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# SEQUENTIAL COMPOUND ESTIMATION[1]

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**1. Introduction.** We consider a sequence of statistical decision problems having the same generic structure with this structure being possessed by what is called the component problem. In the component problem there is a family of probability measures $\{P_\theta \mid \theta \varepsilon \Omega\}$ over a $\sigma$-field $\mathcal{B}$ of subsets of $\mathfrak{X}$, an action space $\mathcal{A}$, and a loss function $L \geqq 0$ defined on $\Omega \times \mathcal{A}$. With $\mathcal{C}$ a $\sigma$-field of subsets of $\mathcal{A}$, a (randomized) decision function $\varphi$ has domain $\mathfrak{X} \times \mathcal{C}$ and is such that $\varphi(x, \cdot)$ is a probability measure on $\mathcal{C}$ for each fixed $x \varepsilon \mathfrak{X}$ and $\varphi(\cdot, C)$ is $\mathcal{B}$ measurable for each fixed $C \varepsilon \mathcal{C}$. The decision procedure $\varphi$ results in an expected loss (risk)

$$R(\theta, \varphi) = \int \int L(\theta, A)\varphi(x, dA)P_\theta(dx). \tag{1.1}$$

In treating the sequence of component problems it is convenient to introduce the notation $\boldsymbol{\theta} = (\theta_1, \theta_2, \cdots)$ and $\boldsymbol{\theta}_i = (\theta_1, \cdots, \theta_i)$; also, we assume that $\mathbf{X}_i \sim P_{\theta_1} \times \cdots \times P_{\theta_i} = \mathbf{P}_i$ for all $i \geqq 1$. The action taken at the $i$th stage (i.e., in the $i$th repetition of the component problem) is allowed to depend on $\mathbf{X}_i$. Formally, a sequential compound procedure $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \cdots)$ is such that for each $i$, $\varphi_i$ is the means by which the $i$th action is taken, $\varphi_i$ is defined on $\mathfrak{X}^i \times \mathcal{C}$ with $\varphi_i(\cdot, C)\mathcal{B}^i$ measurable for each $C$, and $\varphi_i(\mathbf{x}_i, \cdot)$ is a probability measure on $\mathcal{C}$ for each $\mathbf{x}_i$. The average risk up to stage $n$ is

$$R_n(\boldsymbol{\theta}, \boldsymbol{\varphi}) = n^{-1}\textstyle\sum_{i=1}^n \int \int L(\theta_i, A)\varphi_i(\mathbf{x}_i, dA)\mathbf{P}_i(d\mathbf{x}_i). \tag{1.2}$$

In keeping with terminology that is becoming standard, we say a sequential compound procedure $\boldsymbol{\varphi}$ is simple if $\varphi_i(\cdot, C)$ is $x_i$ measurable for each $C$. If in addition all the $\varphi_i$ are identical, say $\varphi_i = \varphi$, we say $\boldsymbol{\varphi}$ is simple symmetric with kernel $\varphi$. Simple symmetric procedures are traditional in case $\Omega$ is not a singleton set. For every simple symmetric procedure $\boldsymbol{\varphi}$ and all $\boldsymbol{\theta}$,

$$R_n(\boldsymbol{\theta}, \boldsymbol{\varphi}) = n^{-1}\textstyle\sum_{i=1}^n R(\theta_i, \varphi) \geqq R(G_n) \tag{1.3}$$

where $G_n$ is the empirical distribution of $\theta_1, \cdots, \theta_n$ and $R(\cdot)$ is the Bayes envelope for the component problem. We also note that for any simple procedure $\boldsymbol{\varphi}$,

$$\begin{aligned} \sup_{\boldsymbol{\theta}} \{R_n(\boldsymbol{\theta}, \boldsymbol{\varphi}) - R(G_n)\} &\geqq \sup_\theta \{n^{-1}\textstyle\sum_{i=1}^n R(\theta, \varphi_i) - \inf_A L(\theta, A)\} \\ &\geqq \inf_\varphi \sup_\theta \{R(\theta, \varphi) - \inf_A L(\theta, A)\}. \end{aligned} \tag{1.4}$$

(Samuel (1965b) gives a necessary condition for the left hand side of (1.4) to be zero.) The right hand side of (1.4) is zero only when the component problem is trivial; otherwise, it is some positive number, say $\epsilon$. Therefore, with a modified

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regret defined by

$$D_n(\boldsymbol{\theta}, \boldsymbol{\varphi}) = R_n(\boldsymbol{\theta}, \boldsymbol{\varphi}) - R(G_n) \tag{1.5}$$

it follows that in a sequence of statistical decision problems involving a non-trivial component problem and with $\boldsymbol{\varphi}$ simple symmetric, $D_n(\boldsymbol{\theta}, \boldsymbol{\varphi}) \geqq 0$ for all $\boldsymbol{\theta}$ and $D_n(\boldsymbol{\theta}, \boldsymbol{\varphi}) \geqq \frac{1}{2}\epsilon > 0$ for some $\boldsymbol{\theta}$, where $\epsilon$ does not depend upon $n$. However, in some problems non-simple sequential compound procedures $\boldsymbol{\varphi}$ have been given for which $D_n(\boldsymbol{\theta}, \boldsymbol{\varphi}) \leqq B_n(\boldsymbol{\theta}) = o(1)$ as $n \to \infty$; and, often, convergence uniform in $\boldsymbol{\theta}$ is proved (e.g., Hannan (1956), Hannan (1957), Samuel (1963), Samuel (1965a), Swain (1965), Van Ryzin (1966b), and Johns (1967)). In some cases lower bounds for $D_n$ are developed and rates of convergence for bounds of $\sup_{\boldsymbol{\theta}} D_n(\boldsymbol{\theta}, \boldsymbol{\varphi})$ proved. Swain (1965) and Johns (1967) discuss an extended version of the sequential compound decision problem where standards asymptotically more stringent than $R(G_n)$ are used to form a modified regret.

The question of where to begin the search for sequential compound procedures with asymptotically small modified regret is now at hand. Robbins (1951) gives an original and general formulation of the compound decision problem. Robbins' formulation involves abeyance of the first $n$ actions until $\mathbf{X}_n$ is observed so that at stage $i$, $1 \leqq i \leqq n$, the action can depend upon $\mathbf{X}_n$ rather than $\mathbf{X}_i$. In this sense it is a set rather than sequence problem. In the set problem it is natural to seek procedures $\boldsymbol{\varphi}$ such that $\varphi_i(\cdot, \cdot)$ approximates $\psi_n(X_i, \cdot)$, a Bayes response versus $G_n$ in the $i$th component problem. It follows from (8.8) and (8.11) of Hannan (1957) that for all $\boldsymbol{\theta}$,

$$n^{-1}\textstyle\sum_{i=1}^{n} R(\theta_i, \psi_i) \leqq R(G_n) \leqq n^{-1}\sum_{i=1}^{n} R(\theta_i, \psi_{i-1}) \tag{1.6}$$

where $\psi_0$ is an arbitrary decision function. Therefore, the structure of natural set compound procedures and the left inequality of (1.6) motivate sequential compound procedures $\boldsymbol{\varphi}$ such that $\varphi_i$ approximates $\psi_i(X_i, \cdot)$. In case $\Omega$ is finite, unbiased estimates $\bar{h}_i$ exist for $G_i$; for example, see Robbins (1964) and Van Ryzin (1966a). Van Ryzin (1966b) proves that in case both $\Omega$ and $\mathfrak{A}$ are finite, the sequential compound procedure $\varphi_i = \psi_i^*(X_i, \cdot)$ where $\psi_i^*$ is a Bayes response versus $\bar{h}_i$ results in a modified regret $O(n^{-\frac{1}{2}})$ uniform in $\boldsymbol{\theta}$; and, under non-degeneracy conditions the same result holds for $\varphi_i = \psi_{i-1}^*(X_i, \cdot)$. The latter sections of this paper deal with a problem where both $\Omega$ and $\mathfrak{A}$ are infinite subsets of the real line and the loss function is squared error. In the case of non-finite $\Omega$, $G_i$ may not be estimable directly. However, with certain discrete exponential families and squared error loss, $\psi_i$ takes a sufficiently simple form to allow approximation in a Cesaro mean sense; and, we will show that under certain conditions the approximating procedures have modified regret $O(n^{-\frac{1}{2}})$ uniform in $\boldsymbol{\theta}$. Empirical Bayes versions of problems involving special discrete exponential families are stated by Robbins (1956) and investigated by Johns (1956). Samuel (1965a) and Swain (1965) investigate the sequential compound estimation (non-Bayesian) problem for a larger class of discrete exponentials. Theorems yet to be stated provide strengthenings of some of their work.

Before turning to the special cases to be studied in detail, it is worthwhile to note how, quite generally, certain empirical Bayes results are immediate consequences of sequential compound results. Suppose $\theta_1, \theta_2, \cdots$ are independent and identically distributed $G$ and let $E$ denote expectation with respect to the infinite product distribution $G^\infty$. For any sequential compound procedure $\boldsymbol{\varphi}$ consider $ER_n(\boldsymbol{\theta}, \boldsymbol{\varphi}) - R(G) = ED_n(\boldsymbol{\theta}, \boldsymbol{\varphi}) + ER(G_n) - R(G)$ which is non-negative since the $E$ expectation of each term of the sum in $R_n(\boldsymbol{\theta}, \boldsymbol{\varphi})$ is no less than $R(G)$. Taking $E$ expectation of (1.3) with $\varphi$ a Bayes response versus $G$, it follows that $R(G) \geqq ER(G_n)$. Therefore,

$$0 \leqq ER_n(\boldsymbol{\theta}, \boldsymbol{\varphi}) - R(G) \leqq ED_n(\boldsymbol{\theta}, \boldsymbol{\varphi}); \tag{1.7}$$

and, we see that rates of convergence for $D_n(\boldsymbol{\theta}, \boldsymbol{\varphi})$ uniform in $\boldsymbol{\theta}$ lead to the corresponding rates for Cesaro sums in the empirical Bayes problem uniform in a priori distributions $G$. Rates of convergence of $ER(G_n)$ to $R(G)$ follow as a corollary. If bounds like $o(1)$ (not necessarily uniform in $\boldsymbol{\theta}$) exist for $D_n(\boldsymbol{\theta}, \boldsymbol{\varphi})$, then the dominated convergence theorem can be used in conjunction with (1.7) to prove $ER_n(\boldsymbol{\theta}, \boldsymbol{\varphi}) - R(G) = o(1)$; for example, when $D_n(\boldsymbol{\theta}, \boldsymbol{\varphi})$ is a bounded function of $\boldsymbol{\theta}$. The extension of sequential compound convergence to the corresponding Cesaro convergence in the empirical Bayes problem has been noted previously in special cases; by Van Ryzin (1966b), Theorem 6.1, for the finite $\Omega \times \mathfrak{A}$ problem and by Swain (1965), Theorem 4, for the extended compound problem in case of squared error loss. After this manuscript was drafted, a paper by Samuel (1965c) in which is reported an extension of sequential compound convergence to convergence in the empirical Bayes problem as conceived by Robbins (1956) came to the author's attention. This extension is incorrect, but the reported proof does include an alternative proof of the Cesaro convergence result.

In the next section we specialize to squared error loss and derive a bound for $|D_n(\boldsymbol{\theta}, \boldsymbol{\varphi})|$ in terms of a Cesaro mean difference of $\varphi_i$ and $\psi_i(X_i, \cdot)$.

**2. Squared error loss.** Suppose that $\Omega$ and $\mathfrak{A}$ are subsets of the line and $L(\theta, A) = (\theta - A)^2$. Only non-randomized procedures need be considered; a non-randomized sequential procedure $\boldsymbol{\varphi}$ has modified regret

$$D_n(\boldsymbol{\theta}, \boldsymbol{\varphi}) = n^{-1}\textstyle\sum_{i=1}^n \mathbf{P}_i(\varphi_i - \theta_i)^2 - R(G_n) \tag{2.1}$$

where we have used $\mathbf{P}_i$ to denote the expectation operator for the similarly labelled product probability measure and $\varphi_i$ is an $\mathbf{x}_i$ measurable function into $\mathfrak{A}$. In this paper we will use the same symbol for a measure and its expectation operator whenever it is convenient to do so. Inequalities (1.6) imply

$$n^{-1}\textstyle\sum_{i=1}^n \mathbf{P}_i((\varphi_i - \psi_{i-1})(\varphi_i + \psi_{i-1} - 2\theta_i)) \leqq D_n(\boldsymbol{\theta}, \boldsymbol{\varphi})$$
$$\leqq n^{-1}\textstyle\sum_{i=1}^n \mathbf{P}_i((\varphi_i - \psi_i)(\varphi_i + \psi_i - 2\theta_i))$$

where, as defined earlier, $\psi_i$ is a decision rule in the component problem which is Bayes versus $G_i$, $i \geqq 1$, and $\psi_0$ is an arbitrary decision rule.

If $\Omega = \mathcal{A} = [-a, a]$ it follows that

$$(2.2) \quad -4an^{-1}\sum_{i=1}^{n} P_i(|\Psi_i|) - 4an^{-1}\sum_{i=1}^{n} \mathbf{P}_i(|\varphi_i - \psi_i|) \leqq D_n(\boldsymbol{\theta}, \boldsymbol{\varphi}) \leqq 4an^{-1}\sum_{i=1}^{n} \mathbf{P}_i(|\varphi_i - \psi_i|)$$

where $\Psi_i = \psi_i - \psi_{i-1}$ and $P_i = P_{\theta_i}$.

THEOREM 2.1. *Let there exist a $\sigma$-finite measure $\mu$ dominating each $P_\theta$, $\theta \varepsilon \Omega = [-a, a]$, and let $p_\theta$ denote a version of the Radon-Nikodym derivative. If $M(x) = \sup_\theta p_\theta(x)$ is $\mu$ integrable, then $n^{-1}\sum_1^n P_i(|\Psi_i|) = O(n^{-1}\log n)$ uniform in $\boldsymbol{\theta}$ for the version of $\psi_i$ given by* (2.3).

PROOF. As the Bayes response we take the version of conditional expectation

$$(2.3) \qquad \psi_i = \sum_{j=1}^{i} \theta_j p_j \left(\sum_{j=1}^{i} p_j\right)^{-1}\left[\sum_{j=1}^{i} p_j > 0\right]$$

where $[\sum_1^i p_j > 0]$ is the indicator function of $\{x \mid \sum_1^i p_j(x) > 0\}$. It is easy to show that

$$|\Psi_i| \leqq 2ap_i\left(\sum_{j=1}^{i} p_j\right)^{-1}\left[\sum_{j=1}^{i-1} p_j > 0\right] + a\left[\sum_{j=1}^{i-1} p_j = 0\right] \quad [p_i > 0]$$

from which

$$(2.4) \quad \sum_{i=1}^{n} P_i(|\Psi_i|) \leqq 2a\mu\left(M\sum_{i=1}^{n} (p_iM^{-1})^2\left(\sum_{j=1}^{i} (p_jM^{-1})\right)^{-1}\right) + a\mu\left(\sum_{i=1}^{n} p_i\left[\sum_{j=1}^{i-1} p_j = 0\right][p_i > 0]\right).$$

The lemma to follow implies that the first term on the right hand side of (2.4) is bounded by $2a\mu(M)\sum_1^n i^{-1}$. Since $M \geqq \sum_1^n p_i[\sum_1^{i-1}p_j = 0] \ [p_i > 0]$, the second term is bounded by $a\mu(M)$ and the theorem is proved.

LEMMA 2.1. $S_n(a_1, \cdots, a_n) = \sum_1^n a_i^2(\sum_1^i a_j)^{-1} \leqq \sum_1^n i^{-1} = S_n(1, \cdots, 1)$ *for all* $0 \leqq a_i \leqq 1$, $1 \leqq i \leqq n$, $n \geqq 1$.

PROOF. Let $A_i = \sum_1^i a_j$ so that $S_n = \sum_1^n a_i^2 A_i^{-1}$, $(\partial S_n/\partial a_n) = (2a_nA_{n-1} + a_n^2)A_n^{-2} \geqq 0$. Therefore, $S_n(a_1, \cdots, a_n) \leqq S_n(a_1, \cdots, a_{n-1}, 1)$ for all $0 \leqq a_i \leqq 1$, $1 \leqq i \leqq n$. Also, for $1 \leqq k < n$

$$(\partial S_n/\partial a_k)(a_1, \cdots, a_k, 1, \cdots, 1) = (2a_kA_{k-1} + a_k^2)A_k^{-2} - \sum_{k+1}^{n} (A_k + i - k)^{-2},$$

$$(\partial^2 S_n/\partial a_k^2)(a_1, \cdots, a_k, 1, \cdots, 1) = 2A_{k-1}^2A_k^{-3} + 2\sum_{k+1}^{n} (A_k + i - k)^{-3} \geqq 0.$$

Since the second partial is non-negative and

$$S_n(a_1, \cdots, a_{k-1}, 1, \cdots, 1) = \sum_1^{k-1} a_i^2A_i^{-1} + \sum_k^n (A_{k-1} + i - k + 1)^{-1} \geqq \sum_1^{k-1} a_i^2A_i^{-1} + \sum_{k+1}^{n} (A_{k-1} + i - k)^{-1} = S_n(a_1, \cdots, a_{k-1}, 0, 1, \cdots, 1),$$

we have $S_n(a_1, \cdots, a_k, 1, \cdots, 1) \leqq S_n(a_1, \cdots, a_{k-1}, 1, \cdots, 1)$ for all $0 \leqq a_i \leqq 1$, $1 \leqq i \leqq k$. The proof is completed by backward induction on $k$.

It is interesting to note that the hypothesis of Theorem 2.1 need not imply

$P_i(|\Psi_i|) = o(1)$ uniformly in $\boldsymbol{\theta}$ no matter what determination of the conditional expectation is used; but, that for a family of normal densities, $P_i(|\Psi_i|) = O(i^{-1})$ uniformly in $\boldsymbol{\theta}$ (Gilliland, (1966)). We now apply the theorem to (2.2).

If $\Omega = \mathcal{A} = [-a, a]$ and the hypothesis of Theorem 2.1 is satisfied, then (2.2) yields

$$(2.5) \qquad |D_n(\boldsymbol{\theta}, \boldsymbol{\varphi})| \leqq 4an^{-1}\sum_{i=1}^n \mathbf{P}_i(|\varphi_i - \psi_i|) + O(n^{-1}\log n)$$

where $O(n^{-1}\log n)$ is uniform in $\boldsymbol{\theta}$.

## 3. Squared error loss and some discrete exponential families.

3.1. *Introduction.* We now consider sequential compound estimation in the case of squared error loss and a family of discrete exponential distributions on the non-negative integers. Let $P_\theta$ have density

$$(3.1) \qquad p_\theta(x) = \theta^x h(\theta) g(x), \qquad x = 0, 1, \cdots,$$

with respect to counting measure $\mu$ where $g(x) > 0$
and

$$(\text{A1}) \qquad \mathcal{A} = \Omega = [0, \beta], \qquad 0 < \beta < \infty.$$

(The condition $g(x) > 0$ is assumed by Samuel (1965a) and is implied by assumptions of Swain (1965).) With this family of distributions the Bayes response (2.3) takes the special form

$$(3.2) \qquad \psi_i = (g\sum_{j=1}^i \tilde{p}_j)(\tilde{g}\sum_{j=1}^i p_j)^{-1}[\sum_{j=1}^i p_j > 0]$$

where $\tilde{f}$ is defined by $\tilde{f}(x) = f(x+1)$. The hypothesis of Theorem 2.1 is satisfied under (A1) since $p_\theta(x) \leqq (h(0)/h(\beta))p_\beta(x)$ for all $x$ and $\theta \in \Omega$. With an adjustment in constant appropriate to the situation, (2.5) reads

$$(3.3) \qquad |D_n(\boldsymbol{\theta}, \boldsymbol{\varphi})| \leqq 2\beta n^{-1}\sum_{i=1}^n \mathbf{P}_i(|\varphi_i - \psi_i|) + O(n^{-1}\log n)$$

where $O(n^{-1}\log n)$ is uniform in $\boldsymbol{\theta}$. In view of (3.2) a natural procedure is provided by $\varphi_i^{**}(X_i)$ with

$$(3.4) \qquad \varphi_i^{**} = \{(g\sum_{j=1}^i \tilde{\delta}_j)(\tilde{g}\sum_{j=1}^i \delta_j)^{-1}\} \wedge \beta$$

where $\delta_j(x) = \delta(X_j, x)$, the Kronecker delta function. (The display of the dependence of $\varphi_i^{**}(X_i)$ upon $\mathbf{X}_{i-1}$ is suppressed above. This new notation will be used for all sequential compound procedures in what follows.) The untruncated version of this procedure is suggested by Robbins (1956) and studied by Johns (1956) in the context of the empirical Bayes problem. A generalized version is applied by Swain (1965) to the extended sequential compound decision problem. Essentially, Samuel (1965a) deduces that under assumption

$$(\text{A1}^+) \qquad \mathcal{A} = [0, \beta], \qquad \Omega = [\alpha, \beta], \qquad 0 < \alpha \leqq \beta < \infty,$$

the variant

$$(3.5) \quad \varphi_i^0 = (g\{((i-1)^{-1}\sum_1^{i-1}\tilde{\delta}_j) \vee \tilde{m}\}/\tilde{g}\{((i-1)^{-1}\sum_1^{i-1}\delta_j) \vee m\}) \wedge \beta$$

where $m(x) = \min_\theta p_\theta(x)$ results in $D_n(\boldsymbol{\theta}, \boldsymbol{\varphi}^0) \leqq B_n(\boldsymbol{\theta}) = o(1)$. (Samuel proves the stronger result involving almost sure convergence.) For this procedure to be realizable $m$ must be known which would be the case if both $\alpha$ and $\beta$ are known. Theorem 3.1 to follow demonstrates that $D_n(\boldsymbol{\theta}, \boldsymbol{\varphi}^*) = o(1)$ uniform in $\boldsymbol{\theta}$ for the procedure

$$(3.6) \qquad \varphi_i^* = \{(g \textstyle\sum_{j=1}^{i-1} \tilde{\delta}_j)(\tilde{g} \sum_{j=1}^{i-1} \delta_j)^{-1}[\sum_{j=1}^{i-1} \delta_j > 0]\} \wedge \beta$$

so that $\alpha$ need not be known. To motivate the introduction of other assumptions in addition to (A1) or (A1$^+$) we give an example to show that the null sequence $D_n(\boldsymbol{\theta}, \boldsymbol{\varphi}^0)$ may be converging arbitrarily slowly. That is, given a null sequence $H_n > 0$, there exists an exponential family and a parameter sequence $\boldsymbol{\theta} \varepsilon [\alpha, \beta]^\infty$ such that $D_n(\boldsymbol{\theta}, \boldsymbol{\varphi}^0) \geqq H_n$ for all large $n$. Of course, the same example works for $\boldsymbol{\varphi}^*$ and $\boldsymbol{\varphi}^{**}$.

EXAMPLE 3.1. Let $0 < \alpha < \beta = 1$ where $g(0) = 1$ and $g(x) \leqq x^{-3}$, $x \geqq 1$, is yet to be specified. It is not difficult to show that $m(x) = p_\alpha(x)$ for $x \geqq 2$. At $\boldsymbol{\theta} = \mathbf{1} = (1, 1, \cdots)$,

$$\begin{aligned} D_n(\mathbf{1}, \boldsymbol{\varphi}^0) &= n^{-1}\textstyle\sum_1^n \mathbf{P}_i((\varphi_i^0 - 1)^2) \\ &\geqq n^{-1}\textstyle\sum_1^n \mathbf{P}_i((\varphi_i^0 - 1)^2[\sum_1^{i-1}\tilde{\delta}_j = 0]) \\ &\geqq n^{-1}\textstyle\sum_1^n \sum_{x=0}^\infty p_1(x) \\ &\quad \cdot \{(g(x)\, m(x+1)/g(x+1)\, m(x)) \wedge \beta - 1\}^2 \mathbf{P}_{i-1}[\textstyle\sum_1^{i-1}\tilde{\delta}_j(x) = 0] \\ &\geqq n^{-1}\textstyle\sum_1^n \sum_2^\infty p_1(x)\{\alpha - 1\}^2 \mathbf{P}_{n-1}[\sum_1^{n-1}\tilde{\delta}_j(x) = 0] \\ &= (\alpha - 1)^2 \textstyle\sum_2^\infty p_1(x)(1 - p_1(x+1))^{n-1} \\ &\geqq \tfrac{1}{4}\textstyle\sum_1^\infty p_1(2x+1)(1 - p_1(2x+2))^{n-1} \end{aligned}$$

where for the last inequality we have taken $\alpha = \frac{1}{2}$. Let $g(x) = x^{-3}$, $x = 1, 3, 5, \cdots$, and $x^{-3} \geqq g(x) = a(x)$, $x = 2, 4, 6, \cdots$ with $a(x)$ strictly decreasing. Then

$$\begin{aligned} D_n(\mathbf{1}, \boldsymbol{\varphi}^0) &\geqq \tfrac{1}{4}\textstyle\sum_1^\infty h(1)(2x+1)^{-3}(1 - h(1)a(2x+2))^{n-1} \\ &\geqq \tfrac{1}{4}\textstyle\sum_{A_n}^\infty h(1)(2x+1)^{-3}(1 - h(1)/(n-1))^{n-1} \end{aligned}$$

where $A_n = \min\{x \mid a(2x+2) \leqq (n-1)^{-1}\}$. Since $\sum_b^\infty (2x+1)^{-3} \geqq \frac{1}{4}(2b+1)^{-2}$ and $h(1) = (\sum_0^\infty g(x))^{-1}$ implies $B = (1 + \sum_1^\infty x^{-3})^{-1} \leqq h(1) \leqq 1$, we can write

$$D_n(\mathbf{1}, \boldsymbol{\varphi}^0) \geqq \tfrac{1}{16}B(1 - (n-1)^{-1})^{n-1}(2A_n + 1)^{-2}.$$

By choice of $a(x)$, $A_n$ can be made to increase arbitrarily slowly. Since $(1 - (n-1)^{-1})^{n-1} \to e^{-1}$ the example is complete.

3.2 *Bounds for the modified regret.* We first introduce procedures $\boldsymbol{\varphi}'$ and $\boldsymbol{\varphi}''$ which are not realizable in the sequential problem as originally stated. Define $\varphi_i'(X_i)$ and $\varphi_i''(X_i)$ by

$$(3.7) \qquad \varphi_i' = \{g[\textstyle\sum_1^{i-1}\tilde{\delta}_j + \tilde{\delta}_i']/\tilde{g}[\sum_1^{i-1}\delta_j + \delta_i']\} \wedge \beta$$

and

$$(3.8)\qquad \varphi_i'' = \{g[\textstyle\sum_1^{i-1}\tilde{\delta}_j + \tilde{\delta}_i' + \sum_1^i \zeta_j]/\tilde{g}[\sum_1^{i-1}\delta_j + \delta_i']\} \wedge \beta$$

where ratios 0/0 are to be interpreted as 0, $\delta_i'(x) = \delta(X_i', x)$, $X_i' \sim P_i$ is independent of $\mathbf{X}_i$, and $\zeta_j$ is any sequence of centered, independent random variables satisfying $|\zeta_j| \leqq 1$ with $\zeta_i$ independent of $(\mathbf{X}_i, X_i')$. With $E_i$ the expectation with respect to the measure induced by $(\mathbf{X}_{i-1}, X_i', \zeta_i)$, we develop bounds for the Cesaro means, $n^{-1}\sum_1^n P_i$, operating on each of the sequences

$$\mathrm{A}_i = E_i(|\varphi_i' - \psi_i|),$$

$$\mathrm{B}_i = E_i(|\varphi_i'' - \psi_i|),$$

$$\mathrm{C}_i = E_i(|\varphi_i^* - \varphi_i'|),$$

and

$$\mathrm{F}_i = E_i(|\varphi_i' - \varphi_i''|).$$

Triangle inequalities together with (3.3) will then yield bounds for the modified regret resulting from the natural sequential compound procedure $\boldsymbol{\varphi}^*$.

We write

$$(3.9)\qquad A_i = \textstyle\int_0^{\psi_i} E_i[\varphi_i' - \psi_i \leqq -t]\,dt + \int_0^\beta E_i[\varphi_i' - \psi_i \geqq t]\,dt$$

and use exponential bounds of Hoeffding (1963) to approximate each integrand displaced on the right hand side of (3.9). For convenience we drop the prime from $\delta_i'$ and the subscript from $E_i$ and have for $0 \leqq t \leqq \beta$

$$E[\varphi_i' - \psi_i \geqq t] \leqq E[g\textstyle\sum_1^i \tilde{\delta}_j/\tilde{g}\sum_1^i \delta_j - (\psi_i + t) \geqq 0]$$

$$= E[\textstyle\sum_1^i Y_j \geqq 0] = E[\sum_1^i (Y_j - EY_j) \geqq -\sum_1^i EY_j]$$

with $Y_j = \tilde{\delta}_j - R_i(t)\delta_j$, $R_i(t) = (\tilde{g}/g)(\psi_i + t)$. Since $-\sum_1^i EY_j = t(\tilde{g}/g)\sum_1^i p_j \geqq 0$ and $-R_i(t) - EY_j \leqq Y_j - EY_j \leqq 1 - EY_j$, Hoeffding's Theorem 2 (1963) yields

$$(3.10)\qquad E[\varphi_i' - \psi_i \geqq t] \leqq \exp\{-2\tilde{g}^2(\textstyle\sum_1^i p_j)^2 (i\,g^2(1 + R_i(t))^2)^{-1}t^2\}.$$

Similarly, we treat the other tail writing for $0 \leqq t \leqq \psi_i$

$$E[\varphi_i' - \psi_i \leqq -t] = E[g\textstyle\sum_1^i \tilde{\delta}_j/\tilde{g}\sum_1^i \delta_j - (\psi_i - t) \leqq 0]$$

$$= E[\textstyle\sum_1^i (Z_j - EZ_j) \leqq -\sum_1^i EZ_j]$$

with $Z_j = \tilde{\delta}_j - S_i(t)\delta_j$, $S_i(t) = (\tilde{g}/g)(\psi_i - t)$. The inequalities $-\sum_1^i EZ_j = -t(\tilde{g}/g)\sum_1^i p_j \leqq 0$ and $-S_i(t) - EZ_j \leqq Z_j - EZ_j \leqq 1 - EZ_j$ imply

$$(3.11)\qquad E[\varphi_i' - \psi_i \leqq -t] \leqq \exp\{-2\tilde{g}^2(\textstyle\sum_1^i p_j)^2(i\,g^2(1 + S_i(t))^2)^{-1}t^2\}.$$

Noting upper bounds for $R_i$ and $S_i$ we combine (3.10) and (3.11) obtaining for all $0 \leqq t \leqq \beta$

$$(3.12)\qquad E[|\varphi_i' - \psi_i| \geqq t] \leqq 2\exp\{-2\tilde{g}^2(\textstyle\sum_1^i p_j)^2(i\,g^2(1 + 2\beta(\tilde{g}/g))^2)^{-1})t^2\}.$$

Since $\int_0^\beta \exp\{-ct^2\}\,dt \leqq \frac{1}{2}(\pi/c)^{\frac{1}{2}}$, (3.9) and (3.12) imply

$$A_i \leqq A(1 + (g/\tilde{g}))(\textstyle\sum_1^i p_j)^{-1} i^{\frac{1}{2}} \tag{3.13}$$

for a constant $A$ not depending upon $\boldsymbol{\theta}$, $i$ and $x$. Inequality (3.13) provides a high rate of convergence at $x = 0$ because $\sum_1^i p_j(0) \geqq h(\beta)g(0)\ i$; and, therefore,

$$A_i(0) \leqq C\, i^{-\frac{1}{2}} \tag{3.14}$$

for a constant $C$ not depending upon $\boldsymbol{\theta}$ and $i$. The approach of Swain (1965) is to partition by $[\sum_1^i p_j > i^{\frac{3}{4}}]$ from which rates $O(n^{-\frac{1}{4}} \log n)$ would follow. However, it turns out that better rates are deducible using another approach. Instead of Hoeffding's theorem one can use normal approximation to bound the integrands displayed on the right hand side of (3.9). This approach leads to a bound for $A_i(x)$, $x \geqq 1$, from which the rate $O(n^{-\frac{1}{2}})$ more readily follows.

As before consider $Y_j = \tilde{\delta}_j - R_i(t)\delta_j$ and let $W_j = (Y_j - EY_j)/(1 + R_i(t))$. Since $|W_j| \leqq 1$, $E(|W_j|^3) \leqq E(W_j^2)$ and it follows with $r_i^2(t) = \operatorname{Var}(\sum_1^i Y_j)$ that

$$\begin{aligned} L_i(t) &= r_i^{-3}(t) \textstyle\sum_1^i E(|Y_j - EY_j|^3) \\ &\leqq (\textstyle\sum_1^i E(W_j^2))^{-\frac{1}{2}} = r_i^{-1}(t)(1 + R_i(t)). \end{aligned}$$

Explicitly, $r_i^2(t) = \sum_1^i \{\tilde{p}_j + R_i^2(t)p_j - (\tilde{p}_j - R_i(t)p_j)^2\}$. Note that $q = \inf\{1 - p_\theta(x) \mid x \geqq 1, \theta \in \Omega\} > 0$, $R_i^2(t) \leqq (2\beta(\tilde{g}/g))^2$, $\tilde{p}_j \leqq \beta(\tilde{g}/g)p_j$ so with $T^2 = \beta(\tilde{g}/g) + 4\beta^2(\tilde{g}/g)^2$, $x \geqq 1$,

$$q\textstyle\sum_1^i \tilde{p}_j + qR_i^2(t)\sum_1^i p_j \leqq r_i^2(t) \leqq T^2 \sum_1^i p_j\,. \tag{3.15}$$

Hence,

$$L_i(t) \leqq (q\textstyle\sum_1^i \tilde{p}_j)^{-\frac{1}{2}} + (q\sum_1^i p_j)^{-\frac{1}{2}}. \tag{3.16}$$

The Berry-Esseen theorem (Loève (1963), p. 288) implies that for a constant $c$

$$E[\varphi_i' - \psi_i \geqq t] \leqq \Phi(-\tilde{g}\textstyle\sum_1^i p_j(g\, r_i(t))^{-1}t) + c\, L_i(t) \tag{3.17}$$

where $\Phi$ denotes the distribution function of $N(0, 1)$. We can write for $a > 0$, $\int_0^\beta \Phi(-at)\,dt \leqq a^{-1}\int_{-\infty}^0 \Phi(\tau)\,d\tau = 2^{-1}a^{-1}\int_0^\infty P[|X| \geqq \tau]\,d\tau = 2^{-1}a^{-1}P(|X|)$ where $X \sim N(0,1)$. Therefore, $\int_0^\beta \Phi(-at)\,dt \leqq (2\pi)^{-\frac{1}{2}}a^{-1}$. Using this result and the upper bound in (3.15) yields

$$\int_0^\beta \Phi(-\tilde{g} \textstyle\sum_1^i p_j(g\, r_i(t))^{-1}t)\,dt \leqq g\, T[(2\pi)^{\frac{1}{2}}\tilde{g}(\sum_1^i p_j)^{\frac{1}{2}}]^{-1}. \tag{3.18}$$

We combine (3.16)–(3.18) to prove

$$\int_0^\beta E[\varphi_i' - \psi_i \geqq t]\,dt \leqq D\{((g/\tilde{g})T + 1)(\textstyle\sum_1^i p_j)^{-\frac{1}{2}} + \sum_1^i \tilde{p}_j)^{-\frac{1}{2}}\} \tag{3.19}$$

for $x \geqq 1$ and a constant $D$. The same method results in

$$\int_0^{\psi_i} E[\varphi_i' - \psi_i \leqq -t]\,dt \leqq D\{((g/\tilde{g})U + 1)(\textstyle\sum_1^i p_j)^{-\frac{1}{2}} + (\sum_1^i \tilde{p}_j)^{-\frac{1}{2}}\} \tag{3.20}$$

for $x \geqq 1$ where $U^2 = \beta(\tilde{g}/g) + \beta^2(\tilde{g}/g)^2$. Since $U \leqq T \leqq \beta^{\frac{1}{2}}(\tilde{g}/g)^{\frac{1}{2}} + 2\beta(\tilde{g}/g)$, (3.19) and (3.20) combine to yield

$$A_i \leqq D\{(1 + (g/\tilde{g})^{\frac{1}{2}}) (\textstyle\sum_1^i p_j)^{-\frac{1}{2}} + (\textstyle\sum_1^i \tilde{p}_j)^{-\frac{1}{2}}\} \tag{3.21}$$

for $x \geqq 1$ and a constant $D$ not depending upon $\boldsymbol{\theta}$, $i$ and $x$. Under ($\mathrm{A1}^+$), $\limsup_i \theta_i > 0$ so that $\sum_1^i p_j(x) \uparrow \infty$ for each $x$ and the normal approximation is good. However, under (A1), var $(\sum_1^i Y_j)$ may be bounded in which case the normal approximation does not lead to a particularly interesting bound in (3.21). With the procedure $\varphi''$ the artificial randomization makes the variance diverge and permits use of Survila's strengthening of the Berry-Esseen theorem (Survila, (1962)) to develop the bound. With $\sigma_i^2 = \text{var}(\sum_1^i \zeta_j) \uparrow \infty$ and $x \geqq 1$,

$$B_i \leqq B\{(g/\tilde{g})\sigma_i(\textstyle\sum_1^i p_j)^{-3/2} + (1 + (g/\tilde{g})^{\frac{1}{2}} + (g/\tilde{g})\sigma_i)(\textstyle\sum_1^i p_j)^{-1} + (1 + (g/\tilde{g})^{\frac{1}{2}})(\textstyle\sum_1^i p_j)^{-\frac{1}{2}}\} \tag{3.22}$$

for some constant $B$ independent of $\boldsymbol{\theta}$, $i$ and $x$. (Gilliland, (1966), pp. 21–23).

We now use the bounds (3.13), (3.14), (3.21) and (3.22) to prove propositions concerning the rates of convergence of $n^{-1}\sum_1^n P_i(A_i)$ and $n^{-1}\sum_1^n P_i(B_i)$.

Proposition 3.1. *Under* ($\mathrm{A1}^+$), $P_i(A_i) = o(1)$ *uniformly in* $\boldsymbol{\theta}$.

Proof. It follows from (3.13) and the fact $\varphi_i'$ and $\psi_i$ take values in $[0, \beta]$ that for each fixed $x$,

$$A_i(x) \leqq f_i(x) \wedge \beta$$

where

$$f_i(x) = A(1 + g(x)/g(x+1))m^{-1}(x)i^{-\frac{1}{2}}$$

and $m(x) = \inf_\theta p_\theta(x) > 0$ under ($\mathrm{A1}^+$). As measures $P_\theta \leqq (h(0)/h(\beta)P_\beta$ so we have $P_i(A_i) \leqq (h(0)/h(\beta))P_\beta(f_i \wedge \beta)$ which converges to zero by the dominated convergence theorem.

Proposition 3.2. *Under* ($\mathrm{A1}^+$),

$$\textstyle\sum_x p_\beta^{\frac{1}{2}}(x) < \infty, \tag{A2}$$

and

$$\textstyle\sum_x (g(x)/g(x+1))^{\frac{1}{2}} p_\beta^{\frac{1}{2}}(x) < \infty, \tag{A3}$$

$n^{-1}\sum_1^n P_i(A_i) = O(n^{-\frac{1}{2}})$ *uniformly in* $\boldsymbol{\theta}$.

Proof. Under ($\mathrm{A1}^+$) the bound (3.21) takes the form

$$A_i \leqq D(1 + (g/\tilde{g})^{\frac{1}{2}})(\textstyle\sum_1^i p_j)^{-\frac{1}{2}} \tag{3.23}$$

where we have used the relation $\tilde{p}_j \geqq \alpha(\tilde{g}/g)p_j$ and $D$ is some constant independent of $\boldsymbol{\theta}$, $i$ and $x \geqq 1$. Writing $\sum_1^n P_i(A_i) = \sum_x \sum_1^n p_i(x)A_i(x)$, (3.14) and (3.23) yield

$$\textstyle\sum_1^n P_i(A_i) \leqq C \sum_1^n i^{-\frac{1}{2}} + D \sum_1^\infty (1 + (g(x)/g(x+1))^{\frac{1}{2}})M^{\frac{1}{2}}(x) \cdot \sum_1^n (p_i(x)/M(x))(\sum_1^i p_j(x)/M(x))^{-\frac{1}{2}} \tag{3.24}$$

where $M(x) = \sup_\theta p_\theta(x) \leqq (h(0)/h(\beta))p_\beta(x)$.

LEMMA 3.1. $\sum_1^n a_i(\sum_1^i a_j)^{-\frac{1}{2}} \leqq 2(\sum_1^n a_i)^{\frac{1}{2}}$ *for all* $a_i \geqq 0$, $1 \leqq i \leqq n$, $n \geqq 1$.

PROOF. Let $A_i = \sum_1^i a_j$. The inequality $0 \leqq A_i - 2A_i^{\frac{1}{2}}A_{i-1}^{\frac{1}{2}} + A_{i-1}$ implies $a_i \leqq 2(A_i - A_i^{\frac{1}{2}}A_{i-1}^{\frac{1}{2}})$ from which $a_iA_i^{-\frac{1}{2}} \leqq 2(A_i^{\frac{1}{2}} - A_{i-1}^{\frac{1}{2}})$. Summing from 1 to $n$ proves the lemma. The lemma applied to (3.24) proves

$$\sum_1^n P_i(A_i) \leqq Cn^{\frac{1}{2}} + Dn^{\frac{1}{2}}\sum_1^\infty (1 + (g(x)/g(x+1))^{\frac{1}{2}})p_\beta^{\frac{1}{2}}(x)$$

for appropriate constants $C$ and $D$.

PROPOSITION 3.3 *Under* (A1), $P_i(A_i) = o(1)$.

PROOF. We can write $P_i(A_i) \leqq A_i(0) + \beta P_i[x > 0] \leqq Ci^{-\frac{1}{2}} + \beta P_i[x > 0]$. Therefore, if $\boldsymbol{\theta}$ is such that $\theta_i \to 0$ then $P_i(A_i) \to 0$. Otherwise, $\limsup_i \theta_i > 0$ and for each $x$, $\sum_1^i p_j(x) \uparrow \infty$ so that (3.14) and (3.21) prove that $A_i(x) \to 0$ for each $x$. Noting $P_i \leqq (h(0)/h(\beta))P_\beta$ the proof is completed by application of the dominated convergence theorem.

PROPOSITION 3.4. *Under* (A1),

$$\text{(A2}^+\text{)} \qquad \beta \in \text{interior } \Omega^*$$

*where* $\Omega^*$ *denotes the natural parameter space, and*

$$\text{(A3}^-\text{)} \qquad \sum_x (g(x)/g(x+1))p_\beta(x) < \infty,$$

$n^{-1}\sum_1^n P_i(B_i) = O(n^{-1/3}\log n)$ *uniformly in* $\boldsymbol{\theta}$ *if* $\sigma_i^2 = \operatorname{var}(\sum_1^i \zeta_j) = O(i^{2/3})$ *and* $\sigma_i^2$ *diverges.*

PROOF. The trivial bound $|\varphi_i'' - \psi_i| \leqq \beta$ and partitioning by $[\sum_1^i p_j > i^{2/3}]$ prove via (3.22) that

$$B_i \leqq \beta[\sum_1^i p_j \leqq i^{2/3}] + B(1 + (g/\tilde{g})^{\frac{1}{2}} + g/\tilde{g})i^{-1/3} \tag{3.25}$$

for $x \geqq 1$ and appropriate constant $B$. The first term is taken care of by a lemma, a weaker version of which is proved by Swain ((1965), pp. 32–34).

LEMMA 3.2. *Let* $1 \leqq b_i \leqq i$ *be an increasing sequence. If* (A1) *and* (A2$^+$) *obtain, then*

$$\sum_1^n P_i[\sum_1^i p_j(x) \leqq b_i] = O(b_n \log n)$$

*uniformly in* $\boldsymbol{\theta}$.

PROOF. For fixed $x$, $\sum_1^n p_i(x)[\sum_1^i p_j(x) \leqq b_i] \leqq b_n$ so $\sum_1^n P_i[\sum_1^i p_j(x) \leqq b_i] \leqq ab_n + \sum_1^n P_i[x \geqq a]$. Since the family has monotone increasing likelihood ratio and $[x \geqq a]$ is increasing in $x$, $P_i[x \geqq a] \leqq P_\beta[x \geqq a]$. But under (A2$^+$) $P_\beta[x \geqq a] \leqq cr^a$ where $c = h(\beta)/h(d)$, $r = \beta/d$ for some $\beta < d$, $d \in \Omega^*$. Therefore, for any $a \geqq 0$

$$\sum_1^n P_i[\sum_1^i p_j(x) \leqq b_i] \leqq ab_n + cnr^a. \tag{3.26}$$

Letting $a$ be such that $nr^a = 1$ implies $a = (\log n)/\log r^{-1}$; and, hence, the lemma follows from (3.26). Since a bound like (3.14) holds for $B_i(0)$, application of the lemma to (3.25) completes the proof of the proposition.

PROPOSITION 3.5. *Under* (A1), (A2$^+$) *and*

$$\text{(A3}') \qquad \textstyle\sum_x (g(x)/g(x+1))p_\beta^{\frac{1}{2}}(x) < \infty,$$

$n^{-1}\sum_1^n P_i(B_i) = O(n^{-\frac{1}{2}})$ *uniformly in* $\boldsymbol{\theta}$ *if* $\sigma_i^2 = O(i^{\frac{1}{4}})$ *and* $\sigma_i^2$ *diverges.*

PROOF. Partitioning (3.22) by $[\sum_1^i p_j > i^{\frac{1}{4}}]$ leads to

$$(3.27) \quad B_i \leqq \beta[\textstyle\sum_1^i p_j \leqq i^{\frac{1}{4}}] + B(1 + (g/\tilde{g})^{\frac{1}{2}} + (g/\tilde{g})\sigma_i i^{-1/8})(\sum_1^i p_j)^{-\frac{1}{2}}$$

for $x \geqq 1$ and a constant $B$. Normalizing the $p_j$ by dividing by $M$ as in (3.24), the proposition is proved by appealing to Lemmas 3.1 and 3.2. Use is also made of the fact that (A2$^+$) implies (A2) which together with (A3′) implies (A3).

Another series of propositions will complete the triangulation and yield five theorems concerning the modified regret. It is easy to verify that

$$(3.28) \quad |\varphi_i^* - \varphi_i'| \leqq \beta[S_{i-1} = 0] + \beta\, S_{i-1}^{-1}[S_{i-1} > 0]$$
$$+ g\,(\tilde{g}S_{i-1})^{-1}[\delta_i' = 1][S_{i-1} > 0]$$

where $S_{i-1} = \sum_1^{i-1} \delta_j$, and, as before, $\delta_i'(x) = \delta(X_i', x)$ and square brackets denote indicator functions. We need the following lemma.

LEMMA 3.3. *For* $i \geqq 2$, $E(S_{i-1}^{-1}[S_{i-1} > 0]) \leqq 4(\sum_1^i p_j)^{-1}$.

PROOF. For each $x$, $[S_{i-1} > 0]S_{i-1}^{-1} \leqq 4(S_{i-1} + \delta_i' + 1)^{-1}$. Since $(S_{i-1} + \delta_i' + 1)^{-1}$ is convex in $S_{i-1} + \delta_i'$, Hoeffding's Theorem 3 (1956) implies $E((S_{i-1} + \delta_i' + 1)^{-1}) \leqq \sum_0^i (j+1)^{-1}\binom{i}{j}p^j(1-p)^{i-j}$ where $ip = \sum_1^i p_j$. This last inequality implies $E((S_{i-1} + \delta_i' + 1)^{-1}) \leqq i/((i+1)\sum_1^i p_j) \leqq (\sum_1^i p_j)^{-1}$ and completes the proof.

PROPOSITION 3.6. *Under* (A1$^+$), $P_i(C_i) = o(1)$ *uniformly in* $\boldsymbol{\theta}$.

PROOF. For fixed $x$ the $E$ expectation of the right hand side of (3.28) is bounded by $\beta \exp\{-\sum_1^{i-1} p_j\} + 4\beta(\sum_1^i p_j)^{-1} + 4\beta p_i(\sum_1^i p_j)^{-1} = g_i$ where Lemma 3.3 and $(g/\tilde{g})\tilde{p}_i \leqq \beta p_i$ have been used. Weakening the bound to $\beta \exp\{-(i-1)m\} + 8\beta i^{-1}m^{-1} = f_i$, we have $P_i(C_i) \leqq h(0)(h(\beta))^{-1}P_\beta(f_i \wedge \beta)$. The right hand side is independent of $\boldsymbol{\theta}$ and converges to zero by the dominated convergence theorem.

In order to deduce rates of convergence it will be necessary to examine the bound (3.28) in the case the family of distributions satisfies conditions in addition to (A1$^+$) or (A1). One such condition is

$$\text{(A2}') \qquad \textstyle\sum_x x p_\beta(x) < \infty$$

which is implied by (A2$^+$).

LEMMA 3.4. *Under* (A1) *and* (A2′), $n^{-1}\sum_1^n \mathbf{P}_i([S_{i-1} = 0]) = O(n^{-\frac{1}{2}})$ *uniformly in* $\boldsymbol{\theta}$.

PROOF. We write $\sum_1^n \mathbf{P}_i[S_{i-1} = 0] \leqq \sum_{x<a} \sum_1^n p_i(x)\mathbf{P}_{i-1}[S_{i-1} = 0] + \sum_1^n P_i[x \geqq a]$. The monotone likelihood ratio property and the fact that the $p_i(x)\mathbf{P}_{i-1}[S_{i-1} = 0] = \mathbf{P}_i[X_1 \neq x, \cdots, X_{i-1} \neq x, X_i = x]$ are probabilities of disjoint events together imply $\sum_1^n \mathbf{P}_i[S_{i-1} = 0] \leqq a + nP_\beta[x \geqq a]$ for any $a \geqq 0$. Under (A2′), $P_\beta[x \geqq a] \leqq P_\beta(X)a^{-1}$ and the choice $a = n^{\frac{1}{2}}$ proves the lemma.

LEMMA 3.5. *Under* (A1) *and* (A2), $n^{-1}\sum_1^n \mathbf{P}_i(S_{i-1}^{-1}[S_{i-1} > 0]) = O(n^{-\frac{1}{2}})$ *uniformly in* $\boldsymbol{\theta}$.

PROOF. From $[S_{i-1} > 0]S_{i-1}^{-1} \leqq ([S_{i-1} > 0]S_{i-1}^{-1})^{\frac{1}{2}}$, Jensen's inequality and Lemma 3.3 it follows that $\mathbf{P}_{i-1}([S_{i-1} > 0]S_{i-1}^{-1}) \leqq 2(\sum_1^i p_j)^{-\frac{1}{2}}$. The proof is completed by application of Lemma 3.1.

LEMMA 3.6. *Under* (A1), $n^{-1}\sum_1^n P_iE(g\tilde{g}^{-1}S_{i-1}^{-1}[\delta_i' = 1][S_{i-1} > 0]) = O(n^{-1} \cdot \log n)$ *uniformly in* $\boldsymbol{\theta}$.

PROOF. The left hand side can be written

$$n^{-1}\sum_1^n \sum_0^\infty p_i(x)p_i(x+1)g(x)g^{-1}(x+1)\mathbf{P}_{i-1}([S_{i-1} > 0]S_{i-1}^{-1}) \leqq 4\beta n^{-1} \sum_0^\infty \sum_1^n p_i^2(x)(\sum_1^i p_j(x))^{-1}$$

where use has been made of Lemma 3.3. The rate $O(n^{-1} \log n)$ follows from Lemma 2.1.

PROPOSITION 3.7. *Under* (A1), (A2) *and* (A2′), $n^{-1}\sum_1^n P_i(C_i) = O(n^{-\frac{1}{2}})$ *uniformly in* $\boldsymbol{\theta}$.

PROOF. The proof follows from (3.28) in view of Lemmas 3.4–3.6.

PROPOSITION 3.8. *Under* (A1), $P_i(C_i) = o(1)$.

PROOF. Consider the bound $g_i$ of the proof of Proposition 3.6. If $\boldsymbol{\theta}$ is such that $\limsup_i \theta_i > 0$ then $\sum_1^i p_j \to \infty$ and $g_i \to 0$ so the dominated convergence theorem implies the result. Write $P_i(C_i) \leqq C_i(0) + \beta P_i[x > 0]$. The sequence $C_i(0) \to 0$ at any $\boldsymbol{\theta}$ since $m(0) > 0$, and therefore, $g_i(0) \to 0$. For $\theta_i \to 0$, $P_i[x > 0] \to 0$ and the result is proved.

PROPOSITION 3.9. (a) *Under* (A1), (A2$^+$) *and* (A3$^-$), $n^{-1}\sum_1^n P_i(F_i) = O(n^{-\frac{1}{2}} \log n)$ *uniformly in* $\boldsymbol{\theta}$ *if* $\sigma_i^2 = O(\log^2 i)$.

(b) *Under* (A1), (A2$^+$) *and* (A3′), $n^{-1}\sum_1^n P_i(F_i) = O(n^{-\frac{1}{4}})$ *uniformly in* $\boldsymbol{\theta}$ *if* $\sigma_i^2 = O(i^{\frac{1}{4}})$.

PROOF. A straight forward computation shows

$$|\varphi_i' - \varphi_i''| \leqq \beta[S_{i-1} + \delta_i' = 0] + g\,|T_i|\,(\tilde{g}(S_{i-1} + \delta_i'))^{-1}[S_{i-1} + \delta_i' > 0] \tag{3.29}$$

where $T_i = \sum_1^i \zeta_j$. By the independence of $T_i$ and $(\mathbf{X}_i, X_i')$, $E\,|T_i| \leqq \sigma_i$ and the proof of Lemma 3.3, we have via (3.29) that

$$F_i \leqq \beta[\sum_1^i p_j \leqq i^\epsilon] + \beta E[S_{i-1} = 0] + 2\beta(g/\tilde{g})\sigma_i(\sum_1^i p_j)^{-1}[\sum_1^i p_j > i^\epsilon] \tag{3.30}$$

for all $\epsilon$. Setting $\epsilon = \frac{1}{2}$ and application of Lemmas 3.2 and 3.4 proves part (a). To prove (b) set $\epsilon = \frac{1}{4}$ and note that the last term of (3.30) is bounded by $2\beta(g/\tilde{g})i^{-1/8}\sigma_i(\sum_1^i p_j)^{-\frac{1}{2}}$. The rate $O(n^{-\frac{1}{4}})$ follows by normalizing the $p_j$ and applying Lemma 3.1.

Inequality (3.3) together with the series of propositions yield the following theorems.

THEOREM 3.1. *Under* (A1$^+$), $D_n(\boldsymbol{\theta}, \boldsymbol{\varphi}^*) = o(1)$ *uniformly in* $\boldsymbol{\theta}$.

PROOF. The proof is carried by Propositions 3.1 and 3.6.

THEOREM 3.2. *Under* ($A1^+$), (A2), ($A2'$) *and* (A3), $D_n(\boldsymbol{\theta}, \boldsymbol{\varphi}^*) = O(n^{-\frac{1}{2}})$ *uniformly in* $\boldsymbol{\theta}$.

PROOF. The proof is carried by Propositions 3.2 and 3.7.

THEOREM 3.3. *Under* (A1), $D_n(\boldsymbol{\theta}, \boldsymbol{\varphi}^*) = o(1)$.

PROOF. The proof is carried by Propositions 3.3 and 3.8.

THEOREM 3.4. *Under* (A1), ($A2^+$) *and* ($A3^-$), $D_n(\boldsymbol{\theta}, \boldsymbol{\varphi}^*) = O(n^{-\frac{1}{3}} \log n)$ *uniformly in* $\boldsymbol{\theta}$.

PROOF. The proof is carried by Propositions 3.4, 3.7 and 3.9 (a) and the fact that ($A2^+$) implies both (A2) and ($A2'$).

THEOREM 3.5. *Under* (A1), ($A2^+$) *and* ($A3'$), $D_n(\boldsymbol{\theta}, \boldsymbol{\varphi}^*) = O(n^{-\frac{1}{2}})$ *uniformly in* $\boldsymbol{\theta}$.

PROOF. The proof is carried by Propositions 3.5, 3.7 and 3.9 (b).

All of the above theorems obtain with $\boldsymbol{\varphi}^*$ replaced by $\boldsymbol{\varphi}^{**}$. This is easily verified by consideration of $n^{-1}\sum_1^n \mathbf{P}_i(|\varphi_i^* - \varphi_i^{**}|)$ and use of a triangle inequality.

3.3. *Examples and remarks*. Samuel's Theorem 4 (1965a) implies the result that under ($A1^+$) with $\alpha$ and $\beta$ known, a natural procedure $\boldsymbol{\varphi}^0$ results in modified regret $D_n(\boldsymbol{\theta}, \boldsymbol{\varphi}^0) \leqq B_n(\boldsymbol{\theta}) = o(1)$. Theorem 3.1 strengthens this result since $\alpha$ is no longer assumed known and $D_n(\boldsymbol{\theta}, \boldsymbol{\varphi}^*) = o(1)$ uniform in $\boldsymbol{\theta}$ is proved. Theorem 3.3 also generalizes the Samuel result since it states $D_n(\boldsymbol{\theta}, \boldsymbol{\varphi}^*) = o(1)$ under the weaker assumption (A1). Theorem 3.4 generalizes a theorem of Swain (1965), Theorem 3, specialized to the unextended sequential compound decision problem. There is the improvement in rate of convergence proved under weaker conditions. (Assumption (iv) of Swain (1965) is not necessary and (iii) implies ($A2^+$) (Gilliland, (1966), p. 25).) Theorems 3.2 and 3.5 are of most interest since they establish the same order $O(n^{-\frac{1}{2}})$ which has been proved for the general finite decision problem (see Hannan (1956) and Van Ryzin (1966b)) and the two-action problem (Johns, (1967)).

Example 3.1 proves that if all that is assumed is ($A1^+$), then $o(1)$ is the best available uniform bound on $D_n(\boldsymbol{\theta}, \boldsymbol{\varphi})$ when $\boldsymbol{\varphi}$ is either $\boldsymbol{\varphi}^*$, $\boldsymbol{\varphi}^{**}$ or $\boldsymbol{\varphi}^0$. The following example shows that the bound in Theorem 3.2 is fairly tight.

EXAMPLE 3.2. Consider the exponential family with $g(0) = g(1) = 1$; $g(x) = x^{-4}, x = 2, 4, 6, \cdots$; $g(x) = x^{-3}\log^{-a-1}x, x = 3, 5, 7, \cdots$; and $\Omega = [\alpha, 1]$, $0 < \alpha < 1$, where $a > 0$ is fixed but otherwise arbitrary. It is not difficult to verify that (A2), ($A2'$) and (A3) are satisfied. Then at $\boldsymbol{\theta} = \mathbf{1}$ with $n \geqq 2$ and any procedure $\boldsymbol{\varphi}$ such that $[\varphi_i \leqq \alpha] \geqq [\sum_1^{i-1} \tilde{\delta}_j = 0]$,

$$\begin{aligned} D_n(\mathbf{1}, \boldsymbol{\varphi}) &= n^{-1} \textstyle\sum_1^n \mathbf{P}_i((1 - \varphi_i)^2) \\ &\geqq n^{-1} \textstyle\sum_1^n \mathbf{P}_i((1 - \varphi_i)^2[\sum_1^{i-1}\tilde{\delta}_j = 0]) \\ &\geqq (1 - \alpha)^2\mathbf{P}_n[\textstyle\sum_1^{n-1} \tilde{\delta}_j = 0] \\ &\geqq (1 - \alpha)^2 \textstyle\sum_1^\infty p_1(2x + 1)(1 - p_1(2x + 2))^{n-1} \\ &\geqq (1 - \alpha)^2(1 - h(1)/(n - 1))^{n-1} \textstyle\sum_{1 \vee A_n}^\infty p_1(2x + 1) \end{aligned}$$

where $A_n = \min\{x \mid (2x + 4)^4 \geqq n - 1\} \leqq \frac{1}{2}(n - 1)^{\frac{1}{4}}$. Comparing the series

$\sum_A^\infty p_1(2x+1)$ with the integral $\int_{2A+1}^\infty x^{-3}\log^{-a-1} x\,dx$ shows via integration by parts that for $A \geqq 1$,

$$\sum_A^\infty p_1(2x+1) \geqq \tfrac{1}{2}h(1)(a+3)^{-1}(2A+1)^{-2}\log^{-a-1}(2A+1).$$

Therefore,

$$D_n(\mathbf{1}, \varphi) \geqq Kn^{-\frac{1}{2}}\log^{-a-1} n$$

for some constant $K > 0$ and all large $n$.

The lower bounds demonstrated in Examples 3.1 and 3.2 also obtain for the natural set compound procedures and illustrate that no significant ratewise improvement results if one observes $\mathbf{X}_n$ and then takes the $i$th action, $1 \leqq i \leqq n$, according to a natural set compound procedure.

Examples of exponential families which satisfy conditions (A1), (A2$^+$), (A3) and (A3$'$) are provided by the Poisson family for $\beta < \infty$ and the negative binomial family $p_\theta(x) = \binom{a+x-1}{x}\theta^x(1-\theta)^a$, $\theta \leqq \beta < 1$, $a > 0$ fixed.

For squared error loss the first inequality of (1.4) yields $\sup_{\boldsymbol\theta} D_n(\boldsymbol\theta, \varphi) \geqq n^{-1}\sup_\theta R(\theta, \varphi_1)$. For the Poisson family and unbounded parameter set $\Omega$, the right hand side is infinite for any procedure $\varphi_1$ (Lehmann, (1950)), p. 4–13). This illustrates the necessity of assumption (A1$^+$) or (A1) in Theorems 3.1, 3.2, 3.4 and 3.5 for the Poisson family.

For the procedures $\varphi^*$ and $\varphi^{**}$ to be realizable $\beta$ must be known. If an upper bound for $\Omega$ is not assumed known, then the natural estimates of $\psi_i$ could be truncated back to $a_i$ where $a_i$ is an arbitrary sequence, $a_i \uparrow \infty$. It can be shown that for the resulting procedures the order of the bound is changed by a factor of $a_n^2$.

We noted in the introduction that the sequential compound results imply Cesaro convergence in the empirical Bayes framework. All of the orders in the theorems obtain for $ER_n(\boldsymbol\theta, \varphi^*) - R(G)$ where the $\theta_i$ are independent and identically distributed $G$ with uniformity in $\boldsymbol\theta$ becoming uniformity in $G$.

For fixed $\mathbf{X}_{i-1}$, $\varphi_i^*$ is not monotone in $X_i$ and is inadmissible. However, it follows from the results of Section 1 and Section 3.2 that $\varphi^*$ has asymptotically lower average risk than any simple symmetric procedure. High rates of convergence have been proved in some cases and the practicality of the compound procedure can be checked by recovering the constant in the bounds. We have not made an attempt to keep track of the constants which are surely quite large in our bounds in view of all the weakened inequalities.

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