2, 0
2	1, 2	2, 3	3, 4	4, 0	0, 1	2, 4	3, 0	4, 1	0, 2	1, 3

We now consider a second class of 4DIB designs, constructed much like the first class. We arrange r $(d + 1) \times s$ arrays lengthwise to obtain an overall array which represents the design. For convenience we now let $w_0 = 0$, and $w_i = c^i$, $i = 1, \cdots, d$. Letting i range from 0 to d, j from 0 to $s - 1$, and y from 0 to $r - 1$, we let the entry in row i and column j of array y be the pair $(g^y w_i + f_j, g^y w_i c + f_j)$. Factors 1, 2, 3, and 4 will be defined as for the first class of designs; the number of levels for these factors will be s, s, rs, and $d + 1$ respectively, and the number of design observations $sr(d + 1)$. Here the matrices H_{14} and H_{24} will equal rJ, and H_{34} will equal J. We show that $H_{13}H_{31} = H_{23}H_{32} = (d + 1)(J + (r - 1)I)$ by proving that the $(s - 1)(d + 1)$ nonzero differences $g^y(w_{i'} - w_i)$, $y = 0, \cdots, r - 1$, $i \neq i'$, $i, i' = 0, \cdots, d$, include $d + 1$ repetitions of each of the nonzero members of F. The proof involves observation that these quantities include the quantities (3.1) and also the $2(s - 1)$ quantities $g^y c^i$ and $-g^y c^i$, $y = 0, \cdots, r - 1$, $i = 1, \cdots, d$.

We now find H_{12}. Suppose that in row i of array y of the overall design, the 1st component x appears in column $j = j(x, y, i)$. The corresponding 2nd component observed in combination with x is $x + g^y(c - 1)w_i$; letting i and y vary and considering the $(d + 1)r$ quantities g^y, $y = 0, \cdots, r - 1$, $i = 0, \cdots, d$, we can conclude that throughout the entire design 1st component x and 2nd ponent $x + a$ are observed together 1 time if $a \neq 0$ and r times if $a = 0$. Thus $H_{12} = J + (r - 1)I$.

The number of times that 1st component x and 2nd component $x + a$ appear in the same column equals the number of occurrences of a among the $r(d + 1)^2$ quantities $g^y(cw_{i'} - w_i)$, $y = 0, \cdots, r - 1$, $i, i' = 0, \cdots, d$. The set of elements $cw_{i'}$, $i' = 0, \cdots, d$, is the same as the coset cT_0 with 0 adjoined, and thus the same as the coset T_0 with 0 adjoined, and thus the same as the set $w_{i'}$, $i' = 0, \cdots, d$. Hence the $r(d + 1)^2$ quantities $g^y(cw_{i'} - w_i)$ are the same as the $r(d + 1)$ quantities $g^y(w_{i'} - w_i)$. Dividing these quantities according to $i = i'$ and $i \neq i'$, we find that 0 occurs $r(d + 1)$ times and $a \neq 0$ occurs $d + 1$ times, so that $H_{13}H_{32} = (d + 1)(J + (r - 1)I)$ and $H_{12} - (1/(d + 1))H_{13}H_{32} = 0$.

Using these matrix relationships, we obtain from equations (1.4)–(1.7) results for factors 1, 2, and 4 entirely like those obtained for the first class of 4DIB designs of this section, with C_{11} now equal to $sI - J$.

A construction can be achieved for $s > 3$ and $d = 1$ so that in terms of d and s, the matrices H_{14}, H_{24}, H_{34}, $H_{13}H_{31}$, $H_{23}H_{32}$, $H_{12} - (1/(d + 1))H_{13}H_{32}$, and C_{11}, and the subsequent results for factors 1, 2, and 4, are the same as for our second class of 4DIB designs. We let $w_{1,1}$, $w_{1,2}$, $w_{2,1}$, and $w_{2,2}$ be four distinct elements of F; we construct $s - 1$ $2 \times s$ arrays which we arrange lengthwise. Letting i range from 1 to 2, j from 0 to $s - 1$, and y from 1 to $s - 1$, we let the entry in row i and column j of array y be the pair $(f_j + f_y w_{1,i}, f_j + f_y w_{2,i})$.

We now present some 3DIB designs. The method of construction is, basically, to interchange field addition and multiplication, and proceed as for the two 4DIB classes of this section. We now suppose that d is a divisor of s, $1 < d < s$. Thus s must be of form a^b and d of form $a^{b'}$, where $a > 1$ is prime and $1 \leqq b' < b$. Regarding F as an additive group under field addition, suppose that T is an (additive) subgroup of F, of order d. For arbitrary a, b, and b', the Galois field F will consist of polynomials of degree $\leqq b - 1$ with (integral) coefficients reduced mod a and with field multiplication based on reduction mod a polynomial of degree b; the subgroup T can consist of those polynomials in F which are of degree $\leqq b' - 1$.

Let us call the elements of T $0 = w_0, w_1, \cdots, w_{d-1}$. The elements of F fall into s/d cosets of T, each containing d elements, with two elements belonging to the same coset, if and only if, their difference belongs to T. Let $t = s/d$; we number these cosets $A_0, \cdots, A_{t-1}$ and let aT denote the coset to which field element a belongs. From each coset A_j we choose a particular member g_j. We form $s - 1$ $d \times t$ arrays, and arrange these lengthwise to form the overall design. Letting i range from 0 to $d - 1$, j from 0 to $t - 1$, and y from 1 to $s - 1$, we let the entry in row i and column j of array y be the pair $(f_y(w_i + g_j), f_y(w_i + e + g_j))$ where e is any field element not belonging to T. We let the 1st and 2nd entry components correspond to the levels of factors 1 and 2 (each factor at s levels), and let columns correspond to the $s(s - 1)/d$ levels of factor 3 (it will be noted that rows do not correspond to a factor here). By showing that the quantities $f_y w_i$, $y = 1, \cdots, s - 1$, $i = 0, \cdots, d - 1$, include $d - 1$ repetitions of each nonzero member of F, we establish that $H_{13}H_{31} = H_{23}H_{32} = (d - 1)J + (s - d)I$. We now want to show that the matrix $H_{12} - (1/d)H_{13}H_{32}$ equals 0, so that the simplification achieved for factors 1 and 2 in the 4DIB designs of this section can be achieved. Observing that the quantities $f_y e$, $y = 1, \cdots, s - 1$, include exactly one occurrence of each nonzero member of F, we can conclude that $H_{12} = J - I$. Since the $(s - 1)d$ quantities $f_y(e + w_i - w_{i'})$, $y = 1, \cdots, s - 1$, $i = 0, \cdots, d - 1$, include exactly d occurrences of each nonzero member of F, we can conclude that $H_{13}H_{32} = d(J - I) = dH_{12}$.

The simplest example (again appropriate as an example but not for practical use) is for $s = 4$, $d = 2$, $e = x$, $F = (0, 1, x, x + 1)$ with reduction of polynomials mod $(x^2 + x + 1)$, $T = (0, 1)$, $g_0 = 0$, and $g_1 = x$:

	0	1	0	1	0	1
0	$0, x$	$x, 0$	$0, x + 1$	$x + 1, 0$	$0, 1$	$1, 0$
1	$1, x + 1$	$x + 1, 1$	x, 1	$1, x$	$x + 1, x$	$x, x+ 1$

4. Variations of Section 3. Suppose, in the notation of Section 3, that $s = 4a + 3$, where $a > 0$ is an integer, and that for $d = 2a + 1$ and $r = 2$ we can find corresponding elements c and g. For these values of s, d, and r, we then can obtain as in Section 3 two 4DIB designs consisting of two $d \times s$ arrays and two $(d + 1) \times s$ arrays respectively. Here, for the same parameters, we present two designs consisting of one $d \times s$ array and one $(d + 1) \times s$ array respectively, instead of two arrays. For practical purposes, only a few values of $s = 4a + 3$ will be powers of primes but not primes: the smallest is 27, and the next smallest is 243.

For this situation we first present some preliminary results. We claim that (1) the set T_0 of $2a + 1$ elements c^i, $i = 0, \cdots, 2a$, does not intersect the set $-T_0$ of $2a + 1$ negatives $-c^i$, $i = 0, \cdots, 2a$, so that each of the $4a + 2$ nonzero field elements belongs to exactly one of these two sets. To prove this, we first show that the additive inverse -1 of 1 is of order 2 under field multiplication, so that since 2 does not divide $2a + 1$, the order of T_0, -1 cannot belong to T_0. Thus the cosets T_0 and $(-1)T_0$ are distinct. It is then straightforward to show that $(-1)T_0$ and $-T_0$ are identical.

We next claim that (2) the set of $(2a + 1)2a$ nonzero differences $c^i - c^{i'}$, $i \neq i'$, $i, i' = 0, \cdots, 2a$, involving the elements of T_0 include exactly a repetitions of each nonzero member of F. Using the fact that $c^{2a+1} = 1$, we may write these differences as $c^i(c^{i'} - 1)$ and $c^i(1 - c^{i'})$, $i = 0, \cdots, 2a$, $i' = 1, \cdots, a$, which include a repetitions, in unspecified order, of each of the members of the two distinct cosets T_0 and $-T_0$, and hence, using (1), of each of the nonzero members of F.

We now generate a first class of designs, with several, say z, factors at s levels $(z \geqq 3)$, one factor at d levels, and ds observations. Suppose that we may choose $z - 1$ distinct elements of T_0—which we number $b_2, \cdots, b_z$—so that $b_v - b_u \in T_0$ for $2 \leqq u < v \leqq z$. We set up a $d \times s$ matrix whose entry will have $z - 1$ components. For notational convenience, we number the matrix rows from 1 to d, the matrix columns from 0 to $s - 1$, and the $z - 1$ entry components from 2 to z; we define component u of the entry in row i and column j to be $b_u c^i + f_j$. We let factor 1 correspond to the s columns of the matrix, factor $z + 1$ to the d rows of the matrix, and factors 2 through z to the respective $z - 1$ components of the entry (also numbered from 2 through z). Arguing as in Section 3, we show that each row appears once in combination with each level of factors 1 through z. We now investigate the marginal matrices involving the s-level factors. Suppose that the levels of factor 1 are numbered from 0 to $s - 1$, with level k corresponding to column k of the matrix, and that the levels of factor u, $u = 2, \cdots, z$, are numbered from 0 to $s - 1$, with level k of factor u corresponding to the occurrence of field element f_k as component u of an entry, $k = 0, \cdots, s - 1$. We first look at H_{1u}, $u = 2, \cdots, z$. For $k, L = 0, \cdots, s - 1$, the (k, L) entry of this matrix will be the number of times (0 or 1) that field element f_L occurs as component u of an entry in column k. For fixed k, field elements appearing as component u in column k are $b_u c^i + f_k$, $i = 1, \cdots, d$. Since b_u belongs to T_0 by assumption, the cosets $b_u T_0$ and T_0 are identical; using this fact, it is easily shown that for each u the field elements appearing as component u in column k are $c^i + f_k$, $i = 1, \cdots, d$. Hence the matrix H_{1u} is equal to a matrix H which has (k, L) entry equal to 1, if and only if, $f_L - f_k$ belongs to T_0 and equal to 0 otherwise.

We now consider the matrix H_{uv} for $2 \leqq u < v \leqq z$. Component u of the entry of the $d \times s$ design matrix will equal a particular element x for exactly one entry in row i, $i = 1, \cdots, d$. Suppose that this entry is in column $j = j(x, i)$; then $x = b_u c^i + f_j$. Letting i vary, the d levels of factor v appearing with the level of factor u corresponding to field element x will correspond to field elements $x + (b_v - b_u)c^i$, $i = 1, \cdots, d$. But since $(b_v - b_u)$ belongs to T_0 by assumption, the cosets $(b_v - b_u)T_0$ and T_0 are identical; we hence conclude for $k, L = 0, \cdots, s - 1$, that the (k, L) entry of H_{uv} is 1 if, again, $f_L - f_k$ belongs to T_0, and is 0 otherwise; thus H_{uv} equals the matrix H defined above. Thus for $1 \leqq u < v \leqq z$ we have that $H_{uv} = H$, and also that $H_{vu} = H'$.

For $k, L = 0, \cdots, s - 1$, we know that H has (k, L) entry 1, if and only if, $f_L - f_k \, \varepsilon \, T_0$, and entry 0 otherwise. Likewise H' has (k, L) entry, if and only if, $f_L - f_k \, \varepsilon \, -T_0$, and entry 0 otherwise. By (1), the sets T_0 and $-T_0$ are disjoint and together include exactly one occurrence of each nonzero member of F, so that $H + H'$ has (k, L) entry equal to 1, if and only if, $f_L - f_k \neq 0$, and entry 0 otherwise. Thus $H + H'$ has entry 0 along the diagonal and entry 1 off the diagonal, and thus equals $J - I$. The matrix product HH' has (k, L) entry equal to the number of integers y, $y = 0, \cdots, s - 1$, for which both the (k, y) and (L, y) entry of H are equal to 1; this number will be just the number of times that $f_L - f_k$ appears among the d^2 differences $c^i - c^{i'}$, $i, i' = 1, \cdots, d$. For $L = k$, $f_L - f_k = 0$ and appears exactly d times; for $L \neq k$, we know from (2) that the nonzero differences $c^i - c^{i'}$ include exactly a repetitions of each nonzero member of F. Hence the matrix HH' will have entry a off the diagonal, and $2a + 1$ along it, so that it can be written $aJ + (a + 1)I$. Using a similar argument, we can show that $H'H$ also equals $aJ + (a + 1)I$.

A second class of designs can be constructed by adding to the first design array a "0 row" visually above the rows 1 through d, to obtain a $(d + 1) \times s$ array. This 0 row will have an entry in column j, $j = 0, \cdots, s - 1$, consisting of $z - 1$ components all equal to f_j. Thus we have that component u of the entry in row i and column j is $w_i b_u + f_j$, $i = 0, \cdots, d$, $j = 0, \cdots, s - 1$, $u = 2, \cdots, z$, where w_i is defined in Section 3. Again we let factor 1 correspond to columns, factor $z + 1$ to rows, and factors 2 through z to components of the array. Again we have that the marginal matrices $H_{u,z+1}$ are all equal to J. For $1 \leqq u < v \leqq z$, H_{uv} equals G (and H_{vu} equals G') where G has (k, L) entry 1 if and only if $f_L - f_k = w_i$ for some $i = 0, \cdots, d$, for $k, L = 0, \cdots, s - 1$. The matrix G is the same as H except that 1 instead of 0 now appears along the diagonal. Thus

$$G = H + I, \quad \text{and} \quad G' = H' + I; \quad \text{and}$$

$$G + G' = H + I + H' + I = H + H' + 2I = J - I + 2I = J + I.$$

Also,

$$GG' = (H+I\ (H' + I) = HH' + H + H' + I$$
$$= aJ + (a + 1)I + J - I + I = (a + 1)J + (a + 1)I;$$

likewise $G'G = (a + 1)J + (a + 1)I$.

The simplest design of our first class is, for $s = 7, d = 3, z = 3, c = 2, b_2 = 1$, and $b_3 = 2$,

		0	1	2	3	4	5	6
(4.1)	1	2, 4	3, 5	4, 6	5, 0	6, 1	0, 2	1, 3
	2	4, 1	5, 2	6, 3	0, 4	1, 5	2, 6	3, 0
	3	1, 2	2, 3	3, 4	4, 5	5, 6	6, 0	0, 1

This design, and the design of our second class for the same parameters, have been constructed by Potthoff [2] in slightly different, but equivalent, form, except without a factor corresponding to the rows of our design. Some further sets of parameters for which designs of both of our classes can be constructed are:

(1) $s = 11, \quad z = 4, \quad c = 4, \quad b_2 = 1, \quad b_3 = 4, \quad b_4 = 5$ (a case also already considered by Potthoff);

(2) $s = 19, \quad z=4, \quad c = 4, \quad b_2 = 1, \quad b_3 = 6, \quad b_4 = 7$;

(3) $s = 23, z = 5, \quad c = 2, \quad b_2 = 1, \quad b_3 = 2, \quad b_4 = 3, \quad b_5 = 4$; and

(4) $s = 31, \quad z = 5, \quad c = 7, \quad b_2 = 1, \quad b_3 = 2, \quad b_4 = 9, \quad b_5 = 10$.

Of course there is no need to use the maximum possible number of factors; we might, for example, have $s = 23$ and $z = 3$, instead of $z = 5$. We have not discovered a general way of determining the maximum possible value of z, for arbitrary s.

We now consider the results obtained for our two classes of designs. In each case the effects of factor $z + 1$ will be estimated as though all other factors were absent; conversely, estimates involving factors 1 through z will not be affected by the presence of factor $z + 1$, so that essentially we have a design for z factors each at s levels. For $z = 3$ (a 3DIB design) we can note that the designs of both our classes belong to "Design Class 2" of [3] for any permutation of the factor indices, and that hence we can obtain a result of form [3], equation (3.3) for each of the three s-level factors–utilizing the simple form of the matrices $H + H'$, $HH' = H'H$, $G + G'$, and $GG' = G'G$. A conditional inverse of C_{11} will be, for each factor in the case of our second class of designs, $(1/(2a + 1))I$; the design efficiency [2], [3] for estimating all contrasts for each factor will be $(2a + 1)/(2a + 2)$. For the first class of designs the conditional inverse will be $(a/(2a^2 - 1))I$, and the efficiency $(2a^2 - 1)/(a(2a + 1))$.

For $z = 4$, the (4DIB) designs of both of our classes belong to "Design Class 3" of [4] so that, again, results can be obtained for all s-level factors. The efficiency for all contrasts for each factor is $(a(4a + 3))/((2a + 1)(2a + 2))$ for the second class of designs, and $((4a+3)(a^2-a-1))/((2a^2-1)(2a+1))$ for the first class.

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