The Annals of Mathematical Statistics
1968, Vol. 39, No. 6, 2141–2144

COMPARISON TESTS FOR THE CONVERGENCE OF MARTINGALES

By Burgess Davis

Rutgers–The State Universtiy

1. Introduction. If $f = (f_1, f_2, \cdots)$ is a sequence of real valued functions on a probability space and $d_1 = f_1$, $d_i = f_i - f_{i-1}$, $i > 1$, let

$$f_n^* = \max(|f_1|, \cdots, |f_n|), \qquad f^* = \sup_n f_n^*,$$
$$S_n(f) = (\textstyle\sum_1^n d_i^2)^{\frac{1}{2}}, \quad \text{and} \quad S(f) = S_\infty(f) = \sup_n S_n(f).$$

In [1], Burkholder proved that if f and g are martingales relative to the same sequence of σ-fields, f is L^1 bounded, and $S_n(g) \leqq S_n(f)$, $n \geqq 1$, then g converges almost everywhere. It will be shown here that the condition $S_n(g) \leqq S_n(f)$, $n \geqq 1$, can be replaced by the weaker condition $S(g) \leqq S(f)$. Using this it requires almost no alteration of Burkholder's proofs to make the same replacement in Theorems 6 and 7 of [1].

Using essentially the same method, a theorem will be proved for L^1-bounded martingales f which gives among other things the convergence of g and finiteness of $S(g)$ if, in place of $S(g) \leqq S(f)$, we have $g^* \leqq f^*$.

2. Comparison tests for martingale convergence. Suppose g is a martingale such that if $\epsilon > 0$ then there is a stopping time t such that $P(t < \infty) < \epsilon$ and $E(S_t(g)) < \infty$. Then g converges almost everywhere by Theorem 2 of [1], which states that if f is a martingale and $E(S(f)) < \infty$ then f converges almost everywhere, since by this theorem g stopped at t will converge almost everywhere and the probability of stopping at a finite time is arbitrarily small.

Lemma 1. *If* $(f_n, \mathcal{A}_n, n \geqq 1)$ *is a nonnegative martingale with difference sequence* $(d_n, n \geqq 1)$, *and* $\lambda > 0$, *then almost everywhere*

$$(1) \qquad P([f_n^2 + d_{n+1}^2 + \cdots]^{\frac{1}{2}} > \lambda f_n \mid \mathcal{A}_n) \leqq M/\lambda$$

where M *is the constant appearing in Theorem* 8 *of* [1], *and almost everywhere*

$$(2) \qquad P(\sup[f_n, f_{n+1}, f_{n+2}, \cdots] > \lambda f_n \mid \mathcal{A}_n) \leqq 1/\lambda.$$

Proof. Let $\lambda > 0$, n be a positive integer, $A \in \mathcal{A}_n$ and $\alpha > 0$. Then

$$P([f_n^2 + d_{n+1}^2 + \cdots]^{\frac{1}{2}} > \lambda[f_n + \alpha], A) = P([(f_n I_A/[f_n + \alpha])^2 + (d_{n+1} I_A/[f_n + \alpha])^2 + \cdots]^{\frac{1}{2}} > \lambda) \leqq (M/\lambda)P(A),$$

using the fact that the partial sums of the series $f_n I_A/[f_n + \alpha] + d_{n+1} I_A/[f_n + \alpha] + \cdots$ form a nonnegative martingale with the L^1 norm of each partial sum equal to $E(f_n I_A/[f_n + \alpha]) \leqq E(I_A) = P(A)$, together with Theorem 8 of [1].

Received 19 February 1968.

Letting $\alpha \to 0$, we obtain

$$P([f_n^2 + d_{n+1}^2 + \cdots]^{\frac{1}{2}} > \lambda f_n, A) \leqq (M/\lambda)P(A),$$

implying (1). (2) is proved similarly using Theorem 3.2 on page 314 of [2].

LEMMA 2. *Suppose that f satisfies the assumptions of Lemma 1. If ϕ_n is an $\mathcal{A}_n$ measurable function satisfying $\phi_n \leqq S(f)$, then almost everywhere*

$$\phi_n \leqq S_n(f) + Mf_n. \tag{3}$$

If ψ_n is an $\mathcal{A}_n$ measurable function satisfying $\psi_n \leqq f^*$, then almost everywhere

$$\psi_n \leqq f_n^*. \tag{4}$$

PROOF. Note that

$$\phi_n \leqq S(f) = (\textstyle\sum_1^\infty d_i^2)^{\frac{1}{2}} \leqq S_n(f) + (\sum_{n+1}^\infty d_i^2)^{\frac{1}{2}}.$$

Let $\lambda > M$. Then, almost everywhere

$$P(\phi_n > S_n(f) + \lambda f_n \mid \mathcal{A}_n) \leqq P((\textstyle\sum_{n+1}^\infty d_i^2)^{\frac{1}{2}} > \lambda f_n \mid \mathcal{A}_n) \leqq M/\lambda < 1,$$

using Lemma 1. Since $A = \{\phi_n > S_n(f) + \lambda f_n\}$ is in $\mathcal{A}_n$, we have that its indicator function I_A satisfies $I_A = P(A \mid \mathcal{A}_n) < 1$ almost everywhere. Consequently, for all $\lambda > M$, $\phi_n \leqq S_n(f) + \lambda f_n$ almost everywhere, so (3) follows.

To prove (4), let $\lambda > 1$. Then almost everywhere

$$\begin{aligned} P(\psi_n > \lambda f_n^* \mid \mathcal{A}_n) &\leqq P(f^* > \lambda f_n^* \mid \mathcal{A}_n) \\ &= P(\sup_{k>n} f_k > \lambda f_n^* \mid \mathcal{A}_n) \\ &\leqq P(\sup_{k>n} f_k > \lambda f_n \mid \mathcal{A}_n) \\ &\leqq 1/\lambda < 1. \end{aligned}$$

This implies (4).

When the assumption $f \geqq 0$ is dropped we can deduce an analogue of (3) as follows. Write ([3], page 144) $f = f' - f''$ where f' and f'' are nonnegative martingales relative to the same sequence of σ-fields, $\|f'\|_1 \leqq \|f\|_1$, $\|f''\|_1 \leqq \|f\|_1$. Then $S(f) \leqq S(f') + S(f'')$ and $\phi_n \leqq S(f)$, where ϕ_n is $\mathcal{A}_n$-measurable, implies that

$$\phi_n \leqq S_n(f') + (\textstyle\sum_{n+1}^\infty (d_i')^2)^{\frac{1}{2}} + S_n(f'') + (\sum_{n+1}^\infty (d_i'')^2)^{\frac{1}{2}},$$

so that if $\lambda > 2M$, then almost everywhere

$$\begin{aligned} &P(\phi_n > S_n(f') + S_n(f'') + \lambda f_n' + \lambda f_n'' \mid \mathcal{A}_n) \\ &\quad\leqq P((\textstyle\sum_{n+1}^\infty d_i'^2)^{\frac{1}{2}} + (\sum_{n+1}^\infty d_i''^2)^{\frac{1}{2}} > \lambda f_n' + \lambda f_n'' \mid \mathcal{A}_n) \\ &\quad\leqq P((\textstyle\sum_{n+1}^\infty d_i'^2)^{\frac{1}{2}} > \lambda f_n' \mid \mathcal{A}_n) + P((\sum_{n+1}^\infty d_i''^2)^{\frac{1}{2}} > \lambda f_n'' \mid \mathcal{A}_n) \\ &\quad\leqq M/\lambda + M/\lambda = 2M/\lambda < 1. \end{aligned}$$

Thus $\phi_n \leqq S_n(f') + S_n(f'') + 2Mf_n' + 2Mf_n''$ almost everywhere.

THEOREM 1. *If $f = (f_1, f_2, \cdots)$ and $g = (g_1, g_2, \cdots)$ are martingales relative* *to the same sequence of σ-fields, f is L^1 bounded, and $S(g) \leqq S(f)$, then g converges almost everywhere.*

Proof. Clearly, $S_n(g) \leqq S(f)$, so that almost everywhere $S_n(g) \leqq S_n(f') + S_n(f'') + 2Mf_n' + 2Mf_n''$, f' and f'' as above. In view of the remarks at the beginning of this section, it will suffice to find a stopping time t such that $ES_t(g) < \infty$ and $P(t < \infty)$ is arbitrarily small.

Let K be a positive number. Let t be the first time one of the terms $S_n(f')$, $S_n(f'')$, f_n', f_n'' exceeds K.
Then on $\{t = \infty\}$,

$$S_t(g) = S(g) \leqq S(f) \leqq S(f') + S(f'') \leqq 2K$$

and on $\{t < \infty\}$,

$$S_t(g) \leqq S_t(f') + S_t(f'') + 2Mf_t' + 2Mf_t'' \leqq (K + |d_t'|) + (K + |d_t''|)$$
$$+ (2MK + 2M|d_t'|) + (2MK + 2M|d_t''|) \leqq C + C|f_t'| + C|f_t''|$$

for some constant C, since either $d_t' > 0$ and thus $d_t' \leqq f_t'$ or $d_t' \leqq 0$ and thus $d_t' \leqq K$ since $f_{t-1}' \leqq K$ and $f_t' \geqq 0$. Thus $E(S_t(g)) < \infty$, and $P(t < \infty)$ can be made arbitrarily small by making K large.

Theorem 2. *Let $f = (f_1, f_2, \cdots)$ be an L^1 bounded martingale and g a martingale relative to the same sequence of σ-fields. Let $e = (e_1, e_2, \cdots)$ be the difference sequence of g. Then if $e^* \leqq f^*$ the three sets $\{\sup_n g_n < \infty\}$, $\{g$ converges$\}$, and $\{S(g) < \infty\}$ are equal almost everywhere.*

Proof. Again writing $f = f' - f''$, where f' and f'' are nonnegative martingales, we have $f^* \leqq (f' + f'')^*$, and $f' + f''$ is a nonnegative martingale, so from now on we may and do assume that f is nonnegative. Since $e^* \leqq f^*$ implies that $e_n^* \leqq f^*$, and e_n^* is $\mathfrak{A}_n$ measurable, we have, by (4), that $e_n^* \leqq f_n^*$ almost everywhere.

Now let t be the first time that f_n exceeds $K > 0$. We have that on $\{t = n\}$, $e_n^* \leqq f_n^* = f_n$, and on $\{t = \infty\}$, $e^* \leqq f^* \leqq K$. Thus if $\hat{g}$ is the martingale g stopped at t, and if $\hat{e}$ is the corresponding difference sequence, then $E(\hat{e}^*) < \infty$. Thus by Theorem 4 of [1] and a result on page 320 of [2], the three sets $\{\hat{g}$ converges$\}$, $\{S(\hat{g}) < \infty\}$, $\{\sup_n \hat{e}_n < \infty\}$ are equivalent. By making K large we make $P(t < \infty)$ arbitrarily small. Therefore, since

$$P(\{\sup_n \hat{g}_n < \infty\} \triangle \{\sup_n g_n < \infty\}) \leqq P(t < \infty),$$
$$P(\{S(\hat{g}) < \infty\} \triangle \{S(g) < \infty\}) \leqq P(t < \infty)$$
$$P(\{\hat{g} \text{ converges}\} \triangle \{g \text{ converges}\}) \leqq P(t < \infty)$$

where $A \triangle B = (A \cup B) - (A \cap B)$, we have the result.

In particular, if f is a martingale and $E(d^*) < \infty$ then f satisfies the conditions of Theorem 2, since if $h_n = E(d^* \mid \mathfrak{A}_n)$ then $(h_n, \mathfrak{A}_n, n \geqq 1)$ is a martingale and $d^* \leqq h^*$ a.e.

Corollary. *If f and g are martingales relative to the same sequence of σ-fields, $g^* \leqq f^*$, and f is L^1 bounded, then g converges almost everywhere and $S(g) < \infty$ almost everywhere.*

PROOF. The condition $\sup_n |g_n| \leqq f^*$ implies that $\sup_n |g_n - g_{n-1}| \leqq 2f^* = (2f)^*$. Thus g satisfies the hypotheses of Theorem 2, and $P(\sup_n g_n < \infty) \geqq P(f^* < \infty) = 1$.

The following example shows that the hypothesis that f and g are martingales relative to the *same* sequence of σ-fields cannot be entirely removed. First we give an L^1 bounded martingale f and then a martingale g which although satisfying $S(g) \leqq S(f)$ diverges on a set of positive measure.

Let $\Omega = \{1, 2, \cdots\}$ and $P(\{k\}) = 1/k - 1/(k+1)$.
Define f by

$$\begin{aligned} f_n(k) &= n && \text{if } n < k, \\ &= -1 && \text{if } n \geqq k, \end{aligned}$$

and g by

$$\begin{aligned} g_n(1) &= -2 \textstyle\sum_{j=1}^{n} 1/(j+4), \\ g_n(2) &= 2, \\ g_n(k) &= g_n(1) && \text{if } n < k-1, \\ &= g_{k-2}(1) + k && \text{if } n \geqq k-1, \end{aligned}$$

for $k > 2$.

Now

$$\textstyle\sum_{j=1}^{\infty} [-2/(j+4)]^2 < 4 \sum_{j=1}^{\infty} [1/(j+3) - 1/(j+4)] = 1.$$

Therefore, at 1, $S(g)^2 < 1 = S(f)^2$, and at $k > 1$, $S(g)^2 < 1 + k^2 \leqq S(f)^2$, implying $S(g) \leqq S(f)$.

It is easily checked that f and g are martingales. Now f is L^1 bounded since it is bounded below. But $g_n \to -\infty$ at 1. If $e_n = g_n - g_{n-1}$, $n > 1$, then by adding $-e_k$ instead of e_k for selected k we can make the resulting martingale oscillate between $-\infty$ and ∞ at 1, still keeping $S(g) \leqq S(f)$, and similar transforms provide counterexamples for Theorem 2 and the corollary, if the same σ-field condition is removed.

Acknowledgment. I am greatly indebted to Professor D. L. Burkholder for much helpful advice and criticism.

REFERENCES

[1] BURKHOLDER, D. L. (1966). Martingale transforms. *Ann. Math. Statist.* **37** 1494–1504.
[2] DOOB, J. L. (1953). *Stochastic Processes*. Wiley, New York.
[3] KRICKEBERG, K. (1965). *Probability Theory*. Addison Wesley, Reading.