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CONTAGION IN STOCHASTIC MODELS FOR EPIDEMICS[1]

By Grace Lo Yang

University of California, Berkeley

1. Introduction. The present paper originated in an effort to build appropriate stochastic models for the spread of epidemics and in an attempt to study their properties. The main results consist in the description of the properties of a test for contagion in an homogeneous population subject to a non-Markovian type of epidemic.

Owing to mathematical intractability, the models actually studied in the literature are always oversimplified in many respects. The present work uses also an oversimplified model. However, we do hope that the method developed in this restricted situation will be found susceptible of applications in more realistic cases.

The model considered here assumes that one observes a finite population consisting of susceptibles, infectives and immunes. A susceptible individual may be changed to infective either through general causes independent of the status of the disease in the population or through infection transmitted by the infectives of the population. It is assumed that during the time of observation the same individual will not recover enough to become infected twice.

The main difference between the model studied here and models used by other investigators is that the infectiousness of a diseased individual depends not only on the length of time he has been sick but also on the time at which he was infected with the disease. Thus to describe the state of the population at time τ it is necessary to give not only the number of infectives but also the times at which they became infected.

A precise description of the actual model studied in detail is given in Section 2. For this particular model we construct an asymptotically optimal test for contagion and investigate its properties, including asymptotic evaluations of its power. Two different sampling procedures are considered. In Section 3 the sampling method consists of observing the population until a predetermined number of cases have been observed. For this sampling procedure one obtains an asymptotic distribution for the logarithm of the likelihood ratio. This is done both under the assumption that no contagion is actually present and under the assumption that the contagion within the population is detectable but not overwhelmingly obvious.

In Section 4 similar results are obtained but the sampling procedure is different. It is assumed there that the length of time of observation is predetermined.

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In Section 5 an asymptotic distribution for the logarithm of the likelihood ratio is obtained under the condition that the rate of incidence of disease coming from causes other than contagion within the population is regarded as an unknown nuisance parameter. It is shown that in this case one can obtain asymptotically similar optimal tests. We also give an evaluation of the limiting power of these tests.

It is rather surprising that even under much more severe restrictions very few results had previously been obtained in this area. The problem was considered by Greenwood [3] in 1946. He suggested a test based on the sum of squares of the time interval between successive occurrences of cases of disease. A modified procedure was proposed by Bartlett [1] in 1949. No information on the efficiency of the tests proposed is presently available. In 1955 Gaffey [2] considered the problem of testing the hypothesis of existence of within family contagion. In his model the contagion rate is assumed to be proportional to the number of infectious individuals present in the population. Furthermore, once infected, an individual remains in that state forever. For this model which is a very particular case of the model studied here, precise information on the power of the test derived there could not be obtained in general.

From a technical point of view it will be seen that the asymptotically optimal test is based on certain double sums and integrals of functions of *pairs* of times of occurrence. Because of the lack of independence and the unusual form of the test statistics no "central limit theorem" seemed directly applicable. In fact we have used none. In spite of this it will be shown that the test statistics are asymptotically normal for the hypotheses and for the alternatives considered. Since the procedure relies almost entirely on the properties of likelihood functions details of structure of the test criteria are rather unessential, a feature which holds good promise for possible extensions.

2. Description of the model. Consider a finite population consisting of $N + 1$ susceptible individuals. It is assumed that at time $t_0 = 0$ a first event takes place changing one of the $N + 1$ individuals from susceptible state S to infectious state I. The process continues from that point on, the basic observable variables being the times $t_0 \leqq T_1 \leqq \cdots \leqq T_N$ at which the N remaining individuals change from state S to state I. No returns from I to S are permitted.

Let Ω_N be the set of all possible outcomes. This is the set of N-tuples $t^{(N)} = [t_1, t_2, \cdots, t_N]$ with $t_1 \leqq t_2 \leqq \cdots \leqq t_N$. Thus Ω_N is a certain cone in the N-dimensional Euclidean space. The σ-field of Borel sets in Ω_N will be called $\mathfrak{F}_N$. Assume that a certain parameter set Θ has been given together with a probability measure P_θ on $\mathfrak{F}_N$ for each $\theta \varepsilon \Theta$.

Two different sampling procedures will be considered. The first one, called n-procedure for short, consists in fixing an integer n and observing the first n variables $T_1, T_2, \cdots, T_n$. The second, called T-procedure, consists in observing those variables T_j such that $T_j \leqq T$, for a predetermined duration T. These two experiments induce two sub-σ-fields of $\mathfrak{F}_N$ which will be called $\mathcal{A}_n$ and $\mathcal{B}_T$

respectively. The σ-field $\mathcal{A}_n$ is generated by $T_1, T_2, \cdots, T_n$. The σ-field $\mathcal{B}_T$ is generated by the variables min (T_j, T) for $j = 1, 2, \cdots, N$.

The probability measures corresponding to the two kinds of experiments will be denoted $P_\theta^{(n)}$ and $P_\theta^{(T)}$ respectively. The measure $P_\theta^{(n)}$ is the restriction of P_θ to $\mathcal{A}_n$ and $P_\theta^{(T)}$ is the restriction of P_θ to $\mathcal{B}_T$. In particular, if a set S belongs to both $\mathcal{A}_n$ and $\mathcal{B}_T$ then $P_\theta^{(n)}(S) = P_\theta^{(T)}(S) = P_\theta(S)$.

For the present paper the structural assumptions made for the construction of the measures P_θ are as follows.

(a) The set Θ is the set of pairs (λ, β) with $\lambda > 0$ and $\beta \geqq 0$.

(b) There is given a function f defined on $R^+ \times R^+$ and taking values in $R^+ = [0, \infty)$. It is assumed that f is bounded by a constant c, jointly measurable and such that $f(t, \tau) = 0$ for $t > \tau$.

Suppose that the times of occurrence of the first j events are $0 = t_0 \leqq t_1 \leqq \cdots \leqq t_i \leqq \cdots \leqq t_{j-1}$. Then the conditional probability, given $T_1 = t_1, \cdots, T_{j-1} = t_{j-1}$ that no new event will take place in the interval $[t_{j-1}, t]$ where $t > t_{j-1}$ is equal to

$$P_\theta\{T_j > t \mid t_0, t_1, \cdots, t_{j-1}\} = \exp\{-(N-j+1) \textstyle\int_{t_{j-1}}^{t} [\lambda + \beta G_j(\tau)]\, d\tau\} \tag{2.1}$$

with

$$G_j(\tau) = \textstyle\sum_{i=0}^{j-1} f(t_i, \tau). \tag{2.2}$$

For epidemic models the foregoing mathematical formulation possesses the following interpretation.

It is assumed that if an individual is infected with the disease at time t he will henceforth contribute at each time $\tau > t$ an amount $f(t, \tau)$ of pathogenic material to the environment.

Thus if j cases of infectives have occurred at times $0 = t_0 \leqq t_1 \leqq \cdots \leqq t_{j-1}$ the total amount of pathogenic material contributed by those cases at time $t > t_{j-1}$ is given by the $G_j(t)$. The amount of pathogenic material $f(t_i, t)$ can be interpreted in various ways. One possibility is that $f(t_i, t)$ is at time t the rate of emission of viruses or bacteria by the infective cases started at t_i.

A main difficulty in the application of this model to actual cases may be the ascertainment of the times t_i at which the organisms became infected. In some diseases, this may be directly possible. In some other diseases the period elapsed from the time of infection to the time of detection may be considered relatively constant so that t_i may be computed by subtracting that constant from the time of detection.

The parameter λ represents a rate of influx of infection from sources which are supposed to be unaffected by the actual state of the population under study. For instance λ may represent a rate of infection by contact with organisms which are not included in the population under study. Another interpretation could be that the pathogenic organism is always present in the population but that it becomes active only occasionally, owing to mutations. The rate λ would then be the mutation rate. Another interpretation is possible. In some cases, such as leukemia, the disease might be initiated by exposure to the normal background of radiation. The parameter λN would then give the average number of cases initiated per unit time in a population size N by this background radiation. Keeping these interpretations in mind, it will be convenient to label λ the "mutation" rate and reserve to the parameter β the name of contagion parameter.

The formula (2.1) which provides the conditional distribution of T_j given $T_0, T_1, \cdots, T_{j-1}$ is the simplest mathematical expression of the following statement. Accepting the interpretation given above, at any time t which follows the occurrence t_{j-1} of the jth case of infection, each of the individuals in the population is subject to a risk of infection $\lambda + \beta G_j(t)$ proportional to the total amount of pathogenic material present at time t, including material of outside origin represented by λ. It is then assumed that the expected number of new infectious cases to occur in a short interval $(t, t + h)$ is given by

$$(N - j + 1)(\lambda + \beta G_j(t))h + o(h). \tag{2.3}$$

The explicit formula (2.1) could be derived from this if one adds the assumption that in $(t, t + h)$ the conditional probability of multiple cases is of order $o(h)$.

Concerning the assumptions made on the function f, one could have assumed that $f(t, \tau)$ is in fact a function of the difference $\tau - t$. This would mean that the effect and evolution of each case of disease is independent of the time of infection. However one may conceive of cases in which there are definite seasonal effects due to seasonal variations in the population densities or climatic differences. Since this is a possibility, it seems preferable to let f depend on the pair (t, τ). Another reason for keeping this general form of f is the following. Suppose that one would want to test for contagion in a disease which is known to be induced by radiation exposure. Suppose that one would also have a time record of the actual amount of radiation exposure. One can then introduce a fictitious time scale in which radiation exposure occurs at the uniform rate λ. The contagion function f which may have been invariant under shift in the original time scale is no longer invariant in the new scale.

A structural assumption which certainly needs modification in practical applications is the assumption made here that the history of each infectious case does not involve any random element except for the time t_i of inception. Natural incubation periods are random, durations of illnesses are random and the rate of emission of pathogenic material is also random. These supplementary random features could be introduced; however they were not considered in this paper.

To terminate this section let us mention some of the formulae which can easily be derived from the assumptions (a) and (b).

It follows immediately from assumption (b) and formula (2.1) that the joint probability distribution of the times $T_1, T_2, \cdots, T_N$ possesses a density with respect to the Lebesgue measure on set Ω_N. This density is given by the formula

$$\begin{aligned} dP_{\lambda,\beta} &= N!\textstyle\prod_{j=1}^{N} [\lambda + \beta G_j(t_j)] \\ &\quad \cdot \exp \{-\textstyle\sum_{j=1}^{N} (N - j + 1) \int_{t_{j-1}}^{t_j} [\lambda + \beta G_j(\tau)]\, d\tau\}\, dt_1 \cdots dt_N, \end{aligned} \tag{2.4}$$

for $0 = t_0 < t_1 < \cdots < t_N$. This can be obtained by repeated application of formula (2.1). It follows also that

$$(2.5) \quad dP_{\lambda,\beta}^{(n)} = [N!/(N-n)!]\prod_{j=1}^{n}[\lambda + \beta G_j(t_j)] \cdot \exp\{-\sum_{j=1}^{n}(N-j+1)\int_{t_{j-1}}^{t_j}[\lambda + \beta G_j(\tau)]\,d\tau\}\,dt_1\cdots dt_n,$$

for $0 = t_0 < t_1 < \cdots < t_n$.

When no contagion takes place within the population, that is, when $\beta = 0$ the preceding formulae take simple forms as follows. The joint density of the N times of occurrence of infection is given by

$$(2.6) \qquad N!\lambda^N \exp\{-\sum_{j=1}^{N}(N-j+1)\lambda(t_j - t_{j-1})\},$$

for $0 < t < \cdots < t_N$.

From this expression one can easily derive the following statements.

(i) The T_j have the same distribution as the ordered observations from a sample of size N taken from the distribution with cumulative distribution function $1 - \exp(-\lambda\tau)$, $\tau \geqq 0$.

(ii) The T_j may be represented as sums

$$(2.7) \qquad T_j = Z_1/N + Z_2/(N-1) + \cdots + Z_j/(N-j+1)$$

where $Z_1, Z_2, \cdots, Z_N$ are independently and identically distributed variables having the cumulative distribution function $1 - \exp(-\lambda z)$, $z \geqq 0$.

(iii) One has

$$(2.8) \qquad ET_j = \sum_{k=1}^{j}[\lambda(N-k+1)]^{-1} < \infty.$$

Therefore, if $\beta = 0$ every T_j is almost surely finite and

$$(2.9) \qquad -\log(1-(j-1)/N) \leqq \lambda ET_j \leqq -\log(1-j/N).$$

(iv) The variance of T_j is given by the formula

$$(2.10) \qquad \operatorname{Var} T_j = \sum_{k=1}^{j}[\lambda^2(N-k+1)^2]^{-1}.$$

(v) Let M be the number of occurrences of infectious cases in the interval $(0, T)$. If no contagion takes place then M is a binomial random variable $B[N, (1-\exp(-\lambda T))]$.

In the model described by assumptions (a) and (b) the hypothesis H_0 that no contagion takes place is equivalent to the hypothesis that $\beta = 0$. The alternative H_1 of existence of contagion is the hypothesis that $\beta > 0$.

For fixed values of λ and β the Neyman-Pearson fundamental lemma shows that the optimal tests of the simple hypothesis $(\lambda, 0)$ against the simple alternative (λ, β) should be derived from the logarithm of the likelihood ratio which can be written

$$\Lambda_n = \log(dP_{\lambda,\beta}^{(n)}/dP_{\lambda,0}^{(n)}),$$

for the n-procedure and

$$\Lambda^{(T)} = \log (dP_{\lambda,\beta}^{(T)}/dP_{\lambda,0}^{(T)}),$$

for the T-procedure. However the actual distribution of Λ_n or $\Lambda^{(T)}$ is extremely complicated. Thus we shall look for limiting distributions for Λ_n or $\Lambda^{(T)}$ as either n or $N(1 - \exp(-\lambda T))$ and N tend to infinity.

Under these passages to the limit Λ_n computed for fixed values of β will not usually have a limiting distribution on the line. To insure that the limiting distribution exists, we shall let β approach zero as n increases and take instead of fixed values for β numbers of the type $\delta_n v$ where $v > 0$ and δ_n is a certain sequence such that $\lim_{n\to\infty} \delta_n = 0$. It will be shown in the next section that an appropriate choice of δ_n is the value

$$\delta_n = 1/nn^{\frac{1}{2}}. \tag{2.11}$$

This choice insures that under assumptions (a) and (b) the logarithm

$$\Lambda_n\{(\lambda, \delta_n v); (\lambda, 0)\} = \log (dP_{\lambda,\delta_n v}^{(n)}/dP_{\lambda,0}^{(n)})$$

has a non-degenerate limiting distribution under H_0 as well as under the sequence of alternatives $\beta_n = \delta_n v$ taken in H_1.

It will also be shown that a similar result is correct for the T-procedure, for a rate of convergence $(\nu)^{-\frac{3}{2}}$ where ν is approximately the expected number of cases of infection occurring in $(0, T)$.

This method will reduce our testing problem to a local one in which normal approximations will be usable.

3. Limiting distributions under n-procedure. In this section, we consider the first sampling system in which we predetermine the number of cases, say n, to be observed and then record the times of occurrence of these n cases, say $T_1, \cdots, T_n$, where $T_1 < \cdots < T_n < \infty$. We assume that λ is known; hence $\Theta = [(\lambda, \beta); \beta \geqq 0] = H \cup H^c$, where $H = (\lambda, 0)$, $H^c = [(\lambda, \beta); \beta > 0]$. We will proceed to show that $\mathcal{L}\{\Lambda_n[(\lambda, \delta_n v); (\lambda, 0)]\}$ converges to a normal distribution as n (and therefore N) tends to infinity.

The existence of a limiting distribution relies on the following conditions:

(A_1) Existence of a sequence Δ_n of $\mathcal{A}_n$ measurable functions and a numerical function A such that

$$\Lambda_n[(\lambda, \delta_n v); (\lambda, 0)] - v\Delta_n + (v^2/2\lambda^2)A$$

converges to zero in $P_{\lambda,0}^{(n)}$ probability.

(A_2) The sequences $\{P_{\lambda,0}^{(n)}\}$ and $\{P_{\lambda,\delta_n v}^{(n)}\}$ are contiguous.

The definition of contiguity [5] used here is the following.

DEFINITION 3.1. Let $\{P_n\}$ and $\{Q_n\}$ be two sequences of probability measures on the σ-fields $\{\mathcal{A}_n\}$. The sequences are called contiguous if the sequences $\{S_n\}$ of $\{\mathcal{A}_n\}$ measurable functions which tend to zero in probability are the same for $\{P_n\}$ and $\{Q_n\}$.

Equivalently, if $\Lambda_n = \log (dQ_n/dP_n)$, the sequences $\{P_n\}$ and $\{Q_n\}$ are contiguous if and only if both $\mathcal{L}\{\Lambda_n \mid P_n\}$ and $\mathcal{L}\{\Lambda_n \mid Q_n\}$ are relatively compact sequences.

The contiguity condition (A_2) will play an important role. The proof that it is satisfied here is one of the main parts of the present section.

Since the logarithm of the likelihood ratio Λ_n will occur repeatedly in the sequel we shall first record here its explicit expression.

$$\text{(3.1)}\quad \begin{aligned}\Lambda[(\lambda, \delta_n v); (\lambda, 0)] &= \log (dP^{(n)}_{\lambda,\delta_n v}/dP^{(n)}_{\lambda,0}) \\ &= \sum_{j=1}^n \log [1 + (\delta_n v/\lambda) G_j(t_j)] - \sum_{j=1}^n (N - j + 1) \int_{t_{j-1}}^{t_j} \delta_n v G_j(\tau)\, d\tau.\end{aligned}$$

Applying a Taylor expansion to the logarithmic term gives

$$\text{(3.2)}\quad \begin{aligned}&\Lambda_n[(\lambda, \delta_n v); (\lambda, 0)] \\ &= (\delta_n v/\lambda) \sum_{j=1}^n [G_j(T_j) - \lambda(N - j + 1) \int_{T_{j-1}}^{T_j} G_j(\tau)\, d\tau] \\ &\quad - (\delta_n^2 v^2/2\lambda^2) \sum_{j=1}^n G_j^2(T_j)/[1 + (\delta_n \eta_j/\lambda) G_j(T_j)]^2,\end{aligned}$$

with $0 \leqq \eta_j \leqq v$ for $j = 1, 2, \cdots, n$.

This will also be written

$$\text{(3.3)}\quad \Lambda_n[(\lambda, \delta_n v); (\lambda, 0)] = v\Delta_n(\lambda) - v^2 R_n(\lambda),$$

with

$$\begin{aligned}&\Delta_n(\lambda) \\ &= (1/\lambda n^{\frac{3}{2}}) \sum_{j=1}^n \sum_{i=0}^{j-1} [f(T_i, T_j) - (N - j + 1)\lambda \int_{T_{j-1}}^{T_j} f(T_i, \tau)\, dt],\end{aligned}$$

$$\text{(3.4)}\quad \begin{aligned}&R_n(\lambda) \\ &= (2\lambda^2 n^3)^{-1} \sum_{j=1}^n [\sum_{i=0}^{j-1} f(T_i, T_j)]^2 \\ &\quad \cdot \{1 + (\eta_j/\lambda n^{\frac{3}{2}}) \sum_{i=0}^{j-1} f(T_i, T_j)\}^{-2}.\end{aligned}$$

We proceed to show that conditions (A_1) and (A_2) are satisfied. First we will establish the contiguity condition (A_2). The following lemma is used.

LEMMA 3.1. *Assume that the conditions* (a) *and* (b) *are satisfied. Let* $\Delta_n(\lambda)$ *be given by equation* (3.4) *and let* E_v *denote expectations taken for the probability measures* $P^{(n)}_{\lambda,\delta_n v}$ *for* $\delta_n = n^{-\frac{3}{2}}$.
Then

$$E_0 \Delta_n(\lambda) = 0.$$

Furthermore there is a constant c_1 *such that*

$$0 \leqq E_v \Delta_n(\lambda) = c_1 v c^2/\lambda^2.$$

PROOF. Consider first the conditional expectation of the term

$$(N - j + 1)\lambda \int_{t_{j-1}}^{T_j} f(t_i, \tau)\, d\tau$$

given the values $t_0, t_1, \cdots, t_{j-1}$. This may be written as

$$(3.5) \quad E_v\{(N-j+1)\lambda \int_{t_{j-1}}^{T_j} f(t_i, \tau)\, d\tau \mid t_0, t_1, \cdots, t_{j-1}\} = (N-j+1)\lambda \int_{t_{j-1}}^{\infty} p_j(t)\{\int_{t_{j-1}}^{t} f(t_i, \tau)\, d\tau\}\, dt$$

where $p_j(t)$ is the density of T_j given $t_0, t_1, \cdots, t_{j-1}$, for the measure $P_{\lambda,\delta_n v}^{(n)}$.

By application of Fubini's theorem, this can also be written in the form

$$(3.6) \quad (N-j+1)\lambda \int_{t_{j-1}}^{\infty} f(t_i, \tau) Q_j(\tau)\, d\tau$$

with

$$(3.7) \quad Q_j(\tau) = \int_{\tau}^{\infty} p_j(t)\, dt.$$

In addition $p_j(\tau) = (N-j+1)[\lambda + \delta_n v G_j(\tau)]Q_j(\tau)$. It follows that

$$(3.8) \quad E_v\{f(t_i, T_j) - (N-j+1)\lambda \int_{t_{j-1}}^{T_j} f(t_i, \tau)\, d\tau \mid t_0, t_1, \cdots, t_{j-1}\} = (N-j+1)\delta_n v \int_{t_{j-1}}^{\infty} f(t_i, \tau) G_j(\tau) Q_j(\tau)\, d\tau.$$

When $v = 0$ this conditional expectation is equal to zero. It follows that $E_0 \Delta_n(\tau) = 0$.

For arbitrary values of $v > 0$ the conditional expectation expressed by the integral in (3.8) is obviously non-negative. Furthermore, since $0 \leqq f \leqq c$ one has $f(t_i, \tau)G_j(\tau) \leqq jc^2$.

Hence

$$(3.9) \quad \delta_n v \int_{t_{j-1}}^{\infty} f(t_i, \tau) G_j(\tau) Q_j(\tau)\, d\tau \leqq jc^2 v / [(N-j+1)\lambda n^{\frac{3}{2}}].$$

It follows by addition that

$$(3.10) \quad 0 \leqq E_v \Delta_n(\lambda) \leqq [(c^2 v)/(\lambda^2 n^3)] \sum_{j=1}^{n} \sum_{i=0}^{j-1} j.$$

Hence the result, since $3 \sum_{j=1}^{n} j^2 \leqq (n+1)^3$.

THEOREM 3.1. *For the sequence $\delta_n = n^{-\frac{3}{2}}$ the sequences of measures $\{P_{\lambda,0}^{(n)}\}$ and $\{P_{\lambda,\delta_n v}^{(n)}\}$ are contiguous.*

PROOF. Denote the logarithm of the likelihood ratio $\Lambda_n[(\lambda, \delta_n v); (\lambda, 0)]$ by Λ_n for short.

Let E_v be defined as in Lemma 3.1. It will be sufficient to prove that for given numbers a, b such that $0 < a < b < \infty$ there is a constant $c_2(a, b)$ such that

$$(3.11) \quad E_0 |\Lambda_n| \leqq c_2(a, b) \quad \text{and} \quad E_v |\Lambda_n| \leqq c_2(a, b)$$

whenever $a \leqq \lambda$ and $0 \leqq v \leqq b$, the desired contiguity property follows then from Markov's inequality. Note then that $E_0 \Lambda_n^+ \leqq E_0 \exp(\Lambda_n^+) \leqq 2$, and $E_v \Lambda_n^- \leqq E_v \exp(\Lambda_n^-) = E_v \exp\{[-\Lambda_n]^+\} \leqq 2$ with $\Lambda_n^+ = \max(0, \Lambda_n)$ and $\Lambda_n^- = \min(0, \Lambda_n)$. Thus, if $E_0 \Lambda_n > -c'$ we have

$$(3.12) \quad E_0 |\Lambda_n| = E_0 \Lambda_n^+ + E_0 \Lambda_n^- = -E_0 \Lambda_n + 2E_0 \Lambda_n^+ \leqq c' + 4.$$

A similar inequality applies to $E_v |\Lambda_n|$. It is therefore sufficient to show that there exists a constant c' such that $E_0 \Lambda_n > -c'$ and $E_v \Lambda_n < c'$. For the expectation taken under the hypothesis we can write

$$(3.13) \qquad E_0\Lambda_n = vE_0\Delta_n(\lambda) - v^2E_0R_n(\lambda).$$

Since by Lemma 3.1 we have $E_0\Delta_n(\lambda) = 0$ and since

$$R_n(\lambda) \leqq c^2/2\lambda^2,$$

it follows that $E_0\Lambda_n \geqq -v^2c^2/2\lambda^2$.

Similarly, the second statement of Lemma 3.1 implies

$$(3.14) \qquad E_v\Lambda_n \leqq c_1v^2c^2/\lambda^2 - v^2E_vR_n(\lambda) \leqq c_1v^2c^2/\lambda^2.$$

This implies the desired result.

Note that the bounds obtained here depend only on the ratio (cv/λ) of the contagion effect to the mutation rate λ.

The next lemma is essential for the establishment of the fact that there exists a numerical function A such that condition (A_1) is satisfied. More explicitly, the lemma is used to prove the convergence in $P_{\lambda,0}^{(n)}$ measure of $2\lambda^2R_n$ (see (3.4)) to A.

Let us recall that for the measure $P_{\lambda,0}^{(n)}$, the times $T_1, T_2, \cdots, T_n$ have the same joint distribution as the first n order statistics from a sample of size N extracted from a negative exponential distribution with density $\lambda \exp(-\lambda\tau)$, $\tau > 0$. Since the σ-fields $\mathcal{A}_n$ and the measures $P_{\lambda,0}^{(n)}$ change if n and N vary, almost sure convergence statements for n (and hence N) tending to infinity, can be made only by introduction of other probability spaces. It is clear that one could introduce a sequence $\{T_i^*\}$ of independently and identically distributed variables, having density $\lambda \exp(-\lambda\tau)$ and let $T_{N,j}$ be the jth order statistic for $\{T_i^*; i = 1, 2, \cdots, N\}$.

For such a space the following lemma is essentially a statement to the effect that the empirical cumulative distribution of the $T_1, \cdots, T_n$ converges almost surely. However, it is stated here in such a way that the result need not be attached to any specific probability space.

Let

$$F_n(\tau) = (1/n)\{\text{number of } T_j\text{'s such that } T_j \leqq \tau, j = 1, 2, \cdots, n\}.$$

The inequalities $\lambda T_j \leqq \lambda\tau$ and $[1 - \exp(-\lambda T_j)][1 - \exp(-\lambda T_{n+1}]^{-1} \leqq [1 - \exp(-\lambda\tau)][1 - \exp(-\lambda T_{n+1})]^{-1}$ are equivalent.

It can be shown easily that conditionally given the value of T_{n+1} the variables

$$U_j = (1 - \exp(-\lambda T_j))(1 - \exp(-\lambda T_{n+1}))^{-1}, \quad j = 1, \cdots, n,$$

have the same joint distribution as the order statistics of a sample of size n from a uniform distribution (0, 1). Therefore given T_{n+1} the process $F_n(\tau)$ behaves exactly in the same way as the empirical cumulative $F_n^*(\tau)$ defined by

$$F_n^*(\tau) = (1/n)\{\text{number of } U_j\text{'s such that } U_j \leqq (1 - e^{-\lambda\tau})(1 - e^{-\lambda T_{n+1}})^{-1}\}$$

in a sample of size n from a uniform distribution (0, 1).

Let Φ_n be the function defined by

$$\Phi_n(\tau) = [1 - \exp(-\lambda\tau)][1 - \exp(-\lambda T_{n+1})]^{-1} \quad \text{for} \quad 0 < \tau < T_{n+1}.$$

Let $\Phi(\tau) = 1$ for $\tau \geqq T_{n+1}$.

According to a theorem of Kiefer and Wolfowitz [4] there are absolute constants K_1 and $K_2 > 0$ such that

$$P\{\sup_\tau |F_n^*(\tau) - \Phi(\tau)| > \epsilon \mid T_{n+1}\} \leqq K_1 \exp(-K_2 n\epsilon^2).$$

LEMMA 3.2. *Let $\Phi_{N,n}^*(\tau)$ be the cumulative distribution function defined by*

$$\Phi_{N,n}^*(\tau) = [(N+1)/(n+1)](1 - \exp(-\lambda\tau))$$

on the range of values of τ where this is less than unity.

There exist absolute constants K_1 and K_2 such that

$$P\{\sup_\tau |F_n(\tau) - \Phi_{N,n}^*(\tau)| > 2\epsilon\} \leqq K_1\{1/n\epsilon^2 + \exp(-K_2 n\epsilon^2)\}$$

for every $n \geqq 1$ and every $\epsilon > 0$.

PROOF. The variable $(1 - \exp(-\lambda T_{n+1}))$ has the same distribution as the $(n+1)$st order statistic U_{n+1} in a sample of size N from a uniform distribution on (0, 1). If follows that $EU_{n+1} = (n+1)/(N+1)$ and variance of $U_{n+1} = (n+1)(N-n)/(N+1)^2(N+2)$.

The maximum difference between $\Phi_{N,n}^*$ and Φ_n occurs at the minimum of U_{n+1} and $(n+1)/(N+1)$. It is equal to

$$|U_{n+1} - (n+1)/(N+1)| \{\max [U_{n+1}, (n+1)/(N+1)]\}^{-1}$$
$$\leqq [(N+1)/(n+1)]|U_{n+1} - (n+1)/(N+1)|.$$

This last expression has a second moment equal to

$$(N-n)/[(n+1)(N+2)] \leqq 1/(n+1).$$

Therefore, by Markov's inequality

$$P\{\sup_\tau |\Phi_{N,n}^*(\tau) - \Phi_n(\tau)| > \epsilon\} \leqq \epsilon^{-2}/(n+1).$$

Since, according to the previous argument,

$$P\{\sup_\tau |F_n(\tau) - \Phi_n(\tau)| > \epsilon\} \leqq K_1 \exp(-K_2 n\epsilon^2),$$

the result is completely proved.

Before going further it is convenient to describe in more detail the possible limiting behavior of the distributions $\Phi_{N,n}^*$. Note first that $\Phi_{N,n}^*$ depends not only on the ratio $(N+1)/(n+1)$ but also on the parameter λ itself. Further, if n (and therefore N) tends to infinity, we can replace $(N+1)/(n+1)$ by (N/n) without any essential changes. Thus, it is convenient to introduce the distribution function

$$\Phi_{N,n,\lambda}(x) = \min\{1, nN^{-1}(1 - \exp(-\lambda x))\}$$

for $x \geqq 0$. The ratio $(n/\lambda N)$ may be interpreted as the ratio of the observed number of cases to the expected number of occurences per unit of time. Let N, n and λ vary in an arbitrary way subject to the sole restriction that $n \to \infty$ and $n \leqq N$.

One can interpret the situation where $(n/\lambda N)$ tends to zero as a situation where the total number of observations made is negligible but the disease is very common. Similarly convergence of $n/\lambda N$ to infinity means that the disease is relatively rare but that the observations are taken for a very long period. Under these circumstances, $\Phi_{N,n,\lambda}$ tends to the distribution which is degenerate at zero or infinity respectively.

In practical cases, one is more likely to observe a small, but nonnegligible fraction of the possible cases. This could correspond to the case where $(n/\lambda N)$ stays bounded away from zero and infinity.

If $(n/\lambda N)$ tends to a finite non-zero limit a but $\lambda \to 0$ then $\Phi_{N,n,\lambda}$ tends to the distribution

$$\Phi_a(x) = \min(1, x/a).$$

This is the cumulative of the uniform distribution on the interval $(0, a)$. If $(n/\lambda N)$ tends to a but λ tends to $\lambda_0 > 0$ then $\Phi_{N,n,\lambda}$ converges to the distribution function $\min(1, b(1 - e^{-\lambda_0 x}))$ for $\lambda_0 b = a$.

Since if $n < N$, the conditional distribution of $T_1, \cdots, T_n$ given T_{n+1} is the same as the distribution of order statistics from a sample of size n taken from the cumulative distribution

$$\Phi_n(t) = \min\{1, (1 - \exp(-\lambda t))(1 - \exp(-\lambda T_{n+1}))^{-1}\}.$$

It follows that for any arbitrary non-negative measurable function φ defined on $R^+ \times R^+$ one can write

$$\begin{aligned} E\{(1/n^2)\textstyle\sum_{j=1}^n \sum_{i=1}^n \varphi(T_i, T_j) \mid T_{n+1}\} \\ = (1/n)\textstyle\int \varphi(t, t)\, d\Phi_n(t) \\ + [n(n-1)/n^2]\textstyle\int\int \varphi(s, t)\, d\Phi_n(s)\, d\Phi_n(t). \end{aligned} \tag{3.15}$$

Let $U_{n+1} = 1 - e^{-\lambda T_{n+1}}$. This variable has expectation $(n + 1)/(N + 1)$ and variance $[(n + 1)/(N + 1)]^2(N - n)/[(N + 2)(n + 1)]$, therefore

$$P\{\tfrac{1}{2}[(n + 1)/(N + 1)] \leqq 1 - e^{-\lambda T_{n+1}} \leqq 2(n + 1)/(N + 1)\} \geqq 1 - 4/(n + 1)$$

except for cases having probability at most $4/(n + 1)$ the conditional distribution Φ_n has a density bounded by $2\lambda[(N + 1)/(n + 1)] \exp(-\lambda t)$ for $t < -(1/\lambda) \log[1 - 2(n + 1)/(N + 1)]$. The density is zero for larger values of t.

Suppose then that $n/\lambda N$ converges to a finite non-zero limit a and that either $\lambda \to 0$ or $\lambda \to \lambda_0 > 0$ as $n \to \infty$. Assuming this there is some number $K < \infty$ and some number λ_1 such that

$$d\Phi_n(t)/dt \leqq K \exp(-\lambda_1 t)$$

for every t except possibly for a set of values T_{n+1} which has probability at most $4/(n+1)$.

Thus we have

$$(3.16)\quad P\{E[(1/n^2)\textstyle\sum_{j=1}^n \sum_{i=1}^n \varphi(T_i, T_j) \mid T_{n+1}] \leqq C(\varphi)\} \geqq 1 - 4/(n+1)$$

with

$$(3.17)\quad C(\varphi) = (K/n)\int \varphi(t, t) \exp(-\lambda_1 t)\, dt + K^2(1 - 1/n)\int\int \varphi(s, t) \exp[-\lambda_1(s+t)]\, ds\, dt.$$

Condition C. Let f be a measurable function defined on the set $R^+ \times R^+$. Assume that $0 \leqq f \leqq c$ and that $f(x, y) = 0$ whenever $x > y$. Furthermore if $n/\lambda N$ is allowed to tend to infinity it is assumed that $f(x, y)$ tends to a limit $f(\infty, \infty) > 0$ for $x \to \infty$ and $x \leqq y$. If $n/\lambda N$ is allowed to tend to zero it is assumed that as $(x, y) \to 0$ in the set $0 \leqq x \leqq y$ the values $f(x, y)$ tend to a limit $f(0, 0) > 0$.

Theorem 3.2. *Let f satisfy Condition C. Let*

$$A(n/N, \lambda) = \int |\int f(x, y)\, d\Phi_{N,n,\lambda}(x)|^2\, d\Phi_{N,n,\lambda}(y).$$

If $n \to \infty$ but $0 < \gamma \leqq n/\lambda N < 1/\gamma$ then

$$P_{\lambda,0}^{(n)}\{|(1/n^3) \textstyle\sum_{j=1}^n |\sum_{i=0}^{j-1} f(T_i, T_j)|^2 - A(n/N, \lambda)| > \epsilon\}$$

tends to zero for every $\epsilon > 0$.

Proof. For the two special cases assuming continuity restrictions on f the result is almost immediate. For the case where $n/\lambda N$ stays bounded away from infinity and zero one can assume that $n/\lambda N$ converges to a limit and that either $\lambda \to \lambda_0 > 0$ or $\lambda \to 0$.

In both cases the terms of the sum having indices $i = 0$ contribute a total at most equal to c^2/n. Thus they may be neglected. Let $B_n(f)$ be defined by

$$B_n(f) = (1/n^3)\textstyle\sum_{j=1}^n |\sum_{i=1}^n f(T_i, T_j)|^2.$$

The proof of the theorem consists of two parts. They are:

(i) Approximation of $B_n(f)$ by $B_n(f^*)$ where f^* is obtained in the following way.

If f is an arbitrary bounded measurable function on $R^+ \times R^+$ there exists a function f^* such that

(1) f^* is continuous,

(2) f^* has compact support and $|f^*| \leqq c$,

(3) if $\varphi = |f - f^*|$ then $C(\varphi) \leqq \epsilon^2/2c$,

for the value $C(\varphi)$ given by equation (3.17).

Select such a function f^*. Clearly

$$B_n(f) - B_n(f^*) = (1/n)\textstyle\sum_{j=1}^n [(1/n)\sum_{i=1}^n \psi(T_i, T_j)][(1/n)\sum_{i=1}^n \varphi^*(T_i, T_j)],$$

for $\psi = f + f^*$ and $\varphi^* = f - f^*$.

It follows that if $|f| \leqq c$ and $|f^*| \leqq c$, then

$$|B_n(f) - B_n(f^*)| \leqq 2cn^{-2} \sum_{j=1}^{n} \sum_{i=1}^{n} \varphi(T_i, T_j), \tag{3.18}$$

for $\varphi = |\varphi^*|$.

Then according to (3.16), (3.18) and Markov's inequality

$$P\{|B_n(f) - B_n(f^*)| > \epsilon\} \leqq 4/(n+1) + 2cC(\varphi)/\epsilon \leqq 4/(n+1) + \epsilon.$$

(ii) Proof of the theorem for f^*.

Since the values $A(n/N, \lambda)$ behave similarly it will be sufficient to prove the theorem for f^* instead of f. However since f^* is a continuous function with compact support, it is uniformly continuous. Therefore the averages $n^{-1} \sum_{i=1}^{n} f^*(T_i, y)$ are equicontinuous in y.

Lemma 3.2 and an application of the Helly-Bray theorem shows that for each value of y the difference

$$n^{-1} \sum_{j=1}^{n} f^*(T_i, y) - \int f^*(x, y) \, d\Phi_{N,n,\lambda}(x) = \int f^*(x, y) \, dF_n(x) - \int f^*(x, y) \, d\Phi_{N,n,\lambda}(x)$$

tends to zero in probability. Since f^* is zero for large values of x and y and since the sums are equicontinuous we conclude that

$$\sup_y |n^{-1} \sum_{i=1}^{n} f^*(T_i, y) - \int f^*(x, y) \, d\Phi_{N,n,\lambda}(x)| \to 0$$

in probability. This implies immediately the desired result.

LEMMA 3.3. *Assume that* $n/\lambda N \to a$ *and that* $\lambda \to \lambda_0 \geqq 0$. *Assume also that* f *satisfies the Condition* C. *Then the triple integrals* $A(n/N, \lambda)$ *converge to a limit* $D(a, \lambda_0)$ *which is continuous on the set* $\{0 < a < \infty, 0 \leqq \lambda_0 < \infty\}$.

PROOF. The cumulative distribution $\Phi_{N,n,\lambda}$ possesses a density $p(x, n/\lambda N, \lambda)$ equal to

$$(\lambda N/n)e^{-\lambda x} \quad \text{for} \quad 0 \leqq x \leqq -\lambda^{-1} \log (1 - n/N).$$

If $n/\lambda N \to a$ and $\lambda \to \lambda_0 > 0$ these densities converge pointwise to the density $p(x, a, \lambda_0)$ equal to

$$a^{-1}e^{-\lambda_0 x} \quad \text{for} \quad 0 \leqq x \leqq -\lambda_0^{-1} \log (1 - \lambda_0 a).$$

This remains true if $\lambda_0 = 0$ provided $-\lambda_0^{-1} \log (1 - \lambda_0 a)$ is replaced by its limit namely a.

According to Scheffé's theorem this implies that

$$\delta_n^* = \int |p(x, n/\lambda N, \lambda) - p(x, a, \lambda_0)| \, dx \to 0.$$

Let $D(a, \lambda_0) = \int |\int f(x, y)p(x, a, \lambda_0) \, dx|^2 p(y, a, \lambda_0) \, dy$. This gives

$$|\int f(x, y)p(x, a, \lambda_0) \, dx - \int f(x, y)p(x, n/\lambda N, \lambda) \, dx| \leqq c\delta_n^*.$$

Hence

$$|A(n/N, \lambda) - D(a, \lambda_0)| \leqq 3c^2\delta_n^* \to 0$$

the continuity of the limit $D(a, \lambda_0)$ follows by the same argument.

REMARK. Note that $D(a, \lambda_0) > 0$ unless the function f is almost everywhere equal to zero on the set

$$\{(x, y), 0 \leqq x \leqq y \leqq -\lambda_0^{-1} \log [1 - \lambda_0 a]\}.$$

THEOREM 3.3. *Suppose that assumptions* (a), (b) *and Condition C are satisfied. Assume that* $n \to \infty$ *in such a way that* $n/\lambda N \to a$ *and* $\lambda \to \lambda_0 > 0$. *Let*

$$\Lambda_n\{(\lambda, \delta_n v); (\lambda, 0)\} = v\Delta_n - v^2 R_n \qquad \text{as in} \qquad (3.4)$$

then

(i) R_n tends in probability to $\frac{1}{2}\lambda_0^{-2}D(a, \lambda_0)$,
(ii) *the distributions* $\mathfrak{L}[\Delta_n \mid P_{\lambda,0}^{(n)}]$ *converge to the normal distribution* $\mathfrak{N}[0, D(a, \lambda_0)/\lambda_0^2]$ *and*
(iii) *the distributions* $\mathfrak{L}[\Delta_n \mid P_{\lambda,\delta_n v}^{(n)}]$ *converge to the normal distribution* $\mathfrak{N}[v D(a, \lambda_0)/\lambda_0^2, D(a, \lambda_0)/\lambda_0^2]$.

PROOF. According to Theorem 3.1 the sequences $\{P_{\lambda,0}^{(n)}\}$ and $\{P_{\lambda,\delta_n v}^{(n)}\}$ are contiguous. It follows then from a lemma of L. LeCam [5] that if F is a cluster point of the sequence of distributions $\mathfrak{L}[\Lambda_n \mid P_{\lambda,0}^{(n)}]$ one must have $\int e^{\Lambda}F(d\Lambda) = 1$. Let $\{m\} \subset \{n\}$ be a subsequence such that $\mathfrak{L}[\Lambda_m \mid P_{\lambda,0}^{(m)}] \to F$ for $\Lambda_m = \Lambda_m\{(\lambda, \delta_m v); (\lambda, 0)\}$ and for a fixed value of $v > 0$. Since by Lemma 3.3 $2\lambda_0^2 R_n \to D(a, \lambda_0)$ in probability, the random variables $v\Delta_m$ have a limiting distribution. Therefore $\mathfrak{L}[\Delta_m \mid P_{\lambda,0}^{(m)}]$ converges to a limit G. Hence $\mathfrak{L}\{\Delta_m \mid P_{\lambda,0}^{(m)}\}$ tends to a limit for every $v > 0$. The equation $\int e^{\Lambda}F(d\Lambda) = 1$ can be written as

$$\int \exp [v\Delta - \tfrac{1}{2}v^2\lambda_0^{-2}D(a, \lambda_0)]G(d\Delta) = 1,$$

that is:

$$Ee^{v\Delta} = \exp \{\tfrac{1}{2}v^2\lambda_0^{-2}D(a, \lambda_0)\}.$$

This must be true for every $v \geqq 0$ and therefore the distribution G is $\mathfrak{N}[0, D(a, \lambda_0)/\lambda_0^2]$, since this does not depend on the choice of the subsequence $\{m\} \subset \{n\}$ the sequence $\{\Delta_n\}$ itself must have the same limiting distribution.

To obtain the limiting distribution under the alternatives $P_{\lambda,\delta_n v}^{(n)}$ it is sufficient to note that $dP_{\lambda,\delta_n v}^{(n)} = e^{\Lambda_n}dP_{\lambda,0}^{(n)}$ and that this relation remains true for the limiting distributions because of the contiguity of the sequences $\{P_{\lambda,\delta_n v}^{(n)}\}$, $\{P_{\lambda,0}^{(n)}\}$. Therefore, the limiting distribution of Δ_n under the alternative must be a distribution G_v such that

$$dG_v = e^{\Lambda}dG = \exp [v\Delta - (v^2/2\lambda_0^2)D(a, \lambda_0)]\, dG.$$

The result follows by simple algebra.

For the case where $n/\lambda N \to a \,\varepsilon\, (0, \infty)$ but $\lambda \to 0$ Theorem 3.1 can still be used to prove contiguity but the sequence of alternatives must be different. This is stated in the following theorem whose proof parallels exactly that of Theorem 3.3.

THEOREM 3.4. *Let assumptions* (a) *and* (b) *be satisfied. Assume that* $n \to \infty$ *and that* $n/\lambda N \to a \,\varepsilon\, (0, \infty)$ *but* $\lambda \to 0$. *Consider the sequences* $P_{\lambda,0}^{(n)}$ *and* $P_{\lambda,\delta_n v}^{(n)}$

with $\delta_n = 1/n\, n^{\frac{1}{2}}$ *as before. Write the logarithm of likelihood ratio in the form* $\Lambda_n[(\lambda, \delta_n\lambda v); (\lambda, 0)] = v\Delta_n^* - v^2R_n^*$. *Then, for every fixed value v the sequences*

$$\mathcal{L}[\Delta_n^* \mid P_{\lambda,0}^{(n)}] \to \mathfrak{N}[0, D(a)], \qquad \mathcal{L}[\Delta_n^* \mid P_{\lambda,\delta_n\lambda v}^{(n)}] \to \mathfrak{N}[vD(a), D(a)],$$

with

$$D(a) = a^{-3}\int_0^a |\int_0^a f(x, y)\, dx|^2\, dy.$$

The proof which is the same as that of Theorem 3.3 will be omitted.

To terminate this section, let us apply Theorem 3.3 to the derivation of an asymptotically optimal test of the hypothesis $v = 0$ against alternatives $v > 0$ for the case where the parameters (n/N) and λ are known.

Under the prescribed conditions the hypothesis $v = 0$ is a simple hypothesis H_0. For a fixed value of v we have a simple alternative H_v. According to the Neyman-Pearson fundamental lemma an optimal test $w_n(v)$ of size α is obtained by rejecting H_0 if $\Lambda_n = \Lambda_n[(\lambda, \delta_n v); (\lambda, 0)] = v\Delta_n - v^2R_n \geqq k_n(\alpha)$ where $k_n(\alpha)$ is such that $P_{\lambda,0}^{(n)}[\Lambda_n \geqq k_n(\alpha)] = \alpha$. Let b denote the number defined by the equality $\alpha = (2\pi)^{-\frac{1}{2}} \int_b^\infty e^{-u^2/2}\, du$. Let w_n^* be the test which rejects H_0 if $\Delta_n \geqq \lambda^{-1}b(D(a, \lambda))^{\frac{1}{2}}$. Since Δ_n is asymptotically normal as stated in Theorem 3.3 the probability that the tests $w_n(v)$ and w_n^* lead to different decisions tends to zero under the hypothesis H_0 as well as under the alternatives. It follows that the test w_n^* is asymptotically optimal of size α and that its limiting power is given by

$$\lim_{n\to\infty} P_{\lambda,\delta_n v}^{(n)}[\Delta_n \geqq \lambda^{-1}b(D(a, \lambda))^{\frac{1}{2}}] = (2\pi)^{-\frac{1}{2}} \int_{b-v(D/\lambda^2)^{\frac{1}{2}}}^\infty e^{-u^2/2}\, du.$$

4. Relations between two sampling procedures. In this section we are going to investigate the second sampling procedure, called "T-procedure" for short. Using the first procedure the total time needed to complete the experiment is random. This may sometimes be inconvenient in practice. It would be desirable to use the "T-procedure" in which the amount of time which we are willing to spend is under our control. However, in this case, the total number of infectious cases M occurring in the interval $(0, T)$ will become a random variable which takes values $0, \cdots, N$. The joint density of $T_1, \cdots, T_M$ is easily seen to be

$$\begin{aligned} dP_{\lambda,\beta}^{(T)} &= [N!/(N - M)!]\textstyle\sum_{j=1}^M [\lambda + \beta G_j(t_j)] \\ &\cdot \exp\{-\textstyle\sum_{j=1}^M (N - j + 1) \int_{t_{j-1}}^{t_j} [\lambda + \beta G_j(\tau)]\, d\tau\} \\ &\cdot \exp\{-(N - M)\int_{t_M}^T [\lambda + \beta G_M(\tau)]\, d\tau\}\, dt_1 \cdots dt_M \end{aligned} \tag{4.1}$$

for $0 = t_0 < t_1 \cdots < t_M < T$ and $M = 0, \cdots, N$.

Here again, because of the intractability of the exact formulas, we shall have an asymptotic test criterion for local alternatives. It would appear at first sight that an appropriate passage to the limit consists in letting the period of observation T increase indefinitely. However, it turns out that T itself is not the controlling parameter.

Under the hypothesis H_0 that no contagion takes place, the number of infective cases occurring during the interval $(0, T]$ is a binomial random variable M such that $EM = N[1 - e^{-\lambda T}]$, Variance $M = N[1 - e^{-\lambda T}]e^{-\lambda T}$. We shall obtain the limiting behavior of the system when $N[1 - e^{-\lambda T}]$ tends to infinity.

For convenience we introduce an integer ν related to N and T as follows. Let $\nu = \nu[N, \lambda T]$ be the smallest integer larger than or equal to

$$\min\{N, N[1 - e^{-\lambda T}] + [N(1 - e^{-\lambda T})]^{3/4}[e^{-\lambda T}]^{\frac{1}{2}}\}. \tag{4.2}$$

Consider alternatives corresponding to parameter values (λ, β) of the form $(\lambda, \delta_\nu v)$ where ν is the function of N and λT just defined and δ_ν is the value $\delta_\nu = \nu^{-\frac{3}{2}}$ as in Section 3.

Formula (4.1) gives

$$\begin{aligned}\Lambda_\nu^{(T)} &= \log [dP_{\lambda,\delta_\nu v}^{(T)}/dP_{\lambda,0}^{(T)}] \\ &= \textstyle\sum_{j=1}^{M} \log [1 + v(\lambda\nu\nu^{\frac{1}{2}})^{-1}G_j(T_j)] \\ &\quad - \textstyle\sum_{j=1}^{M+1} (N - j + 1)\int_{T_{j-1}}^{T_j} v(\nu\nu^{\frac{1}{2}})^{-1}G_j(\tau)\, d\tau\end{aligned}$$

with $T_{M+1} = T$ and $\nu = \nu[N, \lambda T]$ defined by formula (4.2).

Expanding the logarithm by Taylor's formula one obtains

$$\Lambda_\nu^{(T)} = v\Delta_\nu^{(T)} - v^2R_M(\lambda) \tag{4.3}$$

with

$$\Delta_\nu^{(T)} = (\lambda\nu\nu^{\frac{1}{2}})^{-1}\{\textstyle\sum_{j=1}^{M} G_j(T_j) - \sum_{j=1}^{M+1} (N - j + 1)\lambda\int_{T_{j-1}}^{T_j} G_j(\tau)\, d\tau\} \tag{4.4}$$

and

$$R_M(\lambda) = (2\lambda^2\nu^3)^{-1}\textstyle\sum_{j=1}^{M} \{G_j(T_j)[1 + \eta_j(\lambda\nu\nu^{\frac{1}{2}})^{-1}G_j(T_j)]^{-1}\}^2 \tag{4.5}$$

for values η_j such that $0 \leqq \eta_j \leqq v$. To obtain results of the same nature as in Section 3 we should have that $\{P_{\lambda,\delta_\nu v}^{(T)}\}$ and $\{P_{\lambda,0}^{(T)}\}$ are contiguous and that R_M tends to a constant as $N[1 - e^{-\lambda T}]$ tends to infinity.

The desired properties are obtained by making two comparisons. First we will compare the pairs of measures $\{P_{\lambda,\delta_\nu v}^{(\nu)}, P_{\lambda,0}^{(\nu)}\}$ with the pairs $\{P_{\lambda,\delta_\nu v}^{(T)}, P_{\lambda,0}^{(T)}\}$. Then, we will compare the coefficients $R_\nu(\lambda)$ and $R_M(\lambda)$ of the term in v^2 in the expression of the logarithm of the likelihood ratio. The value ν is so chosen that $P[T_\nu < T \mid H_\nu] \rightarrow 0$ as ν tends to infinity. Thus in the limit the "ν"-procedure will be more informative than that of T-procedure. Throughout this section we assume that $0 \leqq f \leqq c$ and $f(x, y) = 0$ for $x > y$. Furthermore we assume that f is measurable in (x, y).

THEOREM 4.1. *Let λ, N, T and v vary arbitrarily subject to the restriction that $N[1 - e^{-\lambda T}] \rightarrow \infty$ and that $0 \leqq v \leqq K\lambda$ for some number $K < \infty$. Then the sequences $\{P_{\lambda,\delta_\nu v}^{(T)}\}$ and $\{P_{\lambda,0}^{(T)}\}$ are contiguous.*

PROOF. According to Theorem 3.1 there exists a function $\eta(\delta)$ tending to zero as $\delta \rightarrow 0$ such that under the conditions of the theorem for $A \in \mathcal{A}_\nu$,

$$P_{\lambda,\delta_\nu v}^{(\nu)}(A) < \delta \Rightarrow P_{\lambda,0}^{(\nu)}(A) < \eta(\delta) \quad \text{and} \quad P_{\lambda,0}^{(\nu)}(A) < \delta \Rightarrow P_{\lambda,\delta_\nu v}^{(\nu)}(A) < \eta(\delta).$$

The statement of the theorem is that a similar result holds for the σ-field $\mathcal{B}_T$.

More precisely, we shall show that for every $\epsilon > 0$ there is a $\nu(\epsilon)$ and a $\delta > 0$ such that if $\nu \geqq \nu(\epsilon)$ and $B \varepsilon \mathcal{B}_T$ then

$$P_{\lambda,0}^{(T)}(B) < \delta \Rightarrow P_{\lambda,\delta_\nu v}^{(T)}(B) < 2\epsilon \quad \text{and} \quad P_{\lambda,\delta_\nu v}^{(T)}(B) < \delta \Rightarrow P_{\lambda,0}^{(T)}(B) < 2\epsilon.$$

For this purpose note the following relations. Let $S_\nu = \{T_\nu \geqq T\}$. Then $S_\nu \varepsilon \mathcal{A}_\nu$, but also $S_\nu \varepsilon \mathcal{B}_T$. If $B \varepsilon \mathcal{B}_T$ then $B \cap S_\nu \varepsilon \mathcal{A}_\nu$ since if $T_\nu \geqq T$ the ν-procedure is more informative than the T-procedure. This could be proved formally using the description of $\mathcal{B}_T$ as the σ-field generated by $\{\min(T, T_j); j = 1, 2, \cdots, N\}$. However it is clear that if we know the process up to T_ν and $T_\nu \geqq T$, we know what happened before T.

Select δ so small that $\eta(\delta) < \epsilon$ and then select $\nu(\epsilon)$ so large that $\nu \geqq \nu(\epsilon)$ implies $P_{\lambda,0}(S_\nu^c) < \eta(\delta) < \epsilon$. Let $B \varepsilon \mathcal{B}_T$ be such that $P_{\lambda,0}^{(T)}(B) < \delta$. This implies $P_{\lambda,0}[B \cap S_\nu] < \delta$, hence since $B \cap S_\nu \varepsilon \mathcal{A}_\nu$, $P_{\lambda,\delta_\nu v}[B \cap S_\nu] < \eta(\delta)$. It follows that $P_{\lambda,\delta_\nu v}^{(T)}(B) \leqq P_{\lambda,\delta_\nu v}[B \cap S_\nu] + P_{\lambda,\delta_\nu v}(S_\nu^c) < \eta(\delta) + \epsilon$. Similarly, if $P_{\lambda,\delta_\nu v}^{(T)}(B) < \delta$ one can write $P_{\lambda,\delta_\nu v}[B \cap S_\nu] < \delta$ hence $P_{\lambda,0}^{(\nu)}[B \cap S_\nu] < \eta(\delta)$ and therefore $P_{\lambda,0}^{(T)}(B) < \eta(\delta) + P_{\lambda,0}(S_\nu^c) \leqq \eta(\delta) + \epsilon < 2\epsilon$.

This concludes the proof of the theorem.

The next result concerns the coefficient R_M of v^2 in the expression (4.3) of $\Lambda_\nu^{(T)}$. We shall show that under the same assumption as in Section 3, this coefficient converges to a nonrandom limit.

LEMMA 4.1. *Let ν be related to N and T by the formula* (4.2). *Let λ, N and T vary in such a way that $N[1 - e^{-\lambda T}] \to \infty$. Let η_j be variables such that $0 \leqq \eta_j \leqq v$. Let*

$$Y_j = [\textstyle\sum_{i=0}^{j-1} f(T_i, T_j)]^2[1 + \eta_j(\lambda\nu\nu^{\frac{1}{2}})^{-1}\sum_{i=0}^{j-1} f(T_i, T_j)]^2.$$

Then

$$\nu^{-3}\textstyle\sum_{j=1}^{M} Y_j - \nu^{-3}\sum_{j=1}^{\nu} Y_j$$

tends to zero in $P_{\lambda,0}$ probability as $N[-e^{-\lambda T}] \to \infty$.

PROOF. The proof follows from the fact that Y_j's are bounded and that M converges in probability to the expected value EM which is of the order of magnitude of ν.

This leads to the following theorem.

THEOREM 4.2. *Let $\Phi_{T,\lambda}$ be the distribution function defined for $x \geqq 0$ by $\Phi_{T,\lambda}(x) = \min\{1, [1 - e^{-\lambda x}][1 - e^{-\lambda T}]^{-1}\}$. Let $D^*(T, \lambda)$ be defined by $D^*(T, \lambda) = \int |\int_{x \leqq y} f(x, y)\, d\Phi_{T,\lambda}(x)|^2\, d\Phi_{T,\lambda}(y)$. Let Y_j be as in Lemma 4.2. We assume that f satisfies the continuity restrictions. If $(1/\lambda)[1 - e^{-\lambda T}]$ stays bounded away from zero and infinity or if $(1/\lambda)[1 - e^{-\lambda T}]$ tends to zero and f is continuous at zero on the set $\{(x, y); 0 \leqq x \leqq y\}$ or if $(1/\lambda)[1 - e^{-\lambda T}]$ tends to infinity and f tends to a limit as $x \leqq y$ tends to infinity, the difference $(1/\nu^3)\sum_{j=1}^{M} Y_j - D^*(T, \lambda)$ tends to zero in $P_{\lambda,0}$ probability.*

PROOF. We omit the proof, since it is similar to that of Theorem 3.2 and Lemma 3.3.

Theorem 4.2 enables us to write the logarithm of likelihood ratio $\Lambda_\nu^{(T)}$ as follows

$$\Lambda_\nu^{(T)} = v\Delta_\nu^{(T)} - v^2 D^*(T, \lambda)/2\lambda^2 + \epsilon_M \tag{4.6}$$

where

$$\epsilon_M \to 0 \quad \text{in} \quad P_{\lambda,0}^{(T)} \quad \text{probability.}$$

Under the conditions of (i) contiguity of $\{P_{\lambda,0}^{(T)}\}$ and $\{P_{\lambda,\delta_\nu v}^{(T)}\}$ and (ii) the expansion of $\Lambda_\nu^{(T)}$ in (4.6), Theorem 3.3 and Theorem 3.4 are applicable. The same argument as in Section 3 if λ is known an optimal asymptotic test of size α of testing $H_0 : v = 0$ against $H_1 : v > 0$ is defined by the test $\omega(\Delta_\nu^{(T)})$ where the critical region is defined by rejecting H_0 if $\Delta_\nu^{(T)} \geqq (1/\lambda)b(D^*(T, \lambda))^{\frac{1}{2}}$.

Notice that the asymptotic distribution for Δ_ν and $\Delta_\nu^{(T)}$ are the same. This means that if we observe a large number of cases under the n-procedure the information collected would be about the same as that under the T-procedure provided that expected number $N(1 - e^{-\lambda T})$ of cases in $(0, T)$ is also large and close to n.

5. Asymptotic expressions for variable values of λ. In Sections 3 and 4, we have assumed that λ is known and obtained an optimal asymptotic test for testing $H_0 : \beta = 0$ against $H_1 : \beta > 0$. However, in general λ is not known. The test statistic $\Delta_n(\lambda)$ (or $\Delta_\nu^{(T)}(\lambda)$) derived from the log of likelihood ratio $\Lambda_n[P_{\lambda,v/nn^{\frac{1}{2}}}^{(n)} ; P_{\lambda,0}^{(n)}]$ (or $\Lambda_\nu^{(T)}$) depends on the nuisance parameter λ, hence the test criterion $\Delta_n(\lambda)$ is not computable. Under this circumstance, we would like to substitute for λ in $\Delta_n(\lambda)$ an estimate $\hat{\lambda}_n$. The question arises, whether, after the substitution, $\Delta_n(\hat{\lambda}_n)$ still tends to a normal distribution with appropriate mean and covariance matrix. Investigation (see LeCam [5]) shows that if $\hat{\lambda}_n$ is such that $\mathfrak{L}\{n^{\frac{1}{2}}(\hat{\lambda}_n - \lambda)/P_{\lambda,0}^{(n)}\}$ forms a relatively compact sequence, then $\Delta_n(\hat{\lambda}_n)$ will have a limiting normal distribution. Let $\xi_n = n^{\frac{1}{2}}(\hat{\lambda}_n - \lambda)$. Then "relatively compact" means that ξ_n is bounded in probability. The procedure of deriving the limiting distribution for $\Lambda_n(\hat{\lambda}_n) = \Lambda_n(\lambda + \xi_n/n^{\frac{1}{2}})$ is carried out in two stages. First, we obtain the limiting distribution for $\Lambda_n(\lambda + u/n^{\frac{1}{2}})$ where u is non-random and belongs to a bounded set $E \subset (0, \infty)$. Second, we replace u by ξ_n and show that the substitution does not affect the asymptotic distribution. This provides an asymptotically similar test to test H_0 against H_1 in the presence of the nuisance parameter.

Requirements analogous to those of Section 3 are needed for the existence of limiting normal distribution of log likelihood ratio. We repeat them briefly as follows:

(i) $\{P_{\lambda+\psi_n u,\beta+\delta_n v}^{(n)}\}$ and $\{P_{\lambda,\beta}^{(n)}\}$ are contiguous with $\psi_n = 1/n^{\frac{1}{2}}$ and $\delta_n = 1/nn^{\frac{1}{2}}$;

(ii) for each $\theta = (\lambda, \beta)$ there exists a sequence $\{X_n(\theta)\}$ of 2-dimensional $\mathfrak{A}_n$ measurable variables and a non-random function $H(t, \theta)$ where $t = (u, v) \in R \times R$ such that $\Lambda_n[P_{\lambda+\psi_n u,\beta+\delta_n v}^{(n)} ; P_{\lambda,\beta}^{(n)}] = tX_n(\theta) - H(t, \theta) + \epsilon_n$ where $\epsilon_n \to 0$ in $P_{\lambda,\beta}^{(n)}$ and $H(t, \theta) = \mu(\theta) + (\frac{1}{2})t\Gamma(\theta)t'$.

(iii) Let $t_n = (u_n, v_n)$, $t_n \to t$ as $n \to \infty$. $\Lambda_n[P_{\lambda+\psi_n u_n,\beta+\delta_n v_n}^{(n)} ; P_{\lambda,\beta}^{(n)}] - \Lambda_n[P_{\lambda+\psi_n u,\beta+\delta_n v}^{(n)} ; P_{\lambda,\beta}^{(n)}]$ converges to zero in $P_{\lambda,\beta}^{(n)}$ as $t_n \to t$.

Notice that the rates of convergence for λ and β are different. This choice of rates is made so that the limiting distribution does exist and is not degenerate. As we can see from (ii) the test criterion is derived on the basis of $X_n(\theta) = X_n(\lambda, \beta)$.

Regarding the substitution of λ by $\hat{\lambda}_n$ in $X_n(\lambda, \beta)$ we shall need the following properties:

(iv) the existence of consistent estimate $\hat{\lambda}_n$ of λ such that $\mathcal{L}\{n^{\frac{1}{2}}(\hat{\lambda}_n - \lambda)/\lambda\}$ is relatively compact, and

(v) the convergence of $X_n(\lambda + u/n^{\frac{1}{2}}, \beta) - X_n(\lambda, \beta)$ to $-\Gamma(\lambda)\begin{bmatrix} u \\ 0 \end{bmatrix}$ in $P_{\lambda,\beta}^{(n)}$ probability.

The reason for this will become evident in the proof of asymptotic normality. To obtain manageable formulas we shall denote the logarithm of the likelihood ratio of $P_{\lambda+u/n^{\frac{1}{2}},\beta_0+v/nn^{\frac{1}{2}}}^{(n)}$ to $P_{\lambda,\beta_0}^{(n)}$ by $\Lambda_n[(\lambda + u/n^{\frac{1}{2}}, \beta_0 + v/nn^{\frac{1}{2}}); (\lambda, \beta_0)]$. The number β_0 will be taken equal to zero. However, the symbol β_0 will be kept in the formula to avoid possible confusion.

It is easily seen that

$$\Lambda_n[(\lambda + u/n^{\frac{1}{2}}, \beta_0 + v/nn^{\frac{1}{2}}); (\lambda, \beta_0)] = \Lambda_n[(\lambda + u/n^{\frac{1}{2}}, \beta_0); (\lambda, \beta_0)] + \Lambda_n[(\lambda + u/n^{\frac{1}{2}}, \beta_0 + v/nn^{\frac{1}{2}}); (\lambda + u/n^{\frac{1}{2}}, \beta_0)].$$

For simplicity, the first term in the right hand side will often be called $\Lambda_n^{(1)}$. The second term will then be called $\Lambda_n^{(2)}$. It is more convenient to establish the properties (i), (ii), (iii) described above for $\Lambda_n^{(1)}$ and $\Lambda_n^{(2)}$ separately and then combine the results to obtain the same properties for Λ_n itself.

LEMMA 5.1. *The log likelihood ratio* $\Lambda_n[(\lambda + u/n^{\frac{1}{2}}, \beta_0); (\lambda, \beta_0)]$ *satisfies the requirements* (i), (ii), *and* (iii) *provided only that* u/λ *stays bounded.*

PROOF. Simple calculations and Taylor's expansion show that the log likelihood ratio $\Lambda_n^{(1)}$ is equal to

$$(5.1) \quad \Lambda_n^{(1)} = (u/\lambda)n^{-\frac{1}{2}}[n - \textstyle\sum_{j=1}^{n} (N - j + 1)\lambda(T_j - T_{j-1})] - u^2/2\lambda^2 + \epsilon_n$$

with

$$\epsilon_n = n\{\log(1 + u/n^{\frac{1}{2}}) + (u^2/2\lambda^2 n) - u/\lambda n^{\frac{1}{2}}\}.$$

It follows that (ii) is satisfied.

To show that $\{P_{\lambda+u/n^{\frac{1}{2}},\beta_0}^{(n)}\}$ and $\{P_{\lambda,\beta_0}^{(n)}\}$ are contiguous it is sufficient to proceed as in Theorem 3.1 and show that $E[\Lambda_n^{(1)} \mid P_{\lambda+u/n^{\frac{1}{2}},\beta_0}^{(n)}]$ and $E[\Lambda_n^{(1)} \mid P_{\lambda,\beta_0}^{(n)}]$ are both bounded.

For this purpose, let η_n be the coefficient of u in the expansion of $\Lambda_n^{(1)}$ so that

$$\eta_n = (\lambda n^{\frac{1}{2}})^{-1}[n - \textstyle\sum_{j=1}^{n} (N - j + 1)\lambda(T_j - T_{j-1})].$$

We have

$$E[\eta_n \mid \lambda + u/n^{\frac{1}{2}}, \beta_0] = (u/\lambda n^{\frac{1}{2}})[1 - \lambda/(\lambda + u/n^{\frac{1}{2}})] = u/\lambda(\lambda + u/n^{\frac{1}{2}}).$$

It follows that

$$E[\Lambda_n^{(1)} \mid \lambda, \beta_0] = n\,[\log(1 + u/\lambda n^{\frac{1}{2}}) - u/\lambda n^{\frac{1}{2}}] \geqq (u^2/2\lambda^2)\,\{\min[1, 1 + u/\lambda n^{\frac{1}{2}}]\}^{-1}.$$

Also

$$E[\Lambda_n^{(1)} \mid \lambda + u/n^{\frac{1}{2}}, \beta_0] \leqq E[\eta_n u \mid \lambda + u/n^{\frac{1}{2}}, \beta_0] = u^2/\lambda(\lambda + u/n^{\frac{1}{2}}).$$

Therefore, if there is a number K such that $|u| \leqq K\lambda$, the two sequences $\{P^{(n)}_{\lambda+u/n^{\frac{1}{2}},\beta_0}\}$ and $\{P^{(n)}_{\lambda,\beta_0}\}$ are contiguous. This and the Taylor expression (5.1) immediately imply the validity of the lemma.

LEMMA 5.2. *Assume that the contagion function f is bounded $0 \leqq f \leqq c$ and that there is a number $K < \infty$ such that $|u| \leqq K\lambda$ and $|v| \leqq K\lambda$. Then the sequences $\{P^{(n)}_{\lambda+u/n^{\frac{1}{2}},\beta_0+v/nn^{\frac{1}{2}}}\}$ and $\{P^{(n)}_{\lambda,\beta_0}\}$ are contiguous.*

REMARK. The parameters λ, u and v are allowed to vary arbitrarily with n and N.

PROOF. Let n be so large that $2K < n^{\frac{1}{2}}$. Then $\lambda + u/n^{\frac{1}{2}} \geqq \lambda/2$. It follows that $|v| \leqq 2K(\lambda + u/n^{\frac{1}{2}})$. According to Theorem 3.1 this implies that the sequences $\{P^{(n)}_{\lambda+u/n^{\frac{1}{2}},\beta_0+v/nn^{\frac{1}{2}}}\}$ and $\{P^{(n)}_{\lambda+u/n^{\frac{1}{2}},\beta_0}\}$ are contiguous By Lemma 5.1 that $\{P^{(n)}_{\lambda+u/n^{\frac{1}{2}},\beta_0}\}$ and $\{P^{(n)}_{\lambda,\beta_0}\}$ are contiguous. Since contiguity is a transitive property, the result follows.

To proceed further it will be necessary to expand $\Lambda_n[(\lambda + u/n^{\frac{1}{2}}, \beta_0 + v/n^{\frac{1}{2}}); (\lambda, \beta_0)]$ in terms of u and v. For this purpose let us introduce the following notations. The integral $A(n/N, \lambda)$ of Section 3 takes the form

$$A(n/N, \lambda) = \int |\int f(x, y)p(x; n/\lambda N, \lambda)\, dx|^2 p(y; n/\lambda N, \lambda)\, dy,$$

where $p(x; n/\lambda N, \lambda)$ is the density of $\Phi_{N,n,\lambda}(x)$. We shall need in addition the integral

$$B(n/N, \lambda) = \int\int f(x, y)p(x; n/\lambda N, \lambda)p(y; n/\lambda N, \lambda)\, dx\, dy.$$

Furthermore we shall use the random expressions

$$\eta_n(\lambda) = (1/\lambda n^{\frac{1}{2}})[n - \textstyle\sum_{j=1}^{n} (N - j + 1)\lambda(T_j - T_{j-1})]$$

and

$$\Delta_n(\lambda) = (1/\lambda n n^{\frac{1}{2}}) \textstyle\sum_{j=1}^{n} \sum_{i=0}^{j-1} \{f(T_i, T_j) - (N - j + 1)\lambda \int_{T_{j-1}}^{T_j} f(T_i, \tau)\, d\tau\}.$$

Also, let $X_n(\lambda)$ be the two dimensional random vector

$$X_n(\lambda) = \begin{bmatrix} \eta_n(\lambda) \\ \Delta_n(\lambda) \end{bmatrix}.$$

Let $\Gamma(n/N, \lambda)$ be the covariance matrix

$$\Gamma(n/N, \lambda) = \begin{bmatrix} 1 & B(n/N, \lambda) \\ B(n/N, \lambda) & A(n/N, \lambda) \end{bmatrix}$$

and t be a row vector $t = (u, v)$.

To simplify the statements of the results, let us introduce the following Condition C^+.

CONDTION C^+. The function f satisfies Condition C. In addition to this we assume that if $n/\lambda N$ tends to a limit a, the integral $\int_0^a \int_0^a f(x, y)\, dx\, dy$ is positive.

The following two Lemmas 5.3 and 5.4 are needed to prove that the requirements (ii) and (v) are satisfied.

LEMMA 5.3. *Assume that Condition* C^+ *is satisfied then even if* λ *is allowed to vary arbitrarily with n and N both*

$$(1/n^2) \sum_{j=1}^{n} \sum_{i=0}^{j-1} f(T_i, T_j) - B(n/N, \lambda)$$

and

$$\sup_{|u| \leqq K\lambda} \lambda |\Delta_n(\lambda + u/n^{\frac{1}{2}}) - \Delta_n(\lambda) + (u/\lambda^2) B(n/N, \lambda)|$$

tend to zero in $P_{\lambda,0}^{(n)}$ *probability.*

PROOF. The argument for the first part of the lemma being the same as that of Theorem 3.2 will be omitted.

For the second part, note that

$$\lambda[\Delta_n(\lambda + u/n^{\frac{1}{2}}) - \Delta_n(\lambda)] = -[u/(\lambda + u/n^{\frac{1}{2}})n^2] \sum_{j=1}^{n} \sum_{i=0}^{i-1} f(T_i, T_j).$$

Since $|(1/n^2) \sum_{j=1}^{n} \sum_{i=0}^{j-1} f(T_i, T_j)| \leqq [(n+1)/n]c$ the desired result follows from the first part of the lemma.

LEMMA 5.4. *Assume that Condition* C^+ *is satisfied. Then*

$$\sup_{\lambda>0} \sup_{|u| \leqq K\lambda} |A(n/N, \lambda + u/n^{\frac{1}{2}}) - A(n/N, \lambda)|$$

tends to zero as $n \to \infty$.

PROOF. Cases where $n/\lambda N$ tends to zero or infinity can be handled trivially, using the appropriate continuity assumption made in Condition C^+. Otherwise, suppose that $a \in (0, \infty)$. Then, the density $p(x; n/\lambda N, \lambda)$ tends to a limit equal to

$$\begin{aligned} p(x) &= (1/a)e^{-\lambda_0 x} \qquad 0 \leqq x \leqq -(1/\lambda_0) \log(1 - \lambda_0 a), \\ &= 0, \qquad \text{otherwise.} \end{aligned}$$

When λ_0 itself is zero the limit $p(x)$ is simply the uniform density

$$\begin{aligned} p(x) &= 1/a, \qquad 0 \leqq x \leqq a, \\ &= 0, \qquad \text{otherwise.} \end{aligned}$$

Note that $p(x; n/\lambda N, \lambda + u/n^{\frac{1}{2}})$ also converges to the limit $p(x)$. Hence, according to Scheffé's theorem both $\int |p(x; n/\lambda N, \lambda + u/n^{\frac{1}{2}}) - p(x)|\, dx$ and $\int |p(x; n/\lambda N, \lambda) - p(x)|\, dx$ converge to zero. Let $g(y; n/\lambda N, \lambda)$ be the integral

$$g(y; n/\lambda N, \lambda) = \int f(x, y) p(x; n/\lambda N, \lambda)\, dx.$$

One can write

$$|g(y; n/\lambda N, \lambda) - \int f(x, y) p(x)\, dx| \leqq c \int |p(x; n/\lambda N, \lambda) - p(x)|\, dx.$$

Therefore $g(y; n/\lambda N, \lambda)$ converges uniformly in y to the limit $\int f(x, y) p(x)\, dx$.

It follows that $A(n/N, \lambda)$ converges to $\int |\int f(x, y)p(x)\,dx|^2 p(y)\,dy$. Since the same result holds for $A(n/N, \lambda + u/n^{\frac{1}{2}})$ the lemma is entirely proved.

The matrix $\Gamma(n/N, \lambda)$ is essentially the covariance matrix of the random vector $X_n(\lambda)$. The following lemma is to show that $\Gamma(n/N, \lambda)$ is not degenerate in the limit as $n(N)$ tends to infinity.

LEMMA 5.5. *Assume that Condition* C^+ *is satisfied. Let* λ *vary arbitrarily with* n *and* N *subject to the restriction* $\lambda > 0$. *Then*

$$\liminf_{n\to\infty} A(n/N, \lambda) - B^2(n/N, \lambda) > 0.$$

PROOF. We rewrite $A(n/N, \lambda)$ as

$$A(n/N, \lambda) = \int g^2(y; n/\lambda N, \lambda)p(y; n/\lambda N, \lambda)\,dy.$$

Similarly

$$B(n/N, \lambda) = \int g(y; n/\lambda N, \lambda)p(y; n/\lambda N, \lambda)\,dy.$$

This shows that $A(n/N, \lambda) - B^2(n/N, \lambda)$ is the variance of the bounded random variable $g(Y_n; n/\lambda N, \lambda)$ for a random variable Y_n distributed according to $p(y; n/\lambda N, \lambda)$.

Consider first that $n/\lambda N \to a \in (0, \infty)$ and $\lambda \to \lambda_0 \geqq 0$. Then $p(x; n/\lambda N, \lambda)$ converges to the density $p(x) = (1/a) \exp(-\lambda_0 x)$ for $0 \leqq x \leqq -(1/\lambda_0) \log(1 - \lambda_0 a)$. The function $g(y; n/\lambda N, \lambda)$ converges to the integral

$$g(y) = \int f(x, y)p(x)\,dx = \int_0^y f(x, y)p(x)\,dx.$$

As $y \to 0$ the value $g(y)$ tends to zero. Thus if $g(y)$ is almost everywhere constant in the range $0 \leqq x \leqq a$, its value must be zero almost everywhere in $0 \leqq x \leqq a$. This contradicts the assumption that $\int_0^a \int_0^a f(x, y)\,dx\,dy > 0$. Thus the variance of $g(Y)$ is not zero.

Assume now that $n/\lambda N \to 0$. Then, according to C^+ for every $\epsilon > 0$ there is a δ such that $0 \leqq x \leqq y \leqq \delta$.

$$(1 - \epsilon)f(0, 0) \leqq f(x, y) \leqq (1 + \epsilon)f(0, 0).$$

Letting $f(0, 0) = b$ this gives

$$(1 - \epsilon)b\Phi_{N,n,\lambda}(y) \leqq g(y; n/\lambda N, \lambda) \leqq (1 + \epsilon)b\Phi_{N,n,\lambda}(y).$$

The variable $\Phi_{N,n,\lambda}(Y)$ has a uniform distribution on $(0, 1)$ if Y has distribution $\Phi_{N,n,\lambda}$. Thus

$$A(n/N, \lambda) - B^2(n/N, \lambda) \geqq b^2[(1 - \epsilon)^2\tfrac{1}{3} - (1 + \epsilon)^2\tfrac{1}{4}].$$

This is strictly positive for $8(1 - \epsilon)^2 \geqq 7(1 + \epsilon)^2$. The case where $n/\lambda N \to \infty$ can be treated similarly, hence the result.

From the above lemmas we obtain

THEOREM 5.1. *Assume that Condition* C^+ *is satisfied. Assume that* λ *and* $t = (u, v)$ *are allowed to vary arbitrarily with* n *and* N *subject to the sole restriction that* $|u| < K\lambda$

and $|v| < K\lambda$ *for some* $K < \infty$. *Then*

(i) *the distributions* $\{P^{(n)}_{\lambda+u/n^{\frac{1}{2}},v/n\, n^{\frac{1}{2}}}\}$ *and* $\{P^{(n)}_{\lambda,0}\}$ *are contiguous.*

(ii) *the difference*

$$\Lambda_n[(\lambda + u/n^{\frac{1}{2}}, v/n^{\frac{3}{2}}); (\lambda, 0)] - tX_n(\lambda) + (\tfrac{1}{2}\lambda^{-2})t\Gamma(n/N, \lambda)t'$$

tends to zero in $P^{(n)}_{\lambda,0}$ *probability as* $n \to \infty$.

(iii) *the difference*

$$\sup_{|u|\leq K\lambda} \lambda \left|X_n(\lambda + u/n^{\frac{1}{2}}) - X_n(\lambda) + (1/\lambda^2)\Gamma(n/N, \lambda)\begin{bmatrix} u \\ 0 \end{bmatrix}\right|$$

tends to zero in $P^{(n)}_{\lambda,0}$ *probability.*

PROOF. Statement (i) is a restatement of Lemma 5.2. For statement (ii) note that, according to Lemma 5.1 and Theorem 3.2

$$\begin{aligned}\Lambda_n[(\lambda + u/n^{\frac{1}{2}}; \beta_0 + v/nn^{\frac{1}{2}}); (\lambda, \beta_0)] &= [v\Delta_n(\lambda + u/n^{\frac{1}{2}}) - (v^2/2(\lambda + u/n^{\frac{1}{2}})^2)A(n/N, \lambda + u/n^{\frac{1}{2}})] \\ &\quad - [u\eta_n(\lambda) - u^2/2\lambda^2] + \epsilon_n\end{aligned}$$

where ϵ_n tends to zero in probability. The result is then obtainable by applying Lemma 5.3 to replace $\Delta_n(\lambda + u/n^{\frac{1}{2}})$ by an expression in terms of $\Delta_n(\lambda)$ and applying Lemma 5.4 to replace $A(n/N, \lambda + u/n^{\frac{1}{2}})$ by $A(n/N, \lambda)$.

Concerning statement (iii) note first that

$$\eta_n(\lambda + u/n^{\frac{1}{2}}) - \eta_n(\lambda) = n^{\frac{1}{2}}[1/(\lambda + u/n^{\frac{1}{2}}) - 1/\lambda] = -u/\lambda(\lambda + un^{\frac{1}{2}}).$$

Since

$$\Gamma(n/N, \lambda)\begin{bmatrix} u \\ 0 \end{bmatrix} = \begin{bmatrix} u \\ uB(n/N, \lambda) \end{bmatrix},$$

the result follows immediately from Lemma 5.3.

THEOREM 5.2. *Assume that Condition* C^+ *is satisfied and that* λ, u *and* v *are allowed to vary arbitrarily subject to the conditions* $\lambda \in (0, \infty)$, $|u| \leqq K\lambda$ *and* $0 \leqq v \leq K\lambda$, *for some* $K < \infty$. *Let* $Z_n(\lambda)$ *be the random vector*

$$Z_n(\lambda) = \lambda X_n(\lambda) = \lambda \begin{bmatrix} \eta_n(\lambda) \\ \Delta_n(\lambda) \end{bmatrix}.$$

Let $Q[B; \lambda, t, n, N]$ *be the probability*

$$Q[B; \lambda, t, n, N] = P^{(n)}\{Z_n(\lambda)' \in B \mid (\lambda + u/n^{\frac{1}{2}}, v/nn^{\frac{1}{2}})\}.$$

Let $\bar{Q}[B; \lambda, t, n, N]$ *be the probability of the set* B *for the normal distribution which has expectation* $(1/\lambda)\Gamma(n/N, \lambda)t'$ *and covariance matrix* $\Gamma(n/N, \lambda)$. *Then*

$$\sup_B \{|Q[B; \lambda, t, n, N] - \bar{Q}[B; \lambda, t, n, N]|; B \text{ convex}\}$$

tends to zero as $n \to \infty$.

PROOF. The introduction of $Z_n(\lambda) = \lambda X_n(\lambda)$ is intended to prevent disappearance of the probability measures at infinity. Since the covariance matrix $\Gamma(n/N, \lambda)$ cannot tend to a singular matrix (Lemma 5.5), the normal measures $\bar{Q}$ do not degenerate. Thus, it will be sufficient to show that if the values (λ, t) are made dependent on (n, N) in such a way that the normal measures $\bar{Q}$ have a weak limit then the corresponding measures Q also tend to the same limit.

For this purpose, assume that for $P_{\lambda,0}^{(n)}$ the distribution of Λ_n converges (in the usual sense) to a limit F on the line augmented by a point at infinity. The contiguity statement of Theorem 5.1 implies that $\int \exp(\Lambda)F(d\Lambda) = 1$. Therefore, if the measures $Q = Q[\cdot\,; \lambda, 0, n, N]$ converge in the usual sense on the plane augmented by points at infinity to a limit Q_0 and if $\Gamma(n/N, \lambda) \to \Gamma$ then

$$\int \exp(tz)Q_0(dz) = \exp[(t/2)\Gamma t']$$

for every $t = [u, v]$ with $v \geqq 0$.

It follows that Q_0 is the normal distribution $\eta(0, \Gamma)$. For the measures $Q[\cdot\,; \lambda, t, n, N]$, the contiguity property implies that their limit has density $\exp[(tz/\lambda) - (\frac{1}{2}\lambda^{-2})t\Gamma t']$ with respect to Q_0. The general statement is then a consequence of a simple computation.

6. Asymptotically similar tests. If the observable variables T_j, $j = 1, 2, \cdots, n$, are obtained from the distribution $P_{\lambda,0}^{(n)}$ then the joint density of the T_j is given by $[N!\lambda^n/(N - n)!] \exp\{-\sum_{j=1}^{n} (N - j + 1)\lambda(t_j - t_{j-1})\}$ for $0 < t_1 < t_2 < \cdots < t_n < \infty$. One can obtain an estimate $\hat{\lambda}_n$ of λ by the maximum likelihood method. This gives

$$1/\hat{\lambda}_n = (1/n)\sum_{j=1}^{n} (N - j + 1)(T_j - T_{j-1}).$$

Let χ_{2n}^2 be the random variable

$$\chi_{2n}^2 = 2\lambda\sum_{j=1}^{n} (N - j + 1)(T_j - T_{j-1}).$$

It is well known that χ_{2n}^2 has in fact the standard chi-square distribution with $2n$ degrees of freedom. In particular

$$E(1/\chi_{2n}^2) = \tfrac{1}{2}(n - 1) \quad \text{and} \quad \sigma^2(1/\chi_{2n}^2) = \tfrac{1}{4}(n - 1)^{-2}(n - 2)^{-1}.$$

This gives $E(\hat{\lambda}_n/\lambda) = n/(n - 1)$ and $\sigma^2(\hat{\lambda}_n/\lambda) = n^2/(n - 1)^2(n - 2)$. An application of Chebyshev's inequality gives the following result.

Lemma 6.1. For every $b > 0$ and every $\lambda \varepsilon (0, \infty)$ one has

$$P_{\lambda,0}^{(n)}\{(n^{\frac{1}{2}}/\lambda)\,|\hat{\lambda}_n - \lambda| \geqq b + n^{\frac{1}{2}}/(n - 1)\} \leqq n^3/[(n - 2)^3 b^2].$$

In particular the family of distributions $\mathcal{L}\{(n^{\frac{1}{2}}/\lambda)(\hat{\lambda}_n - \lambda) \mid \lambda\}, \lambda \varepsilon (0, \infty), n > 2$, is relatively compact.

After these preliminaries, we are in a position to use a procedure partially suggested in [5] for the construction of asymptotically similar test. Let $\Gamma(n/N, \lambda) = \Gamma_n(\lambda)$ for short.

Theorem 6.1. *Let Condition* C^+ *be satisfied. Let* S_n *be the two dimensional* *statistic*

$$S_n = \begin{bmatrix} n^{\frac{1}{2}}\hat{\lambda}_n \\ 0 \end{bmatrix} + \hat{\lambda}_n^2 \Gamma_n^{-1}(\hat{\lambda}_n) X_n(\hat{\lambda}_n).$$

Assume that $\lambda \varepsilon (0, \infty)$ *and that* $|v| \leqq K\lambda$ *for some* $K < \infty$. *Otherwise allow* λ *and* v *to vary arbitrarily with* n *and* N. *Let* ρ *denote either Prohorov's distance or the supremum of the difference of probabilities of convex sets. Then*

$$\rho\left\{\mathcal{L}\left\{(1/\lambda)\left[S_n - \begin{bmatrix} n^{\frac{1}{2}}\lambda \\ v \end{bmatrix}\right] \middle| P^{(n)}_{\lambda, v/nn^{\frac{1}{2}}}\right\} - \mathfrak{N}[0, \Gamma_n^{-1}(\lambda)]\right\}$$

tends to zero as n *tends to infinity.*

Furthermore S_n *is "differentially asymptotically sufficient" in the sense that there are families of measures* $Q^{(n)}_{\lambda,v}$ *for which* S_n *is sufficient such that*

$$\sup \{\|Q^{(n)}_{\lambda,v} - P^{(n)}_{\lambda,v/nn^{\frac{1}{2}}}\|; n^{\frac{1}{2}}|\lambda - \lambda_0| \leqq K\lambda_0, 0 \leqq v \leqq K\lambda_0\}$$

tends to zero as $n \to \infty$.

Proof. Taking first $v = 0$ consider the difference

$$(1/\lambda)\left[S_n - \begin{bmatrix} n^{\frac{1}{2}}\lambda \\ 0 \end{bmatrix}\right] = (1/\lambda)\begin{bmatrix} n^{\frac{1}{2}}(\hat{\lambda}_n - \lambda) \\ 0 \end{bmatrix} + (1/\lambda)\hat{\lambda}_n^2 \Gamma_n^{-1}(\hat{\lambda}_n) X_n(\hat{\lambda}_n).$$

It has been shown (Theorem 5.1) that

$$\lambda[X_n(\lambda + u/n^{\frac{1}{2}}) - X_n(\lambda)] \quad \text{and} \quad -(1/\lambda)\Gamma_n(\lambda)\begin{bmatrix} u \\ 0 \end{bmatrix}$$

differ little, uniformly in u, for $|u| \leqq K\lambda$. Using Lemma 6.1 and Lemma 5.4 this implies that

$$\hat{\lambda}_n[X_n(\hat{\lambda}_n) - X_n(\lambda)] + (1/\hat{\lambda}_n)\,\Gamma_n(\hat{\lambda}_n)\begin{bmatrix} u \\ 0 \end{bmatrix}$$

tends to zero in probability. Since by Lemma 5.5 $\Gamma_n(\hat{\lambda}_n)$ cannot tend to a degenerate limit it follows that

$$\hat{\lambda}_n\,\Gamma_n^{-1}(\hat{\lambda}_n) X_n(\hat{\lambda}_n) = (\hat{\lambda}_n \lambda^{-1})\Gamma_n^{-1}(\lambda)\lambda X_n(\lambda) - (1/\lambda)\begin{bmatrix} u \\ 0 \end{bmatrix} + \epsilon_n,$$

with ϵ_n tending to zero in probability. According to Theorem 5.2 and Lemma 6.1 one can also replace $\hat{\lambda}_n/\lambda$ by unity in the right hand side of the preceding expression. Therefore

$$(1/\lambda)\left[S_n - \begin{bmatrix} n^{\frac{1}{2}}\lambda \\ 0 \end{bmatrix}\right] = (1/\lambda)\begin{bmatrix} n^{\frac{1}{2}}(\hat{\lambda}_n - \lambda) \\ 0 \end{bmatrix} + \Gamma_n^{-1}(\lambda) Z_n(\lambda)$$

$$- (1/\lambda)\begin{bmatrix} u \\ 0 \end{bmatrix} + \epsilon_n' = \Gamma_n^{-1}(\lambda) Z_n(\lambda) + \epsilon_n'$$

with ϵ_n' tending in probability to zero as $n \to \infty$. Therefore, for $v = 0$, the result follows from Theorem 5.2.

We have just shown that for the measures $P_{\lambda,0}^{(n)}$ the difference

$$(1/\lambda)\left[S_n - \begin{bmatrix} n^{\frac{1}{2}}\lambda \\ 0 \end{bmatrix}\right] - \Gamma_n^{-1}(\lambda)Z_n(\lambda)$$

tends to zero in probability. Therefore, because of the contiguity properties stated in Lemma 5.2, this same difference also tends in probability to zero for the measures $P_{\lambda,v/nn^{\frac{1}{2}}}^{(n)}$.

For every value of v, Theorem 5.2 can be applied again to $Z_n(\lambda)$. This implies the convergence to normality stated here.

The asymptotic sufficiency property can be proved as in [5] using the fact that around a fixed λ the vector $X_n(\lambda)$ is "approximately sufficient."

If the statistic S_n of Theorem 6.1 was *exactly* normally distributed there would be no difficulty in constructing a uniformly most powerful similar test of the hypothesis $H_0 : v = 0$ against $H_1 : v > 0$. Such a test would be defined by rejecting H_0 if the second coordinate of S_n is too large. Explicitly, let α be a given number $\alpha \in (0, 1)$, and let b represent the solution of the equation

$$(2\pi)^{-\frac{1}{2}} \int_b^\infty \exp(-u^2/2)\, du = \alpha.$$

The second coordinate $S_n^{(2)}$ of S_n is given by

$$S_n^{(2)} = \hat{\lambda}_n^2[\Delta_n(\hat{\lambda}_n) - B(n/N, \hat{\lambda}_n)\eta_n(\hat{\lambda}_n)][A(n/N, \hat{\lambda}_n) - B^2(n/N, \hat{\lambda}_n)]^{-1}.$$

The variance of this coordinate in the normal distribution would be approximately equal to $\hat{\lambda}_n^2[A(n/N, \hat{\lambda}_n) - B^2(n/N, \hat{\lambda}_n)]^{-1}$. Therefore, the test suggested by the normal theory is the test having for critical region the region

$$(6.1)\quad W_n^* = \{\hat{\lambda}_n[\Delta_n(\hat{\lambda}_n) - B(n/N, \hat{\lambda}_n)\eta_n(\hat{\lambda}_n)] \cdot [A(n/N, \hat{\lambda}_n) - B^2(n/N, \hat{\lambda}_n)]^{-\frac{1}{2}} \geqq b\}.$$

To describe the limiting properties of this test in our present case, let us introduce the following definitions.

DEFINITION 6.1. A sequence $\{W_n\}$ of $\mathfrak{A}_n$ measurable tests is called differentially asymptotically similar of size α on H_0 if for every $K < \infty$ and every $\lambda_0 \in (0, \infty)$

$$\lim_{n\to\infty} \sup_\lambda \{|P_{\lambda,0}^{(n)}(W_n) - \alpha|; |n^{\frac{1}{2}}(\lambda - \lambda_0)| \leqq K\lambda_0\} = 0.$$

DEFINITION 6.2. A sequence $\{W_n\}$ of $\mathfrak{A}_n$-measurable tests will be called uniformly asymptotically similar of size α on H_0 if

$$\lim_{n\to\infty} \{\sup_\lambda |P_{\lambda,0}^{(n)}(W_n) - \alpha|; \lambda \in (0, \infty)\} = 0.$$

Let

$$\Phi(x) = (1/2\pi^{\frac{1}{2}}) \int_x^\infty e^{-u^2/2}\, du.$$

THEOREM 6.2. *Let W_n^* be the region defined by formula* (6.1). *Then W_n^* is* *uniformly asymptotically similar of size* α *on* H_0. *Furthermore*

$$P^{(n)}_{\lambda, v/nn^{\frac{1}{2}}}(W_n^*) - \Phi[b - v\lambda^{-1}[A(n/N, \lambda) - B^2(n/N, \lambda)]^{\frac{1}{2}}]$$

tends to zero uniformly for $\lambda \varepsilon (0, \infty)$ *and* $0 \leqq v \leqq K\lambda$, $K < \infty$, *as* $n \to \infty$.

PROOF. This follows immediately from Theorem 6.1 and Theorem 5.2.

THEOREM 6.3. *Let* $\{W_n\}$ *be a differentially asymptotically similar test of size* α. *Then, for every* $\lambda \varepsilon (0, \infty)$ *and* $K < \infty$

$$\lim \inf_{n\to\infty} \inf_{0 \leqq v \leqq K\lambda} [P^{(n)}_{\lambda, v/nn^{\frac{1}{2}}}(W_n^*) - P^{(n)}_{\lambda, v/nn^{\frac{1}{2}}}(W_n)] \geqq 0.$$

PROOF. The "asymptotic sufficiency" of the statistic S_n implies that it is enough to consider tests W_n which depend on S_n only. Consider a range of values of (λ, v) of the type $\{n^{\frac{1}{2}}|\lambda - \lambda_0| \leqq K\lambda_0, 0 \leqq v \leqq K\lambda_0\}$. Let S_n^* be normal with mean $\begin{bmatrix} n^{\frac{1}{2}}\lambda_0 \\ v \end{bmatrix}$ and covariance matrix $\Gamma_n^{-1}(\lambda_0)$. An argument sketched in [5] shows that there are functions ξ_n such that, for this range of values of (λ, v), the actual distribution of S_n and the distribution of $\xi_n(S_n^*)$ differ *in norm* by a quantity which tends to zero as $n \to \infty$.

This implies that the limiting power obtained from W_n cannot be larger than the power obtainable in an actual normal situation. However, for the normal situation, the test W_n^* is uniformly most powerful among similar tests. Hence the result.

Results on the numerical study of the present model have been obtained which will appear as a separate paper. Extension of the model to include the information of the location of each individual has also been investigated. The introduction of the space coordinates into the model brought no additional difficulties in proving the theory.

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