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SOME RESULTS ON THE COMPLETE AND ALMOST SURE CONVERGENCE OF LINEAR COMBINATIONS OF INDEPENDENT RANDOM VARIABLES AND MARTINGALE DIFFERENCES[1]

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1. Introduction. Let $(\Omega, \mathfrak{F}, P)$ be a probability space with $(\mathfrak{F}_{k;k\geqq 1})$ an increasing sequence of σ-fields such that $\mathfrak{F}_k \subset \mathfrak{F}$. Let $(D_k, \mathfrak{F}_{k,k\geqq 1})$ be a martingale difference sequence; i.e., each D_k is $\mathfrak{F}_k$ measurable and $E(D_k \mid \mathfrak{F}_{k-1}) = 0$ a.s. for all $k \geqq 2$. Let a_{nk} be a matrix of real numbers,

$$A_n = \sum_{k=1}^{\infty} a_{nk}^2, \qquad T_{nm} = \sum_{k=1}^{m} a_{nk} D_k \quad \text{and} \quad T_n \text{ be the a.s. limit}$$

of T_{nm} as $m \to \infty$ whenever this limit exists. T_n is said to converge completely to zero in the sense of Hsu and Robbins [8] if $\sum_{n=1}^{\infty} P[|T_n| > \epsilon] < \infty$ for all $\epsilon > 0$. It should be noted that T_n converging completely to zero implies that T_n converges a.s. to zero and that the two types of convergence are equivalent if the T_n's form a sequence of independent random variables. The purpose of this paper is to present various sets of conditions for the complete or a.s. convergence of T_n to zero.

Sections 3 and 4 deal with the special case where the $(D_k, k \geqq 1)$ are independent random variables, Section 3 treating the identically distributed case and Section 4 treating the non-identically distributed case. The results given in these two sections extend or improve results given by Hsu and Robbins [8], Erdös [4], Pruitt [11], and Chow [1]. The double truncation method of proof developed by Erdös [4] and improved by other authors ([1], [5], and [11] for example) is fundamental. The work of Franck and Hanson [5] is closely related to that presented here. The main results are given by Theorems 1 and 3 with more specific applications given by Corollaries 1–3. Theorem 2 is of special interest since it shows that the double truncation method of Erdös used in [4] to obtain sharp results about complete convergence can sometimes be modified to obtain sharp results about almost sure convergence.

According to Chow [1], a random variable D is generalized Gaussian if there exists an $\alpha \geqq 0$ such that for every real t, $E \exp(tD) \leqq \exp(t^2\alpha^2/2)$. The minimum of these numbers α is denoted by $\tau(D)$. Special cases of generalized Gaussian random variables include normal and bounded random variables each with mean zero. (See [1], p. 1482.) In Section 5 we extend to the martingale case a result of Chow ([1], p. 1483) concerning the complete convergence of T_n to zero when the $(D_k, k \geqq 1)$ are independent and generalized Gaussian with $\tau^2(D_k) \leqq 2$.

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2. Preparatory lemmas.

LEMMA 1. *Let* $(D_k, k \geqq 1)$ *be independent with* $ED_k^2 \leqq K < \infty$, $ED_k = 0$, $a_{nk} > 0$, $A_n < \infty$ *and* $\sum_{n=1}^{\infty} \exp(-\lambda/A_n) < \infty$ *for all* $\lambda > 0$. *Let* $D'_{nk} = D_k I[a_{nk}D_k \leqq n^{-\rho}]$ *for some* $\rho > 0$ *and* $T'_{nm} = \sum_{k=1}^{m} a_{nk}D'_{nk}$. *Then* T'_{nm} *converges* a.s. *to a random variable* T_n' *as* $m \to \infty$ *and*

$$\sum_{n=1}^{\infty} P[T_n' > \epsilon] < \infty \quad \textit{for all } \epsilon > 0. \tag{1}$$

PROOF. According to the Kolmogorov convergence theorem, ([10], p. 236) $\lim_{m\to\infty} T_{nm} = T_n$ exists a.s. since $\sum_{k=1}^{\infty} a_{nk}^2 ED_k^2 < \infty$. Since $\sum_{k=1}^{\infty} P[D_k \neq D'_{nk}] = \sum_{k=1}^{\infty} P[D_k > n^{-\rho}/a_{nk}] \leqq KA_n n^{2\rho} < \infty$, it then follows that T'_{nm} converges a.s. to a random variable T_n' as $m \to \infty$. Fix $\epsilon > 0$. Let $t = \min(\epsilon/(2A_n), n^{\rho})$. Since $a_{nk}D'_{nk}t \leqq 1$, it then follows that $E \exp(a_{nk}D'_{nk}t) \leqq \exp(Ea_{nk}D'_{nk}t + Ea_{nk}^2 D'^2_{nk} t^2)$ using the easily established fact ([1], p. 1488) that $E \exp Y \leqq \exp(EY + EY^2)$ for a random variable $Y \leqq 1$. Since $Ea_{nk}D'_{nk}t \leqq 0$, we obtain $E \exp(a_{nk}D'_{nk}t) \leqq \exp(a_{nk}^2 t^2 ED'^2_{nk})$. Assuming without loss of generality that $ED_k^2 \leqq 1$, it then follows by the independence of the D'_{nk}'s in k that $E \exp(tT_n') \leqq \exp(t^2 A_n)$. By the Chebychev inequality, $P[T_n' > \epsilon] \leqq \exp(-\epsilon t) E \exp(tT_n') \leqq \exp(-\epsilon t) \exp(t^2 A_n)$. If $\epsilon/(2A_n) > n^{\rho}$, we obtain $P[T_n' > \epsilon] \leqq \exp(-\epsilon n^{\rho}/2)$. On the other hand, if $\epsilon/(2A_n) \leqq n^{\rho}$, then $P[T_n' > \epsilon] \leqq \exp(-\epsilon^2/(4A_n))$. Hence $\sum_{n=1}^{\infty} P[T_n' > \epsilon] < \infty$ for all $\epsilon > 0$.

LEMMA 2. *Let* $(Z_k, k \geqq 1)$ *be independent with* $0 < |a_{nk}| \leqq Kn^{-\alpha}$ *for some* $\alpha > 0$. *Let either* $Z''_{nk} = Z_k I[|a_{nk}|Z_k > \epsilon/N]$ *or* $Z''_{nk} = Z_k I[|a_{nk}Z_k| > \epsilon/N]$ *for fixed* $\epsilon > 0$ *and positive integer* N. *Let* $T''_{nm} = \sum_{k=1}^{m} a_{nk}Z''_{nk}$ *and assume* T''_{nm} *converges* a.s. *to a random variable* T_n'' *as* $m \to \infty$. *Let* $f_n(j)$ *be the number of subscripts* k *such that* $|a_{nk}| > \epsilon/(Nj)$ *for integers* $n \geqq 1$ *and* $j \geqq 1$. *Let* $g_j = [(jKN/\epsilon)^{1/\alpha}]$ *where* $[\cdot]$ *is the greatest integer function. Then*

$$\sum_{n=1}^{\infty} P[|T_n''| > \epsilon] \leqq \sum_{j=1}^{\infty} \sum_{n=1}^{g_j} f_n(j) \sup_k P[j - 1 \leqq |Z_k| < j] \quad \textit{and} \tag{2}$$

$$\sum_{n=1}^{\infty} P[|T_n''| > \epsilon] \leqq \sum_{j=1}^{\infty} \sum_{n=1}^{g_j} (f_n(j) - f_n(j-1)) \sup_k P[|Z_k| \geqq j - 1]. \tag{3}$$

PROOF. T_n'' is well defined by hypothesis.

$$P[|T_n''| > \epsilon] \leqq P[\exists\, k \ni |a_{nk}Z_k| > \epsilon/N] \leqq \sum_{k=1}^{\infty} P[|Z_k| > \epsilon/(N|a_{nk}|)] \leqq \sum_{j=1}^{\infty} (f_n(j) - f_n(j-1)) \sup_k P[|Z_k| \geqq j - 1]$$

since $f_n(j) - f_n(j-1)$ is the number of subscripts k such that $j > \epsilon/(N|a_{nk}|) \geqq j - 1$ and $P[|Z_k| > \epsilon/(N|a_{nk}|)] \leqq \sup_k P[|Z_k| \geqq j - 1]$ if $\epsilon/(N|a_{nk}|) \geqq j - 1$. Now $|a_{nk}| \leqq Kn^{-\alpha}$ implies that $f_n(j) = 0$ for $n > g_j$. Thus

$$\sum_{n=1}^{\infty} P[|T_n''| > \epsilon] \leqq \sum_{j=1}^{\infty} \sum_{n=1}^{g_j} (f_n(j) - f_n(j-1)) \sup_k P[|Z_k| \geqq j - 1].$$

Thus (3) is established. Similarly,

$$\begin{aligned} P[\exists\, k \ni |a_{nk}Z_k| > \epsilon N^{-1}] &= P[\textstyle\bigcup_{k=1}^{\infty} \bigcup_{j=1}^{\infty} \{|a_{nk}Z_k| > \epsilon N^{-1}, j - 1 \leqq |Z_k| < j\}] \\ &\leqq \textstyle\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} P[|a_{nk}Z_k| > \epsilon N^{-1}, j - 1 \leqq |Z_k| < j] \\ &\leqq \textstyle\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} P[|a_{nk}j| > \epsilon N^{-1}] P[j - 1 \leqq |Z_k| < j] \\ &\leqq \textstyle\sum_{j=1}^{\infty} f_n(j) \sup_k P[j - 1 \leqq |Z_k| < j]. \end{aligned}$$

Hence

$$\sum_{n=1}^{\infty} P[|T_n''| > \epsilon] \leqq \sum_{j=1}^{\infty} \sum_{n=1}^{g_j} f_n(j) \sup_k P[j-1 < |Z_k| \leqq j].$$

Thus (2) is established.

LEMMA 3. *Let* $(Z_k, k \geqq 1)$ *be independent with* $E|Z_k|^\nu \leqq 1$ *for some* $\nu > 0$ *and* $|a_{nk}| > 0$. *Let either* $Z_{nk}''' = Z_k I[n^{-\rho}/|a_{nk}| < Z_k \leqq \epsilon/(N|a_{nk}|)]$ *or* $Z_{nk}''' = Z_k I[n^{-\rho}/|a_{nk}| < |Z_k| \leqq \epsilon/(N|a_{nk}|)]$ *for fixed* $\rho > 0$, $\epsilon > 0$, *and positive integer* N. *Let* $T_{nm}''' = \sum_{k=1}^m a_{nk} Z_{nk}'''$ *and assume that* T_{nm}''' *converges* a.s. *to a random variable* T_n''' *as* $m \to \infty$. *Then*

$$P[|T_n'''| > \epsilon] \leqq \left(\sum_{k=1}^{\infty} |a_{nk}|^\nu n^{\rho\nu}\right)^N. \tag{4}$$

PROOF. T_n''' is well defined by hypothesis.

$$P[|T_n'''| > \epsilon] \leqq P[\exists N k\text{'s} \ni |Z_k| > n^{-\rho}/(|a_{nk}|)]$$
$$\leqq \left(\sum_{k=1}^{\infty} P[|Z_k| > n^{-\rho}/(|a_{nk}|)]\right)^N \leqq \left(\sum_{k=1}^{\infty} |a_{nk}|^\nu n^{\rho\nu}\right)^N$$

using the fact that $P[|Z_k| > x] \leqq x^{-\nu}$ for any $x > 0$.

LEMMA 4. *Let* $(D_k, k \geqq 1)$ *be independent with* $E|D_k|^\nu \leqq K < \infty$ *for some* $1 \leqq \nu < 2$, $|a_{nk}| \leqq Kn^{-\alpha}$ *for some* $\alpha > 0$, *and* $\sum_{k=1}^{\infty} |a_{nk}|^\nu \leqq Kn^{-\lambda}$ *for some* $\lambda > 0$. *Let* $D_{nk}' = D_k I[|a_{nk} D_k| \leqq n^{-\rho}]$ *for some* $0 < \rho < \min(\alpha, \lambda/\nu)$ *and* $T_{nm}' = \sum_{k=1}^m a_{nk} D_{nk}'$. *Let either* $\sum_{k=1}^{\infty} |a_{nk}| \leqq K$ *and* $ED_k = 0$ *or* $\sum_{k=1}^{\infty} |a_{nk}| \to 0$ *as* $n \to \infty$. *Then* T_{nm}' *converges* a.s. *to a random variable* T_n' *as* $m \to \infty$ *and* T_n' *converges completely to zero.*

PROOF. Since $\sum_{k=1}^{\infty} |a_{nk}| E|D_{nk}'| \leqq \sum_{k=1}^{\infty} |a_{nk}| E|D_k| < \infty$, it follows that T_{nm}' converges a.s. to a random variable T_n' as $m \to \infty$. $E|D_k|^\nu \leqq K$ for $\nu > 1$ implies that the D_k's are uniformly integrable. Thus, if $ED_k = 0$ and $\nu > 1$, then

$$|ED_{nk}'| = |ED_k I[|D_k| > n^{-\rho}/|a_{nk}|]| \leqq E[|D_k| I[|D_k| > n^{-\rho+\alpha}/K]] \to 0$$

as $n \to \infty$ uniformly in k. Thus $|\sum_{k=1}^{\infty} a_{nk} ED_{nk}'| \leqq \sum_{k=1}^{\infty} |a_{nk} ED_{nk}'| \to 0$ as $n \to \infty$ for the case $ED_k = 0$ and $\nu > 1$. Since $|ED_{nk}'| \leqq E|D_k| \leqq K$, it follows that $\sum_{k=1}^{\infty} a_{nk} ED_{nk}' \to 0$ as $n \to \infty$ for the case $\sum_{k=1}^{\infty} |a_{nk}| \to 0$ as $n \to \infty$. Since these are the only two cases which may occur under the hypotheses, it follows that $\sum_{k=1}^{\infty} a_{nk} ED_{nk}' \to 0$ as $n \to \infty$. Let $Y_{nk}' = D_{nk}' - ED_{nk}'$ and $t = n^\rho/2$. Since $EY_{nk}' = 0$ and $|a_{nk} Y_{nk}' t| \leqq 1$, it follows by a lemma of Chow ([1], p. 1482) that $E \exp(a_{nk} Y_{nk}' t) \leqq \exp(t^2 a_{nk}^2 E Y_{nk}'^2)$. We decompose

$$\exp(t^2 a_{nk}^2 E(Y_{nk}')^2) = \exp(t^2 |a_{nk}|^\nu E|a_{nk} Y_{nk}'|^{2-\nu} |Y_{nk}'|^\nu).$$

$2 - \nu > 0$ and $|a_{nk} Y_{nk}'| \leqq 2n^{-\rho}$ together imply $|a_{nk} Y_{nk}'|^{2-\nu} \leqq n^{-\rho(2-\nu)} 2^{2-\nu}$. We assume without loss of generality that $E|D_k|^\nu \leqq 1$. Then $E|D_{nk}'| \geqq |ED_{nk}'|$ and the c_r inequality ([10], p. 155) yields $E|Y_{nk}'|^\nu \leqq 2^\nu$. Combining the above yields $\exp(t^2 a_{nk}^2 E(Y_{nk}')^2) \leqq \exp(t^2 |a_{nk}|^\nu n^{-\rho(2-\nu)} 4)$. Hence

$$E \exp\left(t \sum_{k=1}^{\infty} a_{nk} Y_{nk}'\right) \leqq \exp\left(t^2 \sum_{k=1}^{\infty} |a_{nk}|^\nu n^{-\rho(2-\nu)} 4\right) \leqq \exp(t^2 n^{-\rho(2-\nu)-\lambda} 4K).$$

Fix $\epsilon > 0$. A Chebychev argument yields

$$P\left[\left|\sum_{k=1}^{\infty} a_{nk} Y_{nk}'\right| > \epsilon\right] \leqq 2 \exp(-\epsilon t) \exp(t^2 n^{-\rho(2-\nu)-\lambda} 4K).$$

Since $t = n^\rho/2$, it follows that $\sum_{n=1}^{\infty} P[|\sum_{k=1}^{\infty} a_{nk} Y'_{nk}| > \epsilon] < \infty$. Since $\sum_{k=1}^{\infty} a_{nk} ED'_{nk} \to 0$ as $n \to \infty$, it follows that $\sum_{n=1}^{\infty} P[|T_n'| > 2\epsilon] < \infty$. Hence T_n' converges completely to zero.

LEMMA 5. *Let* $E(\exp(tD_k)|\mathfrak{F}_{k-1}) \leqq \exp(t^2)$ a.e. *for every constant t and let* $A_n < \infty$. *Then* T_n *is generalized Gaussian with* $\tau^2(T_n) \leqq 2A_n$.

PROOF. For fixed t and m, let $Y_j = \exp(tT_{nj} + t^2 \sum_{k=j+1}^{m} a_{nk}^2)$ for $j = 1, 2, \cdots, m$, using the convention $\sum_{k=m+1}^{m}(\cdot) = 0$. Choose $j \neq 1$.

$$E(Y_j \mid \mathfrak{F}_{j-1}) = \exp(tT_{n,j-1} + t^2 \textstyle\sum_{k=j+1}^{m} a_{nk}^2) E \exp(ta_{nj}D_j \mid \mathfrak{F}_{j-1})$$
$$\leqq \exp(tT_{n,j-1} + \textstyle\sum_{k=j}^{m} a_{nk}^2 t^2) = Y_{j-1} \text{ a.e.}$$

Hence $EY_j \leqq EY_{j-1}$. By induction, $EY_m \leqq EY_1$, $EY_m = E\exp(tT_{nm})$ and

$EY_1 = E\exp(ta_{n1}D_1)\exp(t^2 \sum_{k=2}^{m} a_{nk}^2) \leqq \exp(t^2 \sum_{k=1}^{m} a_{nk}^2) \leqq \exp(t^2 A_n)$.

Hence $E\exp(tT_{nm}) \leqq \exp(t^2A_n)$. It is easy to see that $ED_k^2 \leqq 2$. Thus $(E|T_{nm}|) \leqq ET_{nm}^2 + 1 = \sum_{k=1}^{m} a_{nk}^2 ED_k^2 + 1 \leqq 2A_n + 1$. Hence T_{nm} converges a.s. to T_n as $m \to \infty$ by the Doob martingale convergence theorem ([2], p. 319). It then follows by an application of the Fatou lemma that $E\exp(tT_n) \leqq \exp(t^2A_n)$.

REMARK. The manner of constructing the Y_j's so that they form a supermartingale was learned from a paper of Dubins and Freedman ([3], p. 804). A slightly different proof can be given which does not use this technique.

3. Convergence in the independent identically distributed case. Let the $(D_k, k \geqq 1)$ be independent identically distributed random variables.

THEOREM 1. *Let* $|a_{nk}| \leqq Kn^{-\alpha}$ *for some* $\alpha > 0$ *and* $E|D_k|^{(1+\alpha+\beta)/\alpha} < \infty$ *where* $\beta > -1-\alpha$.

(i) *If* $(1+\alpha+\beta)/\alpha \geqq 2$, $A_n \leqq Kn^{\beta-\alpha}$, $\sum_{n=1}^{\infty}\exp(-t/A_n) < \infty$ *for all* $t > 0$, $ED_k^2 \log^+ |D_k| < \infty$ *and* $ED_k = 0$, *then* T_n *converges completely to zero.*

(ii) *If* $(1+\alpha+\beta)/\alpha = 2$, $\sum_{k=1}^{\infty} |a_{nk}|^\delta \leqq Kn^{\alpha(2-\delta)-1}$ *for some* $0 < \delta < 2$ *and* $ED_k = 0$, *then* T_n *converges completely to zero.*

(iii) *If* $1 \leqq (1+\alpha+\beta)/\alpha < 2$, $A_n \leqq Kn^{\beta-\alpha}$, $\sum_{k=1}^{\infty} |a_{nk}|^{(1+\alpha+\beta)/\alpha} \leqq Kn^{-\gamma}$ *for some* $\gamma > 0$ *and either* $\sum_{k=1}^{\infty}|a_{nk}| \leqq K$ *and* $ED_k = 0$ *or* $\sum_{k=1}^{\infty}|a_{nk}| \to 0$ *as* $n \to \infty$, *then* T_n *converges completely to zero.*

(iv) *If* $0 < (1+\alpha+\beta)/\alpha < 1$, $A_n \leqq Kn^{\beta-\alpha}$, $\sum_{k=1}^{\infty}|a_{nk}|^{(1+\alpha+\beta)/\alpha} \leqq Kn^{-\gamma}$ *for some* $\gamma > 0$, *and* $a_{nk} = 0$ *for* $k > n^\zeta$ *where* $\zeta < \gamma\alpha/(1+\alpha+\beta)$, *then* T_n *converges completely to zero.*

PROOF. (i) and (ii). Fix $\epsilon > 0$. According to the Kolmogorov convergence theorem, ([10], p. 236) $\lim_{m\to\infty} T_{nm} = T_n$ exists a.s. Since $\sum_{k=1}^{\infty} a_{nk}^2 ED_k^2 < \infty$. We may decompose

$$T_n = \textstyle\sum_{k=1}^{\infty} a'_{nk}D_k - \sum_{k=1}^{\infty} a''_{nk}D_k$$

where $a'_{nk} > 0$ and $a''_{nk} > 0$.

$$P[|T_n| > 2\epsilon] \leqq P[|\textstyle\sum_{k=1}^{\infty} a'_{nk}D_k| > \epsilon] + P[|\sum_{k=1}^{\infty} a''_{nk}D_k| > \epsilon].$$

Hence, without loss of generality, we assume $a_{nk} > 0$ throughout the remainder of the proof of (i) and (ii). Let $D'_{nk} = D_k I[a_{nk} D_k \leqq n^{-\rho}]$ where $\rho > 0$ will be chosen later. Let $T'_{nm} = \sum_{k=1}^{m} a_{nk} D'_{nk}$. By Lemma 1, T'_{nm} converges a.s. to a random variable T_n' as $m \to \infty$ and $\sum_{n=1}^{\infty} P[T_n' > \epsilon] < \infty$.

Let $D''_{nk} = D_k I[a_{nk} D_k > \epsilon/N]$ where N is a positive integer to be chosen later. Let $T''_{nm} = \sum_{k=1}^{m} a_{nk} D''_{nk}$.

$$\sum_{k=1}^{\infty} P[D''_{nk} \neq 0] = \sum_{k=1}^{\infty} P[D_k > \epsilon/(N a_{nk})] \leqq C N^2 A_n/\epsilon^2 < \infty.$$

Thus, by an application of the Borel Cantelli lemma, it follows that T''_{nm} converges a.s. to a random variable T_n'' as $m \to \infty$. Applying (2) of Lemma 2 with $Z_k = D_k$ yields

$$\sum_{n=1}^{\infty} P[T_n'' > \epsilon] \leqq \sum_{j=1}^{\infty} \sum_{n=1}^{g_j} f_n(j) P[j - 1 \leqq |D_k| < j]$$

where $f_n(j)$ and g_j are defined in the statement of Lemma 2. We now consider (i). By the definitions of A_n and $f_n(j)$, $A_n \geqq f_n(j)\epsilon^2/(Nj)^2$. Since $A_n \leqq K n^{\beta - \alpha}$, it follows that $f_n(j) \leqq K n^{\beta-\alpha} N^2 j^2/\epsilon^2$. Thus

$$\sum_{n=1}^{\infty} P[T_n'' > \epsilon] \leqq (KN^2/\epsilon^2) \sum_{j=1}^{\infty} \sum_{n=1}^{g_j} n^{\beta-\alpha} j^2 P[j - 1 \leqq |D_k| < j].$$

Elementary computation shows that $\sum_{n=1}^{g_j} n^{\beta-\alpha} \leqq K' j^{(\beta-\alpha+1)/\alpha}$ if $\beta - \alpha \neq -1$ and $\sum_{n=1}^{g_j} n^{\beta-\alpha} \leqq K' \log j$ for $j \geqq 2$ if $\beta - \alpha = 1$ where K' is a fixed constant independent of j. $ED_k^2 \log^+ |D_k| < \infty$ implies that

$$\sum_{j=1}^{\infty} j^2 \log j P[j - 1 \leqq |D_k| < j] < \infty.$$

Similarly, $E|D_k|^{(1+\alpha+\beta)/\alpha} < \infty$ implies that

$$\sum_{j=1}^{\infty} j^{(1+\alpha+\beta)/\alpha} P[j - 1 \leqq |D_k| < j] < \infty.$$

Hence $\sum_{n=1}^{\infty} P[T_n'' > \epsilon] < \infty$ in the case of (i). We now consider (ii). Since $\sum_{k=1}^{\infty} |a_{nk}|^{\delta} \leqq K n^{\alpha(2-\delta)-1}$, it follows that $f_n(j) \leqq K n^{\alpha(2-\delta)-1} (Nj/\epsilon)^{\delta}$. Thus

$$\sum_{n=1}^{\infty} P[T_n'' > \epsilon] \leqq (KN^{\delta}/\epsilon^{\delta}) \sum_{j=1}^{\infty} \sum_{n=1}^{g_j} n^{\alpha(2-\delta)-1} j^{\delta} P[j - 1 \leqq |D_k| < j].$$

Elementary computation shows that $\sum_{n=1}^{g_j} n^{\alpha(2-\delta)-1} \leqq K' j^{2-\delta}$ where K' is a fixed constant independent of j. $ED_k^2 < \infty$ implies

$$\sum_{j=1}^{\infty} j^2 P[j - 1 \leqq |D_k| < j] < \infty.$$

Thus, $\sum_{n=1}^{\infty} P[T_n'' > \epsilon] < \infty$ in the case of (ii).

Let $D'''_{nk} = D_k - D'_{nk} - D''_{nk}$, i.e., $D'''_{nk} = D_k I[n^{-\rho}/a_{nk} < D_k \leqq \epsilon/(N a_{nk})]$. Let $T'''_{nm} = \sum_{k=1}^{m} a_{nk} D'''_{nk}$. $T'''_{nm} = T_{nm} - T'_{nm} - T''_{nm}$ converges a.s. since T_{nm}, T'_{nm}, and T''_{nm} each converge a.s. Without loss of generality we assume $E|D_k|^{(1+\alpha+\beta)/\alpha} \leqq 1$. Then, applying Lemma 3 with $Z_k = D_k'''$ and $\nu = (1 + \alpha + \beta)/\alpha$ implies that

$$\sum_{n=1}^{\infty} P[|T_n'''| > \epsilon] \leqq \sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} |a_{nk}|^{(1+\alpha+\beta)/\alpha} n^{\rho(1+\alpha+\beta)/\alpha}\right)^N$$
$$\leqq \sum_{n=1}^{\infty} \left(n^{-1+\rho(1+\alpha+\beta)/\alpha} K'\right)^N$$

for some constant K'. By choosing ρ sufficiently small and N sufficiently large, the preceding sum becomes finite. Combining the above results for T_n', T_n'', and T_n''', it follows that $\sum_{n=1}^{\infty} P[T_n > 3\epsilon] < \infty$. Replacing D_k by $-D_k$ in the above arguments yields $\sum_{n=1}^{\infty} P[-T_n > 3\epsilon] < \infty$. This completes the proof of (i) and (ii).

(iii) Since $\sum_{k=1}^{\infty} |a_{nk}|E|D_k| < \infty$, it follows that T_{nm} converges a.s. to a random variable T_n as $m \to \infty$. Fix $\epsilon > 0$. Let $D'_{nk} = D_k I[|a_{nk}D_k| \leqq n^{-\rho}]$ where $0 < \rho < \min(\alpha, \gamma\alpha/(1 + \alpha + \beta))$. Let $T'_{nm} = \sum_{k=1}^{m} a_{nk}D'_{nk}$. By Lemma 4, T'_{nm} converges a.s. to a random variable T_n' as $m \to \infty$ and T_n' converges completely to zero.

Let $D''_{nk} = D_k I[|a_{nk}D_k| > \epsilon/N]$ where N is a positive integer to be chosen later. Let $T''_{nm} = \sum_{k=1}^{m} a_{nk}D''_{nk}$. Since $\sum_{k=1}^{\infty} |a_{nk}|E|D_k| < \infty$, it follows that T''_{nm} converges a.s. to a random variable T_n'' as $m \to \infty$. Applying (2) of Lemma 2 with $Z_k = D_k$ yields $\sum_{n=1}^{\infty} P[|T_n''| > \epsilon] \leqq \sum_{j=1}^{\infty} \sum_{n=1}^{g_j} f_n(j)P[j - 1 \leqq |D_k| < j]$ where $f_n(j)$ and g_j are defined in the statement of Lemma 2. By the definitions of A_n and $f_n(j)$, $A_n \geqq f_n(j)\epsilon^2/(Nj)^2$. Since $A_n \leqq Kn^{\beta-\alpha}$, it follows that $f_n(j) \leqq Kn^{\beta-\alpha}N^2j^2/\epsilon^2$. Thus,

$$\sum_{n=1}^{\infty} P[|T_n''| > \epsilon] \leqq (KN^2/\epsilon^2) \sum_{j=1}^{\infty} \sum_{n=1}^{g_j} n^{\beta-\alpha} j^2 P[j - 1 \leqq |D_k| < j].$$

Elementary computation shows that $\sum_{n=1}^{g_j} n^{\beta-\alpha} \leqq K' j^{(\beta-\alpha+1)/\alpha}$ where K' is a constant independent of j. $E|D_k|^{(1+\alpha+\beta)/\alpha} < \infty$ implies that

$$\sum_{j=1}^{\infty} j^{(1+\alpha+\beta)/\alpha} P[j - 1 \leqq |D_k| < j] < \infty.$$

Hence $\sum_{n=1}^{\infty} P[|T_n''| > \epsilon] < \infty$.

Let

$$D'''_{nk} = D_k - D'_{nk} - D''_{nk}, \quad \text{i.e.,} \quad D'''_{nk} = D_k I[n^{-\rho}/|a_{nk}| < |D_k| \leqq \epsilon/(N|a_{nk}|)].$$

Let $T'''_{nm} = \sum_{k=1}^{m} a_{nk}D'''_{nm}$. T'''_{nm} converges a.s. to a random variable T'''_n since $\sum_{k=1}^{\infty} |a_{nk}|E|D_k| < \infty$. Without loss of generality, we may assume

$$E|D_k|^{(1+\alpha+\beta)/\alpha} \leqq 1.$$

Applying Lemma 3 with $Z_k = D_k$ and $\nu = (1 + \alpha + \beta)/\alpha$ implies that

$$P[|T_n'''| > \epsilon] \leqq (\sum_{k=1}^{\infty} |a_{nk}|^{(1+\alpha+\beta)/\alpha} n^{\rho(1+\alpha+\beta)/\alpha})^N.$$

Since $\sum_{k=1}^{\infty} |a_{nk}|^{(1+\alpha+\beta)/\alpha} \leqq Kn^{-\gamma}$, it follows that

$$P[|T_n'''| > \epsilon] \leqq (Kn^{-\gamma+(1+\alpha+\beta)\rho/\alpha})^N.$$

By choosing ρ sufficiently small and N sufficiently large, it follows that $\sum_{n=1}^{\infty} P[|T_n'''| > \epsilon] < \infty$. Combining the above results for T_n', T_n'' and T_n''', it follows that T_n converges completely to zero thus completing the proof of (iii).

(iv) Fix $\epsilon > 0$. Let $D'_{nk} = D_k I[|a_{nk}D_k| \leqq n^{-\rho}]$ for some $\rho > 0$ to be chosen later and let $T_n' = \sum_{k=1}^{\infty} a_{nk}D'_{nk}$. $|\sum_{k=1}^{\infty} a_{nk}D'_{nk}| \leqq \sum_{k=1}^{\infty} |a_{nk}D'_{nk}| \leqq n^{\zeta-\rho}$. Hence

$\sum_{n=1}^{\infty} P[|T_n'| > \epsilon] < \infty$ by choosing $\rho > \zeta$. Let $D_{nk}'' = D_k I[|a_{nk} D_k| > \epsilon/N]$ where N is a positive integer to be chosen later and let $T_n'' = \sum_{k=1}^{\infty} a_{nk} D_{nk}''$. Applying (2) of Lemma 2 with $Z_k = D_k$ yields

$$\sum_{n=1}^{\infty} P[|T_n''| > \epsilon] \leqq \sum_{j=1}^{\infty} \sum_{n=1}^{g_j} f_n(j) P[j-1 \leqq |D_k| < j]$$

where $f_n(j)$ and g_j are defined in the statement of Lemma 2. By the definitions of A_n and $f_n(j)$, $A_n \geqq f_n(j)\epsilon^2/(Nj)^2$. Since $A_n \leqq Kn^{\beta-\alpha}$, it follows that $f_n(j) \leqq Kn^{\beta-\alpha}N^2j^2/\epsilon^2$. Thus,

$$\sum_{n=1}^{\infty} P[|T_n''| > \epsilon] \leqq (KN^2/\epsilon^2) \sum_{j=1}^{\infty} \sum_{n=1}^{g_j} n^{\beta-\alpha} j^2 P[j-1 \leqq |D_k| < j].$$

Elementary computation shows that $\sum_{n=1}^{g_j} n^{\beta-\alpha} \leqq K' j^{(\beta-\alpha+1)/\alpha}$ where K' is a constant independent of j. $E|D_k|^{(1+\alpha+\beta)/\alpha} < \infty$ implies that

$$\sum_{j=1}^{\infty} j^{(1+\alpha+\beta)/\alpha} P[j-1 \leqq |D_k| < j] < \infty.$$

Hence $\sum_{n=1}^{\infty} P[|T_n''| > \epsilon] < \infty$. Let

$D_{nk}''' = D_k - D_{nk}' - D_{nk}''$, i.e., $D_{nk}''' = D_k I[n^{-\rho}/|a_{nk}| < |D_k| \leqq \epsilon/(N|a_{nk}|)]$.

Without loss of generality, we may assume $E|D_k|^{(1+\alpha+\beta)/\alpha} \leqq 1$. Applying Lemma 3 with $Z_k = D_k$ and $\nu = (1+\alpha+\beta)/\alpha$ yields

$$P[|T_n'''| > \epsilon] \leqq \left(\sum_{k=1}^{\infty} |a_{nk}|^{(1+\alpha+\beta)/\alpha} n^{\rho(1+\alpha+\beta)/\alpha}\right)^N \leqq \left(Kn^{-\gamma+\rho(1+\alpha+\beta)/\alpha}\right)^N.$$

We now choose $\rho < (\gamma\alpha)/(1+\alpha+\beta)$ such that $\rho > \zeta$ is satisfied. It then follows for sufficiently large N that $\sum_{n=1}^{\infty} P[|T_n'''| > \epsilon] < \infty$. Combining the above results for T_n', T_n'', and T_n''', it follows that T_n converges completely to zero. The proof of (iv) and of the theorem is complete.

COROLLARY 1. *Let* $E|D_k|^{2/\alpha} < \infty$ *for some* $\alpha > 0$, $ED_k = 0$ *if* $0 < \alpha \leqq 1$, $|a_{nk}| \leqq Kn^{-\alpha}$, $a_{nk} = 0$ *for* $k > n$, *and* $\sum_{k=1}^{\infty} \exp(-t/A_n) < \infty$ *for all* $t > 0$. *Then* T_n *converges completely to zero.*

PROOF. Let $0 < \alpha < 1$. Then $|a_{nk}| < Kn^{-\alpha}$ and $a_{nk} = 0$ for $k > n$ together imply that $A_n \leqq K^2 n^{1-2\alpha}$. Thus for $\beta = 1-\alpha$, the hypotheses of Theorem 1 (i) are satisfied. Let $\alpha = 1$. Then for $\delta = 1$ the hypotheses of Theorem 1 (ii) are satisfied. Let $1 < \alpha \leqq 2$. Then, letting $\beta = 1-\alpha$, one set of hypotheses of Theorem 1 (iii) is satisfied. Let $\alpha > 2$. Then for $\beta = 1-\alpha$ and $\zeta = \gamma = 1$, the hypotheses of Theorem 1 (iv) are satisfied.

REMARKS. The fundamental result of the above type on complete convergence states that if $E|D_k|^{2/\alpha} < \infty$ for some $\infty > \alpha > \frac{1}{2}$,

$$ED_k = 0 \quad \text{if} \quad 1 \geqq \alpha > \tfrac{1}{2},$$

$$a_{nk} = n^{-\alpha} \quad \text{for} \quad k \leqq n \quad \text{and}$$

$$a_{nk} = 0 \quad \text{for} \quad k > n,$$

then $T_n = \sum_{k=1}^{n} D_k/n^\alpha$ converges completely to zero. This result is due to Hsu and Robbins [8] for $\alpha = 1$ and Erdös [10] for $\alpha \neq 1$. Corollary 1 includes the above result and generalizes it to a triangular matrix of coefficients satisfying certain restrictions on the magnitude of its entries. Theorem 1 generalizes this result still further by replacing the hypothesis of triangularity by more general hypotheses on the a_{nk}'s. In [4] Erdös also states that if $ED_k^4 < \infty$ and $ED_k = 0$, then there exists an $r > 0$ such that for $a_{nk} = 1/(n^{\frac{1}{2}}(\log n)^r)$ when $k \leqq n$ and $a_{nk} = 0$ for $k > n$, it follows that $T_n = \sum_{k=1}^{n} D_k/(n^{\frac{1}{2}}(\log n)^r)$ converges completely to zero. Corollary 1 and Theorem 1 generalize this result also. From Corollary 1 it is easy to see that $r = \frac{1}{2} + \delta$ for any $\delta > 0$ works in the statement of the Erdös result. Obviously the $(\log n)^r$ term in the denominator cannot be dropped entirely since $\sum_{k=1}^{n} D_k/n^{\frac{1}{2}}$ obeys the central limit theorem ([10], p. 247). This shows that the condition $\sum_{n=1}^{\infty} \exp(-t/A_n) < \infty$ for all $t > 0$ cannot be dropped from the statements of Theorem 1 (i) and Corollary 1. For, if it could be dropped, $ED_k^4 < \infty$ would then imply $\sum_{k=1}^{n} D_k/n^{\frac{1}{2}}$ converges completely to zero. For use in applications of Theorem 1 (i) and Corollary 1, it is interesting to note that $A_n = o((\log n)^{-1})$ implies that $\sum_{n=1}^{\infty} \exp(-t/A_n) < \infty$ for all $t > 0$ and that $A_n = O((\log n)^{-1})$ does not imply that $\sum_{n=1}^{\infty} \exp(-t/A_n) < \infty$ for all $t > 0$. The condition $A_n = o((\log n)^{-1})$ is easy to verify in practice and by the previous remark only slightly stronger than $\sum_{n=1}^{\infty} \exp(-t/A_n) < \infty$ for all $t > 0$.

Recently, Chow ([1], p. 1488) has proved that if $E|D_k|^{2/\alpha} < \infty$, $0 < \alpha \leqq 1$, $ED_k = 0$, $|a_{nk}| \leqq KA_n$ for $k \leqq n$, $a_{nk} = 0$ for $k > n$, and $A_n \leqq Kn^{-\alpha}$, then T_n converges completely to zero. It was this particular result which motivated the present work. Corollary 1 improves and generalizes this result by replacing $|a_{nk}| \leqq KA_n$ for $k \leqq n$, and $A_n \leqq Kn^{-\alpha}$ by weaker conditions on the a_{nk} matrix and by extending the result to the case where $E|D_k|^{2/\alpha} < \infty$ for some $\alpha > 1$ but $ED_k^2 = \infty$. Theorem 1 generalizes this result further by replacing the hypothesis of triangularity by more general hypotheses on the a_{nk}'s.

Erdös [4] has established that $\sum_{k=1}^{n} D_k/n^{\alpha}$ converging completely to zero implies that $E|D_k|^{2/\alpha} < \infty$ if $\alpha > \frac{1}{2}$ (and implies that $ED_k = 0$ if $\frac{1}{2} < \alpha < 1$). Hence, Corollary 1 is sharp for $\alpha > \frac{1}{2}$. Even for $\alpha \leqq \frac{1}{2}$, Corollary 1 is rather sharp since it says that if $ED_k^4 < \infty$ and $ED_k = 0$, then $\sum_{k=1}^{n} D_k/(n^{\frac{1}{2}}(\log n)^{\frac{1}{2}+\delta})$ converges completely to zero for all $\delta > 0$. But, taking the D_k's to be normal random variables with $ED_k = 0$, it is easily seen that $\sum_{k=1}^{n} D_k/(n^{\frac{1}{2}}(\log n)^{\frac{1}{2}})$ does not converge completely to zero, even though $ED_k^4 < \infty$ and $ED_k = 0$.

COROLLARY 2. *Let* $E|D_k|^{1+1/\alpha} < \infty$ *and* $|a_{nk}| \leqq Kn^{-\alpha}$ *for some* $\alpha > 0$.

(i) *If* $0 < \alpha < 1$, $ED_k = 0$, *and* $A_n \leqq Kn^{-\alpha}$, *then* T_n *converges completely to* 0.

(ii) *If* $\alpha = 1$, $ED_k = 0$, *and* $\sum_{k=1}^{\infty} |a_{nk}|^{\delta} \leqq Kn^{1-\delta}$ *for some* $0 < \delta < 2$, *then* T_n *converges completely to* 0.

(iii) *If* $\alpha > 1$ *and either* $\sum_{k=1}^{\infty} |a_{nk}| \leqq K$ *and* $ED_k = 0$ *or* $\sum_{k=1}^{\infty} |a_{nk}| \to 0$ as $n \to \infty$, *then* T_n *converges completely to* 0.

PROOF. (i) is immediate from Theorem 1 (i) with $\beta = 0$.

(ii) is immediate from Theorem 1 (ii) with $\alpha = 1$ and $\beta = 0$.

(iii) is immediate from Theorem 1 (iii) with $\beta = 0$.

REMARKS. Pruitt [11] has proved that for matrix conditions somewhat stronger than regularity, i.e. $\sum_{k=1}^{\infty} a_{nk} \to 1$ as $n \to \infty$, $\sum_{k=1}^{\infty} |a_{nk}| \leqq K$ and $|a_{nk}| \leqq Kn^{-\alpha}$ for some $\alpha > 0$, it then follows from $E|D_k|^{1+1/\alpha} < \infty$ that T_n converges completely to ED_k. Corollary 2 implies this result and generalizes it by replacing $\sum_{k=1}^{\infty} |a_{nk}| \leqq K$ by a weaker condition when $\alpha \leqq 1$. Pruitt gives an example to show his result is sharp. Hence, Corollary 2 (i) and (iii) are sharp and Corollary 2 (ii) is sharp for $\delta \geqq 1$.

COROLLARY 3. *Let* $E\,|D_k|^{1/\eta} < \infty$.

(i) *If* $0 < \eta \leqq 1$, $ED_k = 0$, $ED_k^2 \log^+ |D_k| < \infty$, *and* $A_n \leqq Kn^{-\eta}$, *then* T_n *converges completely to* 0.

(ii) *If* $\eta = 1$, $ED_k = 0$, *and* $\sum_{k=1}^{\infty} |a_{nk}|^{\delta} \leqq Kn^{-\delta/2}$ *for some* $0 < \delta < 2$, *then* T_n *converges completely to* 0.

(iii) *If* $1 < \eta \leqq 2$, $A_n \leqq Kn^{-\eta}$, $\sum_{k=1}^{\infty} |a_{nk}|^{2/\eta} \leqq Kn^{-\gamma}$ *for some* $\gamma > 0$, *and either* $\sum_{k=1}^{\infty} |a_{nk}| \leqq K$ *and* $ED_k = 0$ *or* $\sum_{k=1}^{\infty} |a_{nk}| \to 0$ *as* $n \to \infty$, *then* T_n *converges completely to* 0.

(iv) *If* $\eta > 2$, $\sum_{k=1}^{\infty} |a_{nk}|^{2/\eta} \leqq Kn^{-\gamma}$ *for some* $\gamma > 0$, $A_n \leqq Kn^{-\eta}$, *and* $a_{nk} = 0$ *for* $k > n^{\zeta}$ *for some* $\zeta < (\eta\gamma)/2$, *then* T_n *converges completely to* 0.

PROOF. Let $0 < \eta \leqq 1$. $A_n \leqq Kn^{-\eta}$ implies that $|a_{nk}| \leqq K^{\frac{1}{2}} n^{-\eta/2}$. Hence for $\beta = -\eta/2$ and $\alpha = \eta/2$ the hypotheses of Theorem 1 (i) are satisfied and $(1 + \alpha + \beta)/\alpha = 2/\eta$. Let $\eta = 1$. Choosing $\alpha = \frac{1}{2}$, (ii) is then immediate from Theorem 1 (ii), noting that $\sum_{k=1}^{\infty} |a_{nk}|^{\delta} \leqq Kn^{-\delta/2}$ implies that $|a_{nk}| \leqq K^{1/\delta} n^{-\frac{1}{2}}$. Let $1 < \eta \leqq 2$. Then, as above, $|a_{nk}| \leqq K^{\frac{1}{2}} n^{-\eta/2}$. Hence for $\beta = -\eta/2$ and $\alpha = \eta/2$ the hypotheses of Theorem 1 (iii) are satisfied and $(1 + \alpha + \beta)/\alpha = 2/\eta$. Let $\eta > 2$. Again $|a_{nk}| \leqq K^{\frac{1}{2}} n^{-\eta/2}$. Hence for $\beta = -\eta/2$ and $\alpha = \eta/2$, the hypotheses of Theorem 1 (iv) are satisfied and $(1 + \alpha + \beta)/\alpha = 2/\eta$.

REMARK. If we assume that the D_k's are identically distributed, $a_{nn} = 1/n^{\eta/2}$ for some $\eta > 0$, and $a_{nk} = 0$ for $k \neq n$, then $T_n \equiv D_n/n^{\eta/2}$ converging completely (a.s.) to zero implies that $E\,|D_k|^{2/\eta} < \infty$ since $\sum_{k=1}^{\infty} P[|D_k| > k^{\eta/2}] < \infty$ is equivalent to $E\,|D_k|^{2/\eta} < \infty$. Thus Corollary 3 (i) is sharp for $\eta < 1$, Corollary 3 (ii) is sharp, Corollary 3 (iii) is sharp for $\gamma \leqq 1$ and Corollary 3 (iv) is sharp for $\gamma \leqq 1$ and $\zeta \geqq 1$.

THEOREM 2. *Let* $A_n \leqq Kn^{-\delta}$ *for some* $\delta > 0$, $a_{nk}^2 \leqq Kk^{-1}$, $ED_k^2 = 1$, *and* $ED_k = 0$. *Then* T_n *converges* a.s. *to zero.*

PROOF. Fix $\epsilon > 0$. $\lim_{m\to\infty} T_{nm} \equiv T_n$ exists a.s. by the Kolmogorov convergence theorem. Without loss of generality, we assume $a_{nk} > 0$ throughout the remainder of the proof. Let $D'_{nk} = D_k I[a_{nk} D_k \leqq n^{-\rho}]$ where $\rho > 0$ will be chosen later. Let $T'_{nm} = \sum_{k=1}^{m} a_{nk} D'_{nk}$. By Lemma 1, T'_{nm} converges a.s. to a random variable T_n' as $m \to \infty$ and $\sum_{n=1}^{\infty} P[T_n' > \epsilon] < \infty$.

Let $D''_{nk} = D_k I[a_{nk} D_k > \epsilon/N]$ where N is a positive integer to be chosen later. Let

$$T''_{nm} = \sum_{k=1}^{m} a_{nk} D''_{nk}. \quad \sum_{k=1}^{\infty} P[D''_{nk} \neq 0] = \sum_{k=1}^{n} P[D_k > \epsilon/(Na_{nk})] \leqq N^2 A_n/\epsilon^2 < \infty.$$

Hence, T''_{nm} converges a.s. to a random variable T_n'' as $m \to \infty$. By the Holder inequality,

$$|T''_n|^2 \leqq A_n \sum_{k=1}^{\infty} (D''_{nk})^2 = A_n \sum_{k=1}^{\infty} D_k^2 I[D_k > \epsilon/(Na_{nk})] \leqq Kn^{-\delta} \sum_{k=1}^{\infty} D_k^2 I[D_k^2 > \epsilon^2 k/(N^2 K)]$$

since $a_{nk}^2 \leqq Kk^{-1}$. But $X \equiv K \sum_{k=1}^{\infty} D_k^2 I[D_k^2 > \epsilon^2 k/(N^2 K)] < \infty$ a.s. since $\sum_{k=1}^{\infty} P[D_k^2 > \epsilon^2 k/(N^2 K)] < \infty$ follows from the fact that the D_k's are identically distributed with $ED_k^2 < \infty$. Thus $|T_n''|^2 \leqq Xn^{-\delta}$ a.s. and hence T_n'' converges a.s. to zero.

Let

$$D_{nk}''' = D_k - D_k' - D_k'', \quad \text{i.e.,}$$
$$D_{nk}''' = D_k I[n^{-\rho}/a_{nk} < D_k \leqq \epsilon/(Na_{nk})].$$

Let $T_{nm}''' = \sum_{k=1}^{m} a_{nk} D_{nk}'''$. T_{nm}''' converges a.s. since T_{nm}, T_{nm}' and T_{nm}'' each converge a.s. Applying Lemma 3 with $Z_k = D_k$ and $\nu = 2$ implies that

$$\sum_{n=1}^{\infty} P[T_n''' > \epsilon] \leqq \sum_{n=1}^{\infty} (\sum_{k=1}^{\infty} a_{nk}^2 n^{2\rho})^N \leqq \sum_{n=1}^{\infty} (Kn^{-\delta+2\rho})^N.$$

By choosing ρ sufficiently small and N sufficiently large the preceding sum becomes finite. Combining the above results for T_n', T_n'', and T_n''', it follows that $T_n)^+ \equiv \max(0, T_n)$ converges a.s. to zero. By symmetry it follows that $(T_n)^- \equiv -\min(0, T_n)$ and hence T_n converges a.s. to zero.

REMARKS. Chow ([1], p. 1484) has recently proved that

$$(5) \qquad ED_k^2 < \infty, \qquad ED_k = 0, \qquad nA_n \to 1$$

as $n \to \infty$ and $a_{nk} = 0$ for $k > n$, implies that T_n converges a.s. to zero. Theorem 2 includes this result and generalizes it by replacing the assumption of triangularity of the a_{nk} matrix by a weaker condition and weakening the condition on the magnitudes of the A_n's. Unlike the proof given by Chow, the present proof makes no use of the strong law of large numbers.

It is interesting to compare Corollary 3 (i) and (5). Corollary 3 (i) does not imply (5). However if in (5) we were to assume the finiteness of a slightly higher moment than the second, i.e., $ED_k^2 \log^+ |D_k| < \infty$, we could drop the assumption of triangularity and conclude that T_n converges completely to zero by Corollary 3 (i). Chow [1] has established that (5) is sharp; hence, it follows that Theorem 2 is sharp also.

4. Convergence in the non-identically distributed case. Let the $(D_k, k \geqq 1)$ be independent random variables. An examination of the proof of Theorem 1 shows that the identically distributed hypotheses may be dropped by slightly strengthening the moment condition assumed. This yields Theorem 3.

THEOREM 3. *Let* $|a_{nk}| \leqq Kn^{-\alpha}$ *for some* $\alpha > 0$.

(i) *If* $E|Dk|^{(1+\alpha+\beta)/\alpha} (\log^+ |D_k|)^{1+\xi} \leqq K$ *for some* $\xi > 0$,

$$(1 + \alpha + \beta)/\alpha \geqq 2, \qquad ED_k^2(\log^+ |D_k|)^{2+\xi} \leqq K, \qquad ED_k = 0, \qquad A_n \leqq Kn^{\beta-\alpha},$$

and $\sum_{n=1}^{\infty} \exp(-t/A_n) < \infty$ *for all* $t > 0$, *then* T_n *converges completely to zero.*

(ii) *If* $ED_k^2 \leqq K$, $ED_k = 0$, *and* $\sum_{k=1}^{\infty} |a_{nk}|^\delta \leqq Kn^{\alpha(2-\delta)-1-\xi}$ *for some* $0 < \delta < 2$ *and* $\xi > 0$, *then* T_n *converges completely to zero.*

(iii) *If* $E|D_k|^{(1+\alpha+\beta)/\alpha} (\log^+ |D_k|)^{1+\xi} \leqq K$ *for some* $\xi > 0$,

$$1 \leqq (1 + \alpha + \beta)/\alpha < 2, \qquad A_n \leqq Kn^{\beta-\alpha}, \qquad \sum_{k=1}^{\infty} |a_{nk}|^{(1+\alpha+\beta/\alpha} \leqq Kn^{-\gamma}$$

for some $\gamma > 0$ *and either* $\sum_{k=1}^{\infty} |a_{nk}| \leqq K$ *and* $ED_k = 0$ *or* $\sum_{k=1}^{\infty} |a_{nk}| \to 0$ *as* $n \to \infty$, *then* T_n *converges completely to zero.*

(iv) *If* $E|D_k|^{(1+\alpha+\beta)/\alpha}(\log^+ |D_k|)^{1+\xi} \leqq K$ *for some* $\xi > 0$,

$$0 < (1 + \alpha + \beta)/\alpha < 1, \qquad A_n \leqq Kn^{\beta-\alpha}, \qquad \sum_{k=1}^{\infty} |a_{nk}|^{(1+\alpha+\beta)/\alpha} \leqq Kn^{-\gamma}$$

for some $\gamma > 0$ *and* $a_{nk} = 0$ *for* $k > n^{\zeta}$ *where* $\zeta < \gamma(1 + \alpha + \beta)/\alpha$ *then* T_n *converges completely to zero.*

Proof. We establish (i) only, the proofs of (ii), (iii) and (iv) being omitted because of the similarity of their proofs to (i). Fix $\epsilon > 0$ and assume without loss of generality that $a_{nk} > 0$. Let $D'_{nk} = D_k I[a_{nk}D_k \leqq n^{-\rho}]$ where $\rho > 0$ is chosen later in the proof, $D''_{nk} = D_k I[a_{nk}D_k > \epsilon/N]$ where N is a positive integer to be chosen later, and $D'''_{nk} = D_k - D'_{nk} - D''_{nk}$. Let

$$T'_{nm} = \sum_{k=1}^{m} a_{nk}D'_{nk},$$
$$T''_{nm} = \sum_{k=1}^{m} a_{nk}D''_{nk}, \quad \text{and}$$
$$T'''_{nm} = \sum_{k=1}^{m} a_{nk}D'''_{nk}.$$

In Theorem 1 (*i*) the proof of the facts that T_{nm} converges a.s. to T_n as $m \to \infty$, T'_{nm} converges a.s. to T_n' as $m \to \infty$, $\sum_{n=1}^{\infty} P[T_n' > \epsilon] < \infty$, T''_{nm} and T'''_{nm} each converge a.s. to random variables T''_n and T_n''' respectively as $m \to \infty$, and that $\sum_{n=1}^{\infty} P[T'''_n > \epsilon] < \infty$ in no way depended on the additional assumption of the random variables D_k being identically distributed. Thus the proof of these facts is the same here and we therefore omit repeating their proof. We thus need only to establish $\sum_{n=1}^{\infty} P[T_n'' > \epsilon] < \infty$ to complete the proof of (i). Applying (3) of Lemma 2 with $Z_k = D_k$ yields

$$\sum_{n=1}^{\infty} P[T_n'' > \epsilon] \leqq \sum_{j=1}^{\infty} \sum_{n=1}^{g_j} (f_n(j) - f_n(j-1)) \sup_k P[|D_k| \geqq j-1)],$$

where $f_n(j)$ and g_j are defined in the statement of Lemma 2. By the Chebychev inequality,

$$P[|D_k| \geqq j-1] \leqq K(j-1)^{-(1+\alpha+\beta)/\alpha}(\log^+ (j-1))^{-(1+\xi)}.$$

Likewise

$$P[|D_k| \geqq j-1] \leqq K(j-1)^{-2}(\log^+ (j-1))^{-(2+\xi)}.$$

To show that $\sum_{n=1}^{\infty} P[T_n'' > \epsilon] < \infty$, it is sufficient to show that

$$\sum_{j=4}^{\infty} \sum_{n=1}^{g_j} (f_n(j) - f_n(j-1)) \sup_k P[|D_k| > j-1] < \infty$$

since $g_j < \infty$ for $j = 1, 2$, and 3.

We now consider two cases. First, if $(1 + \alpha + \beta)/\alpha = 2$, then

$$\sum_{j=4}^{\infty} \sum_{n=1}^{g_j} (f_n(j) - f_n(j-1)) \sup_k P[|D_k| \geqq j-1]$$
$$\leqq K \sum_{j=4}^{\infty} \sum_{n=1}^{g_i} (f_n(j) - f_n(j-1))(j-1)^{-2}(\log (j-1))^{-2+\xi}.$$

By the definitions of A_n and $f_n(j)$, it is clear that $A_n \geqq f_n(j)\epsilon^2(Nj)^2$. Since $A_n \leqq Kn^{\beta-\alpha}$ by hypothesis, we conclude that $f_n(j) \leqq Kn^{\beta-\alpha}N^2j^2/\epsilon^2$. Inversion of the order of summation and summation by parts thus yields

$$\sum_{j=4}^{\infty} \sum_{n=1}^{g_j} (f_n(j) - f_n(j-1)) \sup_k P[|D_k| > j-1] \leqq K^2N^2/\epsilon^2 \cdot \sum_{j=4}^{\infty} j^2((j-1)^{-2}(\log (j-1))^{-(2+\xi)} - j^{-2}(\log j)^{-(2+\xi)}) \sum_{n=1}^{g_j} n^{\beta-\alpha}.$$

Since $\sum_{n=1}^{g_j} n^{\beta-\alpha} = \sum_{n=1}^{g_j} n^{-1} \leqq K' \log j$, where K' is independent of j, it then follows by elementary computation that the preceding sum is finite. Thus $\sum_{n=1}^{\infty} P[T_n'' > \epsilon] < \infty$ in the first case. Secondly, if $(1 + \alpha + \beta)/\alpha > 2$, then

$$\begin{aligned} &\sum_{j=4}^{\infty} \sum_{n=1}^{g_j} (f_n(j) - f_n(j-1)) \sup_k P[|D_k| > j-1] \\ &\quad \leqq K \sum_{j=4}^{\infty} \sum_{n=1}^{g_j} (f_n(j) - f_n(j-1))(j-1)^{-(1+\alpha+\beta)/\alpha}(\log (j-1))^{-(1+\xi)} \\ &\quad \leqq K^2N^2/\epsilon^2 \sum_{j=4}^{\infty} j^2((j-1)^{-(1+\alpha+\beta)/\alpha}(\log (j-1))^{-(1+\xi)} - j^{-(1+\alpha+\beta)/\alpha} \\ &\qquad \cdot (\log j)^{-(1+\xi)}) \sum_{n=1}^{g_j} n^{\beta-\alpha}. \end{aligned}$$

Since $\sum_{n=1}^{g_j} n^{\beta-\alpha} \leqq K' j^{(\beta-\alpha+1)/\alpha}$, where K' is independent of j, it then follows by elementary computation that the preceding sum is finite. Hence $\sum_{n=1}^{\infty} P[T_n'' > \epsilon] < \infty$ and (i) is established.

REMARKS. The assumption of the D_k's being identically distributed can obviously be dropped from the statement of Corollaries 1–3 in an analogous manner as done above in Theorem 3. Statements of these corollaries are therefore omitted. Theorem 3 was motivated by a result given by Chow ([1], p. 1489) for the non-identically distributed case. His result states that if $ED_k = 0$, $E|D_k|^{(1+\lambda)/\alpha} \cdot (\log^+ |D_k|)^2 \leqq K$, $|a_{nk}| \leqq KA_n$ for $k \leqq n^\lambda$, $a_{nk} = 0$ for $k > n^\lambda$, and $A_n \leqq Kn^{-\alpha}$ where $\lambda \geqq 1$ and $0 < \alpha \leqq 1$, then T_n converges completely to zero. Theorem 3 extends this result by treating the case where the rows of the a_{nk} matrix may have infinitely many non-zero entries and the case where the second moments of the D_k's may be infinite. Setting $\beta = 0$ shows that the moment condition given in Theorem 3 is sharper (except for the special case $\alpha = \lambda = 1$ where the Chow result is sharper).

5. A martingale convergence result.

THEOREM 4. *Let* $E(\exp (tD_k) | \mathfrak{F}_{k-1}) \leqq \exp (t^2)$ a.e. *for every constant* t, $A_n < \infty$, *and* $\sum_{n=1}^{\infty} \exp (-\lambda/A_n) < \infty$ *for all* $\lambda > 0$. *Then* T_n *converges completely to zero.*

PROOF. By Lemma 5, T_n is generalized Gaussian with $\tau^2(T_n) \leqq 2A_n$. From this, it follows easily by a Chebychev argument (see [1], p. 1483) that $P[|T_n| > \epsilon] \leqq 2 \exp (-\epsilon^2/(4A_n))$ for all $\epsilon > 0$. Thus $\sum_{n=1}^{\infty} P[|T_n| > \epsilon] < \infty$ for all $\epsilon > 0$.

REMARK. Chow ([1], p. 1483) proves that

(6) $(D_k, k \geqq 1)$ independent generalized Gaussian with $\tau^2(D_k) \leqq 2$, $A_n < \infty$, and $\sum_{n=1}^{\infty} \exp (-\lambda/A_n) < \infty$ for all $\lambda > \Rightarrow 0$ T_n converges completely to zero.

Theorem 4 generalizes this result to the martingale case. The key step in the proof of Theorem 4 as well as (6) is the establishment of Lemma 5.

COROLLARY 4. *Let* $|D_k| \leqq K$ a.s. *for some* $K < \infty$, $A_n < \infty$, *and* $\sum_{n=1}^{\infty} \exp (-\lambda/A_n) < \infty$ *for all* $\lambda > 0$. *Then* T_n *converges completely to zero.*

PROOF. Without loss of generality, we assume $K = 1$. By Theorem 4 it is sufficient to show that $E(\exp(tD_k) \mid \mathfrak{F}_{k-1}) \leqq \exp(t^2)$ a.e. for each t and $k \geqq 2$. Consider $t > 0$. If $t \geqq 1$, then $\exp(tD_k) \leqq \exp(t^2)$ a.e. Consider $0 < t < 1$.

$$\operatorname{Exp}(tD_k) \leqq 1 + tD_k + \sum_{n=2}^{\infty} t^n/n! \leqq 1 + tD_k + t^2 \quad \text{a.e.}$$

Hence $E(\exp(tD_k) \mid \mathfrak{F}_{k-1}) \leqq 1 + t^2 \leqq \exp(t^2)$ for $0 < t < 1$. Thus $E(\exp(tD_k) \mid \mathfrak{F}_{k-1}) \leqq \exp(t^2)$ a.e. for each $t > 0$, for each $t < 0$ by symmetry, and for $t = 0$ trivially.

REMARKS. Corollary 4 has been proved by Hill ([7], p. 405) in the special case where $(D_k, k \geqq 1)$ are independent and $P[D_k = 1] = P[D_k = -1] = \frac{1}{2}$. Even for this case, Erdös ([7], p. 404) gives an example which shows that $\sum_{n=1}^{\infty} \exp(-\lambda/A_n) < \infty$ for all $\lambda > 0$ cannot be replaced by $A_n = O(\log^{-1} n)$. Hence the statement of Theorem 4 is rather sharp. As an example of the application of Theorem 4, one may take $a_{nk} = 1/(n^{\frac{1}{2}}(\log n)^{\frac{1}{2}+\delta})$ for $k \leqq n$ and $a_{nk} = 0$ for $k > n$, where $\delta > 0$. Then it follows that $E(\exp(tD_k) \mid \mathfrak{F}_{k-1}) \leqq \exp(t^2)$ a.s. implies that $T_n = \sum_{k=1}^{n} D_k/(n^{\frac{1}{2}}(\log n)^{\frac{1}{2}+\delta})$ converges completely to zero. This example is given by Chow in the independent case ([1], p. 1484). The papers by Hill [7] and Chow [1] may be referred to for other applications of Theorem 4 since the examples given there will apply in the more general martingale case.

6. Concluding remark. In Sections 3 and 4, we can redefine

$$T_{nm} = \sum_{k=1}^{m} a_{nk} D_{nk},$$

where $(D_{nk}, k \geqq 1)$ is a sequence of independent random variables for each $n \geqq 1$. Likewise, in Section 5, we can redefine $T_{nm} = \sum_{k=1}^{m} a_{nk} D_{nk}$ where $(D_{nk}, \mathfrak{F}_{nk}, k \geqq 1)$ is a martingale difference sequence for each $n \geqq 1$. All results stated in Sections 3–5 then remain valid with the exception of Theorem 2. Moreover, this generalization is trivial since all proofs given remain valid without modification. In anticipation of possible application of the results of this paper we have called attention to this slightly more general formulation of the results given in this paper.

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