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ON THE ASYMPTOTIC NORMALITY OF ONE-SIDED STOPPING RULES[1]

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1. Introduction and summary. Assume that $x_1, x_2, \cdots$ are independent random variables with expectation $\mu > 0$ and finite variance σ^2. Let $s_k = x_1 + \cdots + x_k$ $(k = 1, 2, \cdots)$. For any family of positive, non-decreasing, eventually concave functions f_c defined on the positive real numbers and indexed by $c > 0$ such that $f_c \to \infty$ as $c \to \infty$, define

$$\begin{aligned}\tau = \tau(c) &= \text{first} \quad k \geqq 1 \quad \text{such that} \quad s_k > f_c(k)\\ &= \infty \quad \text{if no such} \quad k \quad \text{exists.}\end{aligned}$$

The stopping variable (sv) τ arises in various problems in probability and statistics. For example, the sequential statistical procedures of [2], [3], and [8] involve sv's similar to τ (see also the next to last paragraph of this section).

Suppose that the family $\{f_c : c > 0\}$ is such that we may define $\lambda = \lambda(c)$ by

$$\mu\lambda = f_c(\lambda). \tag{1}$$

In [9] it is shown under conditions on the joint distribution of $x_1, x_2, \cdots$ weaker than the above that

$$E\tau \sim \lambda \qquad (c \to \infty) \tag{2}$$

for a certain class of families $\{f_c\}$. In Section 2 of this note it is shown that if $f_c(x) = cx^\alpha$ for some $0 \leqq \alpha < 1$, then τ (suitably normalized) is asymptotically normally distributed whenever $(s_n - n\mu)/\sigma n^{\frac{1}{2}}$ is. (See also remarks (a) and (b) in Section 4.) This extends Heyde's result [5], [7], valid when the x_k have a common distribution and $\alpha = 0$.

Assume now for simplicity that $f_c(x) \equiv c$ and $x_1, x_2, \cdots$ have a common distribution. If $x_1 \geqq 0$ the random variable $M(c)(N(c))$ defined by

$$M(c) = \sup\{n : s_n \leqq c\}(N(c) = \textstyle\sum_{n=1}^\infty I\{s_n \leqq c\}) \tag{3}$$

is of interest in renewal theory; and the observation that $M(c) = N(c) = \tau(c) - 1$ allows one to study $M(N)$ by studying the stopping variable τ. In the general case it has been noted by several authors that the behavior of τ and N may differ in important respects. (For example, if $E(x_1^-)^2 = \infty$, it is known that $EN = \infty$ [6], whereas it is easy to show that $E\tau < \infty$, $E\tau \sim c/\mu$ (e.g. [9]).) In Section 3 we point out that some knowledge of $M(N)$ can be obtained in a direct fashion from relevant knowledge of τ.

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Our methods throughout involve finding convenient estimates for the probability that s_k crosses the curve f_c for the first time at some index $k < n$ and then falls back below the curve at time n.

2. Asymptotic normality of τ. We use without comment the basic

LEMMA 1. *If F is a distribution function and (z_n) a sequence of random variables such that $\lim_{n\to\infty} P\{z_n \leqq x\} = F(x)$ at all continuity points x of F, then for any sequence (ϵ_n) of random variables tending in probability to 0 as $n \to \infty$,*

$$\lim_{n\to\infty} P\{z_n + \epsilon_n \leqq x\} = F(x)$$

at all continuity points x of F.

THEOREM 1. *Let $(x_n), f_c, \lambda, \tau$ be as above and suppose that for some $0 \leqq \alpha < 1$*

$$f_c(x) = cx^\alpha.$$

If

(4) $$\lim_{n\to\infty} P\{[s_n - n\mu](\sigma n^{\frac{1}{2}})^{-1} \leqq x\} = \Phi(x) \equiv \int_{-\infty}^{x} (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}y^2}\, dy,$$

then

(5) $$\lim_{c\to\infty} P\{[\tau - \lambda][(1-\alpha)^{-1}\lambda^{\frac{1}{2}}\mu^{-1}\sigma]^{-1} \leqq x\} = \Phi(x).$$

PROOF. We shall give a proof for the case $\alpha > 0$. The case $\alpha = 0$ is similar and somewhat easier.

From (1) we have $\lambda = (c/\mu)^{1/(1-\alpha)}$. Let x be arbitrary and assume that n is a function of c such that

(6) $$[cn^\alpha - n\mu](\sigma n^{\frac{1}{2}})^{-1} = -x,$$

so by inversion

(7) $$[n - \lambda][(1-\alpha)^{-1}\lambda^{\frac{1}{2}}\mu^{-1}\sigma]^{-1} \to x.$$

(Note that n is not, in general, an integer. We denote the largest integer $\leqq n$ by $[n]$.)

Now $P\{\tau \leqq n\} \geqq P\{s_{[n]} > cn^\alpha\}$, so by (4), (6), and (7)

(8) $$\liminf P\{[\tau - \lambda][(1-\alpha)^{-1}\lambda^{\frac{1}{2}}\mu^{-1}\sigma]^{-1} \leqq x\} \geqq 1 - \Phi(-x) = \Phi(x).$$

Now suppose that $x > 0$. (The case $x < 0$ follows by a similar argument and the case $x = 0$ by continuity.) Let $0 < \epsilon, \delta < 1$. Then from (6)

(9) $$\begin{aligned} P\{\tau \leqq n\} &\leqq P\{\tau \leqq \epsilon n\} + P\{s_{[n]} > [n]\mu - (1+\delta)x\sigma n^{\frac{1}{2}}\} \\ &\quad + P\{\epsilon n < \tau < n, s_{[n]} \leqq [n]\mu - (1+\delta)x\sigma n^{\frac{1}{2}}\} = p_1 + p_2 + p_3. \end{aligned}$$

By (4)

(10) $$\lim_{c\to\infty} p_2 = 1 - \Phi(-x(1+\delta)) = \Phi(x(1+\delta)).$$

By (6) $ck^\alpha - k\mu = (k^\alpha n^{1-\alpha} - k)\mu - x\sigma k^\alpha n^{\frac{1}{2}-\alpha}$, which is easily seen to be increasing for $k \leqq \epsilon n$ provided that ϵ is so small and n so large that

$$(\alpha\epsilon^{\alpha-1} - 1)\mu - x\sigma\epsilon^{\alpha-1}n^{-\frac{1}{2}} > 0.$$

Thus we may apply the Hájek-Rényi inequality [4] to obtain

$$
\begin{aligned}
p_1 &= P\{s_k - k\mu > ck^\alpha - k\mu, \quad \text{some} \quad k \leqq \epsilon n\} \\
&= P\{s_k - k\mu > (k^\alpha n^{1-\alpha} - k)\mu - x\sigma k^\alpha n^{\frac{1}{2}-\alpha}, \quad \text{some} \quad k \leqq \epsilon n\} \qquad (11) \\
&\leqq \textstyle\sum_1^{[\epsilon n]} \sigma^2/n^{2(1-\alpha)}k^{2\alpha}[(1 - (k/n)^{1-\alpha})\mu - x\sigma n^{-\frac{1}{2}}]^2 \\
&= O(1/n) \to 0 \quad \text{as} \quad c \to \infty.
\end{aligned}
$$

From (6),

$$
\begin{aligned}
p_3 &= \textstyle\sum_{k=[\epsilon n]+1}^{[n]-1} P\{\tau = k, s_{[n]} \leqq [n]\mu - (1 + \delta)x\sigma n^{\frac{1}{2}}\} \\
&\leqq \textstyle\sum_{[\epsilon n]+1}^{[n]-1} P\{\tau = k\}P\{s_{[n]} - s_k \leqq [n]\mu - k^\alpha n^{1-\alpha}\mu - \delta x\sigma n^{\frac{1}{2}}\} \qquad (12) \\
&= \textstyle\sum_{[\epsilon n]+1}^{[n]-1} P\{\tau = k\} \\
&\qquad \cdot P\{s_{[n]} - s_k - ([n] - k)\mu \leqq (k - k^\alpha n^{1-\alpha})\mu - \delta x\sigma n^{\frac{1}{2}}\}.
\end{aligned}
$$

If $k \geqq (1 - \epsilon)n$, i.e., $|k - n| \leqq \epsilon n$, then by Chebyshev's inequality

$$
\begin{aligned}
(13) \quad P\{s_{[n]} - s_k - ([n] - k)\mu &\leqq (k - k^\alpha n^{1-\alpha})\mu - \delta x\sigma n^{\frac{1}{2}}\} \\
&\leqq (n - k)\sigma^2(\delta^2 x^2 \sigma^2 n)^{-1} \leqq \epsilon\delta^{-2}x^{-2}.
\end{aligned}
$$

If $\epsilon n < k < (1 - \epsilon)n$,

$$
\begin{aligned}
&P\{s_{[n]} - s_k - ([n] - k)\mu \leqq (k - k^\alpha n^{1-\alpha})\mu - \delta x\sigma n^{\frac{1}{2}}\} \\
&\qquad \leqq n\sigma^2[(k - k^\alpha n^{1-\alpha})^2\mu^2]^{-1} \leqq \sigma^2[\epsilon^{2\alpha} n(1 - (k/n)^{1-\alpha})^2\mu^2]^{-1} \qquad (14) \\
&\qquad \to 0 \quad \text{uniformly in } k \text{ as} \quad c \to \infty.
\end{aligned}
$$

Thus from (9)–(14) we have

$$\limsup_{c\to\infty} P\{[\tau - \lambda][(1 - \alpha)^{-1}\lambda^{\frac{1}{2}}\mu^{-1}\sigma]^{-1} \leqq x\} \leqq \Phi(x(1 + \delta)) + \epsilon(\delta^2 x)^{-1} \to \Phi(x)$$

as first $\epsilon \to 0$, then $\delta \to 0$, which in conjunction with (8) proves the theorem.

3. The random variable $M = \sup\{n: s_n \leqq c\}$ Now suppose that $x_1, x_2, \cdots$ are independent and identically distributed with positive mean μ and that $f_c(x) \equiv c$. By truncation it follows easily from the equivalence of (a) and (c) of Theorem 3 of [1] (with $r = t = 2$) that with M defined by (3) $EM < \infty$ provided that $E(x_1^-)^2 < \infty$.

THEOREM 2. *If $E(x_1^-)^2 < \infty$, for any $c > 0$*

$$0 \leqq E(M(c) - \tau(c) + 1) \leqq EM(0) < \infty.$$

PROOF. The left hand inequality follows at once from the observation that

$$M(c) \geqq N(c) \geqq \tau(c) - 1.$$

To complete the proof we write

$$
\begin{aligned}
&E(M(c) - \tau(c) + 1) \\
&\qquad = \textstyle\sum_{n=1}^{\infty} (P\{M \geqq n\} - P\{\tau > n\})
\end{aligned}
$$

$$= \sum_{n=1}^{\infty} P\{\tau \leqq n, M \geqq n\} = \sum_{n=1}^{\infty} \sum_{k=1}^{n} P\{\tau = k, s_k > c, \inf_{j \geqq n} s_j \leqq c\}$$
$$\leqq \sum_{k=1}^{\infty} P\{\tau = k\} \sum_{n=k}^{\infty} P\{\inf_{j \geqq n} (s_j - s_k) < 0\}$$
$$\leqq \sum_{i=0}^{\infty} P\{\inf_{j \geqq i} s_j < 0\} \leqq EM(0) < \infty.$$

Corollary 1. *If* $E(x_1^-)^2 < \infty$, *then* $EM(c) \sim EN(c) \sim E\tau(c) \sim c/\mu (c \to \infty)$.

Proof. The corollary follows at once from the theorem and the result for τ, which is well-known.

Corollary 2. *If* $Ex_1^2 - \mu^2 = \sigma^2 < \infty$, *then*

$$\lim_{c \to \infty} P\{[M(c) - c\mu^{-1}][c^{\frac{1}{2}}\sigma\mu^{-\frac{3}{2}}]^{-1} \leqq x\} = \Phi(x) \qquad (-\infty < x < \infty).$$

Proof. By Theorem 2 $(M(c) - \tau(c))/c^{\frac{1}{2}} \to 0$ in probability, and hence the corollary follows from Lemma 1 and Theorem 1.

It is interesting to note that if $\sigma^2 < \infty$ Heyde [5] has shown that Var $\tau \sim c\sigma^2\mu^{-3}$ $(c \to \infty)$. However, it is possible that $EM^2 = \infty$. In fact it is an easy consequence of results in [1] and [6] that $\sum_1^{\infty} nP\{M \geqq n\}$ (and thus EM^2) is finite if and only if $E(x_1^-)^3 < \infty$.

4. Remarks. The following comments suggest some straightforward generalizations of the results of this note.

(a) Theorem 1 remains true if the x_k's do not have the same expectation but $Es_n - n\mu = o(n^{\frac{1}{2}})$ $(n \to \infty)$. The case of non-constant variance can also be treated.

(b) In [3] stopping boundaries $f(n)$ such that $f(n) = O((n \log \log n)^{\frac{1}{2}})$ $(n \to \infty)$ are discussed. A particularly simple parameterization of such boundaries giving an asymptotic result similar to Theorem 1 is

$$f_c(n) = n^{\frac{1}{2}}(c + \log \log n)^{\frac{1}{2}}.$$

The case of a more general slowly varying function L can be treated similarly.

(c) A version of Theorem 1 follows at once for

$$\tau^* = \text{first } n \text{ such that } \quad |s_n| > cn^{\alpha}.$$

By the strong law of large numbers $s_n/n \to \mu$ and hence $s_n \to \infty$ a.s. It follows that $P\{\tau^*(c) \neq \tau(c)\} \to 0$ as $c \to \infty$. Lemma 1 and Theorem 1 now give the limiting distribution of τ^*.

(d) The method of proof of Theorem 2 may be used to relate

$$EM^{r+1}, \sum_{n=1}^{\infty} n^r P\{s_n \leqq c\}, \quad \text{and} \quad E\tau^{r+1}$$

for positive integral values of r. It may also be adapted to the case of non-identically distributed variables.

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