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AN ELEMENTARY METHOD FOR OBTAINING LOWER BOUNDS ON THE ASYMPTOTIC POWER OF RANK TESTS[1]

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1. Introduction and summary. The rapid development of non-parametric rank tests was generated, in part, by the result of Hodges and Lehmann [6] which stated that the asymptotic relative efficiency (ARE) of the Wilcoxon test to the classical two-sample t-test was always $\geqq .86$. They also conjectured that the ARE of the normal scores test to the t-test was always greater than or equal to one. In 1958, Chernoff and Savage [3] proved the validity of this conjecture using variational methods. In this paper we give a simple proof of their result.

Recently Doksum [4] has shown that the Savage test [11] maximizes the minimum asymptotic power for testing for scale change over the family of distributions with increasing failure rate averages (IFRA) [2]. The technique of our proof enables us to obtain a lower bound for the asymptotic power of Savage's test for scale change of any positive random variable possessing a finite second moment. When the positive random variables are restricted to be IFRA, Doksum's [4] results follow.

2. The proof of the result of Chernoff and Savage. It is known ([3], [7]) that the asymptotic efficacy of the normal scores two-sample test for change in location is given by

$$\left(\int J'(F(x))f^2(x)\,dx\right)^2, \tag{2.1}$$

where $J(u) = \Phi^{-1}(u)$ (the inverse of the cdf of the standard normal rv) and $J'(u) = [\varphi(\Phi^{-1}(u))]^{-1}$, provided that the density $f(x)$ exists and satisfies mild regularity conditions ([7], p. 313). Also, the asymptotic efficacy of the two-sample t-test is $1/\sigma^2$, where σ^2 is the variance of the underlying cdf $F(x)$. Thus the asymptotic relative efficiency (ARE) of the normal scores test to the t-test for samples from $F(x)$ is

$$A_{N,t}(F) = \sigma^2 I^2(F), \tag{2.2}$$

where

$$I(F) = \int J'(F(x))f^2(x)\,dx. \tag{2.3}$$

We refer the reader to [3] and [6] for formal definitions of efficacy and ARE. The Chernoff-Savage theorem is

THEOREM 2.1. *If F is a* cdf *with density f and variance $\sigma^2 < \infty$, then $A_{N,t}(F) \geqq 1$ and $A_{N,t}(F) = 1$ if and only if F is normal.*

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Our development rests on *Jensen's inequality* ([8], *p.* 159 *or* [10], *p.* 46): Let $g(x)$ be measurable and convex on an open interval S. Let X be a rv with $EX < \infty$ and $P(X \varepsilon S) = 1$. Then $E[g(X)] \geqq g(EX)$.

PROOF OF THEOREM 2.1 Without loss of generality we may assume that $\int xf(x)\,dx = 0$. Since $g(x) = 1/x$ is convex on the open interval $(0, \infty)$ and the rv $J'(F(X))f(X) = f(X)/\varphi(\Phi^{-1}(F(X)))$ is positive wp 1 with respect to the measure with density $f(x)$, Jensen's inequality can be applied to the integral in expression (2.3) yielding

$$I(F) = \int J'(F(x))f^2(x)\,dx \geqq [\int \varphi(\Phi^{-1}(F(x)))\,dx]^{-1}. \tag{2.4}$$

Integrating the right side of (2.4) by parts yields

$$[I(F)]^{-1} \leqq x\varphi(\Phi^{-1}(F(x)))|_{-\infty}^{\infty} + \int x\Phi^{-1}(F(x))f(x)\,dx. \tag{2.5}$$

The first term on the right side of inequality (2.5) can be seen to vanish by an elementary application of Chebyshev's inequality and a bound on the tail probability of the standard normal cdf ([5], p. 166). Applying the Cauchy-Schwarz inequality, one obtains

$$[I(F)]^{-1} \leqq [\sigma^2 \int [\Phi^{-1}(F(x))]^2 f(x)\,dx]^{\frac{1}{2}}, \tag{2.6}$$

or

$$\sigma^2 I^2(F) \geqq [\int [\Phi^{-1}(F(x))]^2 f(x)\,dx]^{-1} = 1, \tag{2.7}$$

with equality if and only if $\Phi^{-1}(F(x)) = x/\sigma$, i.e., if $F(x) = \Phi(x/\sigma)$.

REMARK. The proof consists of two steps. A lower bound for the efficacy is established in (2.4) and then this lower bound is shown to attain its minimum at the normal cdf. A similar proof can be based on the concavity of the log function and the well-known information-theoretic fact that among all densities with a specified variance σ_0^2, the entropy is maximized by the normal density with variance σ_0^2 ([12], pp. 55–56). We omit the details because the proof of Shannon's inequality is based on variational methods, which we wished to avoid.

3. The scale problem for positive random variables. Doksum [4] proved that if F and G are defined by

$$F(t) = H(t/\theta) \quad \text{and} \quad G(t) = H(t/\gamma),$$

where H is an unknown continuous IFRA distribution with $H(0) = 0$, then for the two-sample problem where one tests the equality of the means of F and G, the Savage statistic maximizes the minimum power over IFRA distributions asymptotically. In this section we show that a lower bound for the efficacy of the Savage test for positive random variables with a finite second moment can be derived using Jensen's inequality.

Doksum [4] has essentially shown that if $H(0) = 0$ and if H has a density h then the efficacy of the Savage test [11] is given by

$$e = \int_0^\infty th(t)(1 - H(t))^{-1}h(t)\,dt. \tag{3.1}$$

Actually, Doksum showed that (3.1) is the key parameter in the expression for the asymptotic power of the Savage test. It equals the efficacy under mild regularity conditions, analogous to Lemma 3 of [7], which justify differentiating under an integral sign in calculating Pitman efficiency. Since the referee has kindly informed us that such conditions have been given in a recent paper by Govindarajulu, we shall not discuss them in detail as our result holds whenever (3.1) is valid. We now prove

THEOREM 3.1. *For any positive random variable with density h, mean* μ, *and second moment* μ_2, *such that the efficacy of the Savage test is given by* (3.1), *the efficacy of the Savage test is always* $\geqq 2\mu^2/\mu_2$.

PROOF. Since $th(t)/\mu$ is a probability density (3.1) can be expressed as

$$e = \mu \int_0^\infty h(t)(1 - H(t))^{-1} th(t)\mu^{-1}\, dt. \tag{3.2}$$

Applying Jensen's inequality with $X = (1 - H(T))(h(T))^{-1}$ and $g(x) = 1/x$, one sees that (3.2) implies

$$e \geqq \mu E[g(X)] \geqq \mu^2/\{\int_0^\infty [1 - H(t)t\, dt]\} = 2\mu^2/\mu_2 . \tag{3.3}$$

REMARK 1. Since Barlow, Marshall and Proschan [1] have essentially shown that for IFRA cdf's, $\mu_2 \leqq 2\mu^2$ with equality only at the exponential distribution, Doksum's Theorem 2.1 is a consequence of our Theorem 3.1.

REMARK 2. While this paper dealt with the two-sample problem, the results of Puri [9] show that the lower bounds given extend to both corresponding c-sample problems.

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