The Annals of Mathematical Statistics
1969, Vol. 40, No. 1, 319–321

NEW CONDITIONS FOR CENTRAL LIMIT THEOREMS[1]

BY PERCY A. PIERRE[2]

University of Michigan

1. Introduction. A general formulation of the central limit problem for sums of independent random variables is the following (see Loève [3], p. 291). Let

$$S_n = \sum_k X_{nk}$$

where $k = 1, \cdots, k_n, k_n \to \infty$ as $n \to \infty$, and for each n $\{X_{nk}\}$ are independent random variables with probability distribution functions F_{nk} and $EX_{nk} = 0$. Let $\{F_n\}$ be the distribution functions of $\{S_n\}$ and let $\Phi(x)$ be the distribution function of a normal random variable with zero-mean and variance σ^2. Under these conditions it is possible to show the following:

THEOREM 1.1. *Let* $\max_k \text{Var } X_{nk} \to 0$ *and* $\sum_k \text{Var } X_{nk} \to \sigma^2 < \infty$ *where* σ^2 *is a positive constant. The sums* S_n *are asymptotically normal* (i.e., $F_n(x) \to \Phi(x)$) *if and only if for every* $\epsilon > 0$

$$g_n(\epsilon) = \sum_k \int_{|x| \geqq \epsilon} x^2 \, dF_{nk} \to 0. \tag{1.1}$$

Except in special cases, the application of condition (1.1) is difficult because of the integrals involved. By assuming the existence of fourth-order moments, we are able to prove new necessary and sufficient conditions for both normal and Poisson convergence which involve only moments. The proof of the theorem makes use of a characterization of the normal distribution among infinitely divisible (ID) laws which was perhaps first recognized by Borges [1] and later independently by the author [4].

2. Normal convergence.

THEOREM 2.1. *Let* $E|S_n|^{(4+\delta)}$ *be uniformly bounded for some* $\delta > 0$. *Let* $\max_k \text{Var } X_{nk} \to 0$, $\sum_k \text{Var } X_{nk} \to \sigma^2 < \infty$ *where* σ^2 *is a positive constant. Then* S_n *is asymptotically normal if and only if*

$$ES_n^4 - 3\{ES_n^2\}^2 \to 0. \tag{2.1}$$

PROOF. The asymptotic normality of S_n implies condition (2.1) by the moment convergence theorem (see Loève [3], p. 184) and the fact that for a zero-mean normal random variable S, $ES^4 - 3\{ES^2\}^2 = 0$.

To prove the converse it is sufficient to show that every convergent subsequence $\{F_{n'}\}$ of $\{F_n\}$ converges to $\Phi(x)$ (see Feller [2], p. 261).

Received 1 July 1968.

[1] This research was done at the University of Michigan while the author was on a Post-Doctoral Research Fellowship in the Program of Information and Control Engineering.

[2] Currently with the System Sciences Department of The RAND Corporation, Santa Monica.

Let F be the limit of $F_{n'}$, then F is an infinitely divisible law with characteristic function $f(u)$ such that (see Loève [3], p. 293),

$$\log f(u) = \int (e^{iux} - 1 - iux)x^{-2}\, dK(x) \tag{2.2}$$

where $K(x)$ is monotone increasing and of bounded variation, $K(-\infty) = 0$, and $K(\infty) = \sigma^2 < \infty$. The integrand is defined by continuity at the origin. It is known that $F(x)$ is a normal law if and only if $K(x)$ increases only at $x = 0$.

Since $E|S_n|^{(4+\delta)}$ are uniformly bounded, all moments of $S_{n'}$ of order 4 or less converge to those of F. Since $ES_n^4 - 3\{ES_n^2\}^2 \to 0$,

$$\begin{aligned} 0 &= \int x^4\, dF(x) - 3\,\{\int x^2\, dF(x)\}^2 \\ &= (d^4/du^4)\log f(u)\,|_{u=0} \\ &= \int x^2\, dK(x). \end{aligned} \tag{2.3}$$

This last equation is obtained by differentiating the right hand side of equation (2.2) under the integral sign. This is justified in the following way. That

$$-(d^2/du^2)\log f(u) = \int e^{iux}\, dK(x) \tag{2.4}$$

is shown by Loève [1], p. 293. Thus the left hand side of equation (2.4) is a characteristic function. Since this characteristic function is twice differentiable, its second derivative is given by (Loève [3], p. 200)

$$(d^4/du^4)\log f(u) = \int x^2 e^{iux}\, dK(x)$$

and equation (2.3) follows. Thus $K(x)$ increases only at a $= 0$.

Finally, since $\sum_k \operatorname{Var} X_{nk} \to \sigma^2$ we have shown that $F_{n'} \to \Phi(x)$ and the proof is complete.

Condition (2.1) becomes even simpler when we note that

$$ES_n^4 - 3\{ES_n^2\}^2 = \sum_k [EX_{nk}^4 - 3\{EX_{nk}^2\}^2].$$

The left hand side is usually called the fourth cumulant of S_n. This identity says that the fourth cumulant of a sum of independent random variables equals the sum of the fourth cumulants.

If the distributions F_n are known to be infinitely divisible (ID), then moments higher than 4 are not required.

THEOREM 2.2. *If F_n are* ID, *then $F_n(x) \to \Phi(x)$ if and only if $ES_n^2 \to \sigma^2$ and*

$$ES_n^4 - 3\{ES_n^2\}^2 \to 0.$$

PROOF. The characteristic functions $f_n(u)$ are given by

$$\log f_n(u) = \int (e^{iux} - 1 - iux)x^{-2}\, dK_n(x).$$

For any $\epsilon > 0$

$$\int_{|x|>\epsilon} dK_n(x) \leqq \int x^2\epsilon^{-2}\, dK_n(x) \to 0$$

as $n \to \infty$. Thus $K_n(x)$ converges to a step function at the origin of size σ^2.

The converse is obtained from the moment convergence theorem.

3. Poisson convergence. The key features of the results above is that the fourth cumulant (i.e., $ES_n^4 - 3\{ES_n^2\}^2$) corresponds to $\int x^2\,dK(x)$ and is a good test for a jump of $K(x)$ at the origin. However, this method can be used to test for jumps at other points also. For example, the equation $\int (x-1)^2\,dK(x) = 0$ implies that $K(x)$ can only have a jump at $x = 1$. This leads to the following results. First we note that

$$\begin{aligned}\int (x-1)^2\,dK_n(x) &= ES_n^4 - 3\{ES_n^2\}^2 - 2ES_n^3 + ES_n^2 \\ &= \textstyle\sum_k [EX_{nk}^4 - 3\{EX_{nk}^2\}^2 - 2EX_{nk}^3 + EX_{nk}^2].\end{aligned}$$

THEOREM 3.1. *Let $E|S_n|^{(4+\delta)}$ be uniformly bounded for some $\delta > 0$. Then S_n is asymptotically Poisson* (i.e., $\log f_n(u) \to [\sigma^2(e^{iu} - 1) - iu\sigma^2]$) *if and only if*

$$ES_n^4 - 3\{ES_n^2\}^2 - 2ES_n^3 + ES_n \to 0.$$

In a similar way we obtain:

THEOREM 3.2. *If F_n are* ID, *then F_n is asymptotically Poisson if and only if $ES_n^2 \to \sigma^2$ and*

$$ES_n^4 - 3\{ES_n^2\}^2 - 2ES_n^3 + ES_n \to 0.$$

REFERENCES

[3] BORGES, R. (1966). A characterization of the normal distribution (A note on a paper by Kozin). *Z. Wahrsheinlichkeitstheorie und Verew. Geite.* **5** 244–246.

[2] FELLER, W. (1966). *An Introduction to Probability Theory and Its Applications* **2.** Wiley, New York.

[1] LOÈVE, M. (1963). *Probability Theory* (3rd edition). Van Nostrand, New York.

[4] PIERRE, P. (1967). Properties of non-Gaussian, continuous parameter, random processes as used in detection theory. Dissertation, The Johns Hopkins University.