The Annals of Mathematical Statistics
1969, Vol. 40, No. 1, 197–202

EQUIVALENCE OF GAUSSIAN STATIONARY PROCESSES[1]

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Let G be the real line, and $R(s, t) = r(s - t)$ and $S(s, t) = s(s - t)$, where r and s are continuous positive definite functions, and hence by Bochner's theorem can be written as Fourier transforms of finite regular Borel measures (spectral measures) $d\rho$, $d\sigma$. There are unique Gaussian measures P and Q such that the processes defined by $x_t(\omega) = \omega(t)$ have means respectively m and n and covariances respectively R and S. It is known (cf. Grenander [4]) that P and Q are equivalent if and only if

(a) ρ and σ have the identical non-atomic parts;

(b) they have the same set of atoms and if the masses are a_i and b_i then $\sum \{1 - (a_i/b_i)\}^2$ must be finite.

The purpose of this paper is to investigate the necessary and sufficient conditions for P and Q being equivalent when G is a separable locally compact group. Let G be a separable locally compact group which need not be abelian. Then every positive definite function ρ can be written as an integral of positive definite functions ρ_λ

$$\rho(t) = \int_\Lambda \rho_\lambda(t)\, d\mu$$

where Λ may be taken as the real line and μ a finite regular Borel measure. This representation, being essentially unique in the sense that we shall discuss in the paper, is called spectral representation of ρ. Theorem 2 gives the desired conditions in terms of spectral representations.

1. Decomposition of positive definite functions. Let G be a separable locally compact group; $M(G)$, the set of all finite Radon measures on G; γ, a continuous positive definite function on G. Consider the sesqui-linear functional B_γ on $M(G) \times M(G)$ defined by

$$B_\gamma(\alpha, \beta) = \int \int \gamma(s^{-1}t)\alpha(dt)\bar{\beta}(ds) \cdots . \tag{1}$$

It is clear that $B_\gamma(\alpha, \alpha) \geqq 0$ for all $\alpha \varepsilon M(G)$, and the subspace $\mathfrak{N}_\gamma = \{\alpha \varepsilon M(G) | B_\gamma(\alpha, \alpha) = 0\}$ is invariant under the left translations. Let H_γ be the Hilbert space gotten by completing $M(G)/\mathfrak{N}_\gamma$ with the inner product defined by (1). The left translations induce a unitary representation U of G on H_γ in a canonical way.

Let $\mathfrak{U}$ be the von Neumann algebra[2] generated by $\{U_s, s \varepsilon G\}$. Then there is a

Received 27 May 1968.

[1] This work was supported by NSF Grant No. GP 7474.

[2] A subalgebra of the Banach algebra of all continuous operators in a Hilbert space is called a von Neumann algebra if it is closed in the weak operator topology, and the ajoint operation. For the general theory in this section we refer to Dixmier [2].

separable locally compact space Λ, which may be taken as the real line, and a direct integral representation of H_γ (confer the appendix for definition)

$$H_\gamma = \int_\Lambda^\oplus H_\gamma(\lambda)\mu(d\lambda) \tag{2}$$

so that $\mathfrak{U}$ has central decomposition

$$\mathfrak{U} = \int_\Lambda^\oplus \mathfrak{U}(\lambda)\mu(d\lambda), \tag{3}$$

i.e., the centers of μ—a.a. algebras $\mathfrak{U}(\lambda)$ are the algebras of scalar multiples of the identity transformation on $H_\gamma(\lambda)$. Accordingly,

$$U_s = \int_\Lambda^\oplus U_s(\lambda)\mu(d\lambda) \tag{4}$$

where μ—a.a. $U_s(\lambda)$ are unitary on $H_\gamma(\lambda)$, and the cyclic vector ξ is the direct integral of cyclic vectors $\xi(\lambda)$ in $H_\gamma(\lambda)$

$$\xi = \int_\Lambda^\oplus \xi(\lambda)\mu(d\lambda). \tag{5}$$

Since the inner product on the direct integral of Hilbert spaces is defined as

$$(x, y) = \int_\Lambda (x_\lambda, y_\lambda)_\lambda \mu(d\lambda) \tag{6}$$

where $x = \int_\Lambda^\oplus x_\lambda \, d_\mu$, $y = \int_\Lambda^\oplus y_\lambda \, d_\mu$ and $(\cdot, \cdot)_\lambda$ denotes the inner product of $H(\lambda)$, it follows that

$$\begin{aligned} \gamma(s) = (U_s\xi, \xi) &= \left(\int_\Lambda^\oplus U_s(\lambda)\xi(\lambda)\, d\mu, \int_\Lambda^\oplus \xi(\lambda)\, d\mu\right) \\ &= \int_\Lambda (U_s(\lambda)\xi(\lambda), \xi(\lambda))_\lambda \, d\mu \\ &= \int_\Lambda \gamma_\lambda(s)\, d\mu \end{aligned} \tag{7}$$

where $s \longrightarrow \gamma_\lambda(s) = (U_s(\lambda)\xi(\lambda), \xi(\lambda)_\lambda)$ is positive definite for μ almost every λ. Normalizing γ_λ so that $\gamma_\lambda(e) = 1$, we obtain an integral representation

$$\gamma(s) = \int_\Lambda \gamma_\lambda(s)\, d\mu. \tag{8}$$

The change of the measure μ within the equivalence class will not change the central decomposition. Therefore the integral representation is unique.

DEFINITION. The finite regular Borel measure μ in (8) is called the central Radon measure of γ. Identity (8) is called the spectral representation of γ.

2. Gaussian stationary processes. Let G be a separable locally compact group. A real (or complex) valued stochastic process $\{X_t, t \in G\}$ is called a Gaussian process if all finite subsets of X_t's are jointly Gaussian or equivalently if every finite linear combinations of $\{X_t\}$ are Gaussian. Mean functions and covariance functions can be defined similarly as in real line case.

Let $\{x(t), t \in G\}$ be a Gaussian stationary process with mean zero and covariance function ρ;

$$H_\rho = (M(G)/N_\rho)^-$$

as defined in Section 1; and ξ, the cyclic vector in H_ρ. Let M be the Hilbert space spanned by $\{x(t), t \in G\}$ with inner product defined by $(U, V) = E(U\bar{V})$. Since

$E\{x(t)\bar{x}(s)\} = (U_t\xi, U_s\xi)$, there is an isometry $\Psi: M \to H_\rho$. According to the decomposition theory, H_ρ can be centrally decomposed as

$$\int_\Lambda^\oplus H_\rho(\lambda)\, d\mu$$

where Λ may be taken as the real line (cf. J. T. Schwartz, [5]), and μ is the central Radon measure so that

$$\rho(t) = \int_\Lambda^\oplus \rho_\lambda(t)\, d\mu$$

with $\rho_\lambda(e) = 1$, and μ—a.a. ρ_λ's are positive definite. For each positive definite function ρ_λ, there corresponds a stationary Gaussian process $\{x_\lambda(t), t \in G\}$ so that $H_\rho(\lambda)$ is isometrically isomorphic to $M(\lambda)$, the Hilbert space spanned by $\{x_\lambda(t), t \in G\}$.

Let $\psi(\lambda): M(\lambda) \to H_\rho(\lambda)$ be the isometry such that

$$\psi(\lambda)x_\lambda(t) = U_t(\lambda)\xi(\lambda) \qquad \text{and}$$

$$E\{x_\lambda(t)\bar{x}_\lambda(s)\} = \rho_\lambda(s^{-1}t) = (U_t(\lambda)\xi(\lambda), U_s(\lambda)\xi(\lambda))_\lambda .$$

It is clear that $\lambda \to M(\lambda)$ is a measurable field of Hilbert spaces. $x_\lambda(t)$'s are certainly μ-"integrable"[3] since $\lambda \to \rho_\lambda(t)$ is a μ-integrable function. We see that $\int_\Lambda^\oplus M(\lambda)\, d\mu$ is well defined and is isometrically isomorphic to $\int_\Lambda^\oplus H(\lambda)\, d\mu$ with $\int_\Lambda^\oplus x_\lambda(t)\, d\mu$ corresponding to $\int_\Lambda^\oplus U_t(\lambda)\xi(\lambda)\, d\mu$. M and $\int_\Lambda^\oplus M(\lambda)\, d\mu$ are thus isometrically isomorphic and the isometry carries $x(t)$ to $\int_\Lambda^\oplus x_\lambda(t)\, d\mu$. By identifying them, we have

$$x(t) = \int_\Lambda^\oplus x_\lambda(t)\, d\mu \tag{9}$$

and

$$\Psi = \int_\Lambda^\oplus \psi(\lambda)\, d\mu. \tag{10}$$

Definition. We say $x(t)$ in the formula (9) admits a direct integral representation.

From what we discussed above, we conclude the following theorem:

Theorem 1. *Any stationary Gaussian process on a locally compact group admits a direct integral representation.*

3. The equivalence of Gaussian stationary processes.

Definition. Two stochastic processes are said to be mutually equivalent (or perpendicular) if and only if their induced probability measures are equivalent (or totally singular respectively).

Definition. Let G be a separable locally compact group. Two positive definite functions ρ and σ are said to be equivalent ($\rho \sim \sigma$ symbolically), if there is an equivalence operator T from H_ρ onto H_σ; i.e., T is a linear homeomorphism such that $I - T^*T$ is Hilbert-Schmidt.

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[3] $f(\lambda)$ is said to be μ-"integrable" if and only if $\lambda \to (f(\lambda), (f(\lambda))_\lambda$ is μ-integrable and μ-measureable.

THEOREM 2. *Let G be a separable locally compact group. Two stationary processes $\{X(t), t \varepsilon G\}$ and $\{Y(t), t \varepsilon G\}$ with means zero, covariances ρ and σ respectively are equivalent if and only if they admit the direct integral representations*

$$x(t) = \int_{\Lambda}^{\oplus} x_{\lambda}(t)\, d, \qquad y(t) = \int_{\Lambda}^{\oplus} x_{\lambda}(t)\, d$$

where ρ and σ are the central Radon measures of ρ and σ respectively such that

(i) *μ and ν have the identical non-atomic parts, and*

(ii) *they have the same set of atoms which is countable such that*

$$\sum_{a\varepsilon A} d(a)(1 - \mu(a)/\nu(a))^2 < \infty \tag{11}$$

where $d(a)$ is the dimension of $H_{\rho}(a)$ if $H_{\rho}(a)$ is finite dimensional and ∞ otherwise, and A is the set of all atoms.

The theorem is an immediate consequence of a theorem of Feldman (cf. Feldman [3]), together with one proved by the author (cf. Chow [1]).

THEOREM. (Feldman)[4] *$\{X_t, t \varepsilon G\}$ and $\{Y_t, t \varepsilon G\}$ are mutually equivalent if and only if $\rho \sim \sigma$.*

THEOREM. (Chow) *$\rho \sim \sigma$ if and only if* (i) *there exists a locally compact Hausdorff space Λ such that both ρ and σ admit spectral representations*

$$\rho(t) = \int_{\Lambda} \rho_{\lambda}(t)\, d\mu, \qquad \sigma(t) = \int_{\Lambda} \rho_{\lambda}(t)\, d\nu;$$

(ii) *the central Radon measure μ and ν have the identical non-atomic parts*; *and* (iii) *they have the same set of atoms such that* (11) *is satisfied.*

Our result can be slightly generalized. Suppose X is a separable metric space; G, a locally compact group on X which has a dense orbit. For instance, G acts ergodicly on X and ergodic measure satisfies $\mu(O) > 0$ for every open set $O \subset X$. It is known that in such cases G possesses a dense orbit in X. It has been proved (cf. Chow [1]) that for every continuous group invariant[5] positive definite function there corresponds a positive definite function in G, and vice versa. So our theorem can be carried over by natural embedding.

4. Examples. We shall use following examples in this section to demonstrate how our results imply the known results.

EXAMPLE 1. Let G be the abelian group of real numbers with usual topology. Then the spectral representation of a position definite function is given by Bochner's theorem. The central Radon measure is nothing but the ordinary spectral measure in stationary analysis.

$$\rho(t) = \int_{-\infty}^{\infty} e^{-i\lambda t}\, d\mu(\lambda).$$

To each covariance function $e^{-i\lambda t}$, there corresponds a stationary process

$$x_{\lambda}(t) = e^{-i\lambda t}\xi_{\lambda}(0)$$

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[4] Feldman originally proved the real valued case and the theorem was stated more generally. For our purpose, we transform his theorem into complex form and state it in such a way which fits our special case.

[5] A function on $X \times X$ is said to be group invariant if $f(x_1, x_2) = f(gx_1, gx_2)$.

or as spectral representation,

$$x_\lambda(t) = \int_{-\infty}^{\infty} e^{-i\omega t}\, dZ_\lambda(\omega)$$

where $\{Z_\lambda(\omega), -\infty < \omega < \infty\}$ is a process with orthogonal increment, such that $E\{|dZ_\lambda|^2\}$ = point mass at λ. Let

$$d\xi(\omega) = \int_{(-\infty,\infty)}^{\oplus} dz_\lambda(\omega)\, d\mu(\lambda),$$

then $d\xi(\omega)$ can be identified with a stationary Gaussian process as discussed in Section 2 such that

$$E\{|d\xi(\omega)|^2\} = \int_{-\infty}^{\infty} E\{|dz_\lambda|^2\}\, d\mu(\lambda) = d\mu(\lambda).$$

By a modified argument of Fubini's theorem, it follows that

$$\begin{aligned} x(t) &= \int_{(-\infty,\infty)}^{\oplus} x_\lambda(t)\, d\mu = \int_{(-\infty,\infty)}^{\oplus} \int_{-\infty}^{\infty} e^{-i\omega t}\, dZ_\lambda(\omega)\, d\mu \\ &= \int_{-\infty}^{\infty} e^{-i\omega t} \int_{(-\infty,\infty)}^{\oplus} dZ_\lambda(\omega)\, d\mu = \int_{-\infty}^{\infty} e^{-i\omega t}\, d\xi(\omega) \end{aligned}$$

which is the classical spectral representation of the Gaussian stationary process. Theorem 2 translated into spectral language provides the exact result mentioned in the introduction.

EXAMPLE 2. Let G be a compact group, ρ, a positive definite function on G, (U, H_ρ), the canonical unitary representation of G. Then the direct integral decomposition reduces to a direct sum.

$$H_\rho = \oplus_{\lambda \varepsilon \Lambda} H_\rho(\lambda)$$

where $H_\rho(\lambda)$ is the multiple of some irreducible invariant subspace; $U_s(\lambda)$ is some multiple of irreducible representation; and μ—a.a. $t \to \rho_\lambda(t) = (U_s(\lambda)\xi(\lambda), \xi(\lambda)) \cdot \mu(\lambda)$ are positive definite. Therefore any Gaussian stationary process $\{x(t), t \varepsilon G\}$ can be decomposed as a direct sum:

$$x(t) = \sum_{\lambda \varepsilon \Lambda} x_\lambda(t)$$

which corresponds to sum $\rho(t) = \sum_{\lambda\varepsilon\Lambda} \rho_\lambda(t)$. Since $x(t)$'s are mutually orthogonal, this direct sum can actually be considered as an ordinary sum of stochastic processes. Then it follows from our theorem that two stationary processes are equivalent if and only if conditions (i), (ii) in Theorem 2 are satisfied and

(iii) $\sum d(x)(1 - \sigma\lambda(e)/\rho_\lambda(e))^2 < \infty$ where $d(\lambda)$ is the dimension of $H_\rho(\lambda)$.

APPENDIX

Let Λ be a locally compact space, and $\Sigma = \{\tilde{e}_n, 1 \leqq n \leqq \infty\}$ be a collection of disjoint Borel subsets of Λ such that $\Lambda = \bigcup_{n=1}^{\infty} \tilde{e}_n$. Let $h_1 \subseteq h_2 \subseteq \cdots \subseteq h_\infty$ be a sequence of Hilbert spaces, h_n having dimension n. The *direct integral pre-hilbert space* with Radon measure μ and dimension sets Σ is the set of all functions $f: \Lambda \to h_\infty$ such that

(i) $f(\lambda)\ \varepsilon\ h_n \subseteq h_\infty$ if $\lambda\ \varepsilon\ \tilde{e}_n$.

(A.1) (ii) f is a μ-measurable vector-valued function in h_∞,
(iii) $\int_\Lambda \|f(\lambda)\|^2 \, d\mu < \infty$, and
(iv) $(f, g) = \int_\Lambda (f(\lambda), g(\lambda))_\lambda \mu(d\lambda)$
where $(\cdot, \cdot)_\lambda$ is the inner product in h_λ.

If we identify two such functions when they differ only by a μ-null function, we then obtain a complete space with an inner product given by (A.1). We call this Hilbert space the *direct integral hilbert space*, denoted by $\int_\Lambda^\oplus h(\lambda)\mu(d\lambda)$. For any element $f \varepsilon \int_\Lambda^\oplus h(\lambda)\mu(d\lambda)$, we write $f = \int_\Lambda^\oplus f(\lambda)\mu(d\lambda)$.

Similarly, we can define the direct integral von Neumann algebra. Let $\mathfrak{A}$ be a von Neumann algebra on a Hilbert space H. Then there exists a self-adjoint bounded operator A in the center $Z_\mathfrak{A}$ of $\mathfrak{A}$ such that $Z_\mathfrak{A}$ is identical with the collection of all bounded Borel functions of A. Let Λ be the space of the spectral representation of A (which may be taken as the real line) and μ its spectral measure. Then H can be decomposed into a direct integral Hilbert space $\int_\Lambda^\oplus H(\lambda)\mu(d\lambda)$ such that $\mathfrak{A}$ is accordingly decomposed into a central decomposition $\int_\Lambda^\oplus \sigma(\lambda)\mu(d\lambda)$; i.e., $\mathfrak{A}(\lambda)$ acts on $H(\lambda)$ and μ-almost every von Neumann algebras $\mathfrak{A}(\lambda)$ has center isomorphic to the algebra of scalar multiples of identity operator on $H(\lambda)$.

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