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PARTIALLY BALANCED INCOMPLETE BLOCK DESIGNS WITH TWO-WAY CLASSIFICATION OF TREATMENTS

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1. Introduction. After Shah's [6] generalisation of the partially balanced incomplete block (PBIB) designs the author [5] gave a generalisation in a different direction. In this later generalisation, the association relation between treatments played a prominent part. This paper gives yet another genalisation of the PBIB designs. When the treatments to be administered are given, we may wish to classify them into groups with reference to each of the treatments, the classification being done on the basis of some characteristic of the treatments. Then we may insist that each of the treatments belonging to a particular group should occur together with the reference treatment of classification a particular number of times. From the point of view of easiness of analysis all such classifications of treatments are not desirable. A PBIB design insists on certain other conditions to be satisfied by these groups; for example, the constancy of the p^i_{jk} for any two treatments which are ith associates. We shall call the classification of treatments described above the classification according to association relation. After classifying treatments in this way, there still may exist some reason for classifying them into different groups independently of the former classification and then insisting that due consideration be given to this grouping in the lay-out of the design. In this paper, we shall consider one such type of PBIB design with two-way classification of treatments, in short written as PBIB (TW).

2. Notations and definition. Let α and β be two treatments. Let $\alpha \varepsilon G(a)$ denote "α belongs to group a" and $(\alpha, \beta) = l$ denote "α and β are lth associates". Then we may define our PBIB(TW) as follows:

Definition. Given v treatments we group them into g groups containing $v_1, v_2, \cdots, v_g$ treatments such that the following conditions hold:
(1) Two treatments are either 1st, 2nd, $\cdots$, or mth associates;
(2) Each treatment belonging to group a has exactly $n(a, i)$ ith associates, $a = 1, 2, \cdots, g; i = 1, 2, \cdots, m$;
(3) Given two treatments, say α and β, $\alpha \varepsilon G(a)$, $\beta \varepsilon G(b)$, $(\alpha, \beta) = l$, the number of treatments common to the cth associates of α and dth associates of β in the order mentioned is given by $p(a, b; l; c, d)$ and is independent of the treatments α and β. If $\alpha \varepsilon G(a)$, $\beta \varepsilon G(b)$, $(\alpha, \beta) \neq l$, we define $p(a, b; l; c, d) = 0$. In general, $p(a, b; l; c, d) \neq p(a, b; l; d, c)$

Given such an association scheme we shall say that a PBIB(TW) of v treatments exists if the treatments can be arranged in b blocks of size k $(k < v)$ each such that

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(i) each treatment belonging to group a occurs r_a times in the design ($a = 1, 2, \cdots, g$),

(ii) two treatments which are lth associates occur together in λ_l blocks, $l = 1, 2, \cdots, m$.

EXAMPLE. Consider the following design with parameters

$$v_1 = 15, \quad v_2 = 10, \quad b = 10, \quad r_1 = 2, \quad r_2 = 3, \quad k = 6, \quad \lambda_1 = 0,$$
$$\lambda_2 = 1, \quad n(1, 1) = 14, \quad n(1, 2) = 10, \quad n(2, 1) = 9, \quad n(2, 2) = 15.$$

$$p(1, 1; 1; i, j) = \begin{pmatrix} 7 & 6 \\ 6 & 4 \end{pmatrix}, \qquad p(1, 1; 2; i, j) = \begin{pmatrix} 10 & 4 \\ 4 & 5 \end{pmatrix},$$
$$p(2, 2; 1; i, j) = \begin{pmatrix} 2 & 6 \\ 6 & 9 \end{pmatrix}, \qquad p(2, 2; 2; i, j) = \begin{pmatrix} 3 & 6 \\ 6 & 8 \end{pmatrix},$$
$$p(1, 2; 1; i, j) = \begin{pmatrix} 4 & 9 \\ 4 & 6 \end{pmatrix}, \qquad p(1, 2; 2; i, j) = \begin{pmatrix} 6 & 8 \\ 3 & 6 \end{pmatrix}.$$

PLAN

(2.1)

2_1	4_1	10_1	3_2	7_2	6_2
4_1	6_1	12_1	5_2	9_2	8_2
3_1	5_1	11_1	4_2	8_2	7_2
5_1	7_1	13_1	6_2	10_2	9_2
7_1	9_1	2_1	8_2	12_2	11_2
8_1	10_1	3_1	9_2	13_2	12_2
9_1	11_1	4_1	10_2	1_2	13_2
10_1	12_1	5_1	11_2	2_2	1_2
11_1	13_1	6_1	12_2	3_2	2_2
13_1	2_1	8_1	1_2	5_2	4_2

ASSOCIATION RELATION. The first associates of a treatment are those not occurring with it in the same block of the design (2.1) and the second associates are those occurring with it in the same block.

GROUPS. Those treatments which occur twice in (2.1) belong to group 1 and the rest to group 2.

3. Relations between the parameters. We shall consider each treatment as the 0th associate of itself and of no other treatment. We have obviously

$$\textstyle\sum_{i=1}^{g} v_i = v, \qquad \sum_{i=1}^{g} r_i v_i = bk,$$
$$\textstyle\sum_{j=1}^{m} n(a, j) = v - 1, \qquad a = 1, 2, \cdots, g; \tag{3.1}$$
$$\textstyle\sum_{j=1}^{m} n(a, j)\lambda_j = r_a(k - 1), \qquad a = 1, 2, \cdots, g.$$

Consider $p(a, b; l; c, d)$. Let $\alpha \varepsilon G(a)$, $\beta \varepsilon G(b)$, $(\alpha, \beta) = l$. There are $n(a, c)$ members in the cth associate class of α. If $l \neq c$, β is not one of them. Hence the $n(a, c)$ treatments should occur among the associate classes of β.

$$\therefore \sum_{d=1}^{m} p(a, b; l; c, d) = n(a, c), \quad \text{if } l \neq c; \qquad = n(a, c) - 1, \quad \text{if } l = c. \tag{3.2}$$

By a similar argument we have

$$\sum_{c=1}^{m} p(a, b; l; c, d) = n(b, d), \quad \text{if } l \neq d; \qquad = n(b, d) - 1, \quad \text{if } l = d. \tag{3.3}$$

We shall call $n(a, i)$, $a = 1, 2, \cdots, g$; $i = 1, 2, \cdots, m$ and $p(a, b; l; c, d)$, $a, b = 1, 2, \cdots, g$; $l, c, d = 1, 2, \cdots, m$; the secondary parameters of the PBIB(TW). To make the discussion easier we shall assume throughout the following that $\alpha \varepsilon G(\alpha^*)$ where α is any treatment.
Define

$$B_i = (b(\alpha, \beta; i)) = \begin{pmatrix} b(1, 1; i) b(1, 2; i) \cdots b(1, v; i) \\ b(2, 1; i) b(2, 2; i) \cdots b(2, v; i) \\ b(v, 1; i) b(v, 2; i) \cdots b(v, v; i) \end{pmatrix} \tag{3.4}$$

where

$$(b(\alpha, \beta); i) = 1, \quad \text{if } (\alpha, \beta) = i, \qquad = 0, \quad \text{otherwise} \tag{3.5}$$

B_i, $i = 0, 1, 2, \cdots, m$, are called association matrices [1].
Then

$$\sum_{\alpha=1}^{v} b(\alpha, \beta; i) = n(\beta^*, i), \qquad \beta^* = 1, 2, \cdots, g;$$
$$\sum_{\beta=1}^{v} b(\alpha, \beta; i) = n(\alpha^*, i), \qquad \alpha^* = 1, 2, \cdots, g \tag{3.6}$$

$\sum_{i=0}^{m} B_i = E_{vv}$, where E_{vv} denotes a matrix of 1's of order $v \times v$.
It follows that the linear form $\sum_{i=0}^{m} c_i B_i = 0$ if and only if $c_i = 0$, $i = 0, 1, 2, \cdots, m$. It can be deduced that

$$\sum_{y=1}^{v} b(\alpha, y; i) b(y, \beta; j) = \sum_{l=0}^{m} p(\alpha^*, \beta^*; l; i, j) b(\alpha, \beta; l). \tag{3.7}$$

Equation (3.7) gives a correct expression for the element in (α, β) position of matrix $B_i B_j$, where α^* and β^* are determined by the particular treatments α and β. However, no fixed values of α^* and β^* are valid for all entries of $B_i B_j$. The arrangement of treatments into groups gives a partitioning of matrix B_i into g^2 submatrices. For the case $g = 2$ this may be represented as:

$$B_i = \left(\begin{array}{c|c} B_i^{(1,1)} & B_i^{(1,2)} \\ \hline B_i^{(2,1)} & B_i^{(2,2)} \end{array}\right).$$

Then

$$B_i B_j = \left[\begin{array}{c|c} \sum_{l=0}^{m} p(1, 1; l; i, j) B_l^{(1,1)} & \sum_{l=0}^{m} p(1, 2; l; i, j) B_l^{(1,2)} \\ \hline \sum_{l=0}^{m} p(2, 1; l; i, j) B_l^{(2,1)} & \sum_{l=0}^{m} p(2, 2; l; i, j) B_l^{(2,2)} \end{array}\right].$$

4. Analysis.

LEMMA. *Let $S_q(t_i)$ denote the sum of yields of the* qth *associates of treatment i and $S_j\{S_q(t_i)\}$ denote the sum of the* jth *associates of all the* qth *associates of treatment i. Then*

$$\begin{aligned} S_j\{S_q(t_i)\} &= \textstyle\sum_{h=1}^{m} p(\Phi^*, i^*; h; j, q) S_h(t_i), \qquad \text{if } j \neq q, \\ &= n(i^*, j) t_i + \textstyle\sum_{h=1}^{m} p(\Phi^*, i^*; h; j, q) S_h(t_i), \qquad \text{if } j = q; \end{aligned}$$

where Φ^* denotes the group to which the hth *associates of i belong, $\Phi^* = 1, 2, \cdots, g$.*

PROOF. Consider the set of jth associates of all qth associates of treatment i. Let Φ be an hth associate of i. How many times Φ occurs in this set? How many times i itself occurs?

Let $\Psi_1, \Psi_2, \cdots, \Psi_{n(i^*,q)}$ be the $n(i^*, q)$ qth associates of treatment i. Suppose Φ occurs in the set of jth associates of $\Psi_1, \Psi_2, \cdots, \Psi_x$ only, $1 \leqq x \leqq n(i^*, q)$. Then $(\Phi, \Psi_1) = (\Phi, \Psi_2) = \cdots = \cdots = (\Phi, \Psi_x) = j$ so that $\Psi_1, \Psi_2, \cdots, \Psi_x$ occur in the jth associate class of Φ. Hence the number of treatments common between the qth associate class of i and jth associates of Φ is x.

i.e. $$p(\Phi^*, i^*; h; j, q) = x, \qquad \Phi^* = 1, 2, \cdots, g.$$

case when $j \neq q$: Since $(i, \Psi_1) = \cdots = (i, \Psi_{n(i^*,q)}) = q$, i occurs in the qth associate class of each of $\Psi_1, \Psi_2, \cdots, \Psi_{n(i^*,q)}$ but not in the jth associate class. Hence if $j = q$, i occurs $n(i^*, j)$ times and it occurs 0 times if $j \neq q$. Hence the lemma.

Assuming the usual additive model for a PBIB design we get the well known reduced normal equations for the intrablock estimates of the treatment contrasts as:

$$Q_i = \textstyle\sum_{j=1}^{v} C_{ij} \hat{t}_j, \qquad i = 1, 2, \cdots, v.$$

Where

$$\begin{aligned} C_{ii} &= r_{i^*}(1 - k^{-1}) \\ C_{ij} &= -\lambda_{l/k}, \quad \text{if} \quad (i, j) = l \end{aligned}$$

i.e. $Q_i = r_{i^*}(1 - k^{-1})\hat{t}_i - k^{-1} \sum_{q=1}^{m} \lambda_q S_q(\hat{t}_i)$.

Let $i_1, i_2 \cdots, i_{n(i^*,j)}$ be the $n(i^*, j)$ jth associates of treatment i. Then

$$\begin{aligned} Q_{i_1} &= r_{i_1}^*(1 - k^{-1})\hat{t}_{i_1} - k^{-1} \textstyle\sum_q \lambda_q S_q(\hat{t}_{i_1}), \\ Q_{i_{n(i^*,j)}} &= r_{i_{n(i^*,j)}}^*(1 - k^{-1})\hat{t}_{i_{n(i^*,j)}} \\ &\quad - k^{-1} \textstyle\sum_q \lambda_q S_q(\hat{t}_{i_{n(i^*,j)}}). \end{aligned}$$

Adding

$$\begin{aligned} S_j(Q_i) &= (r_{i_1}^* + \cdots + r_{i_{n(i^*,j)}}^*)(1 - k^{-1})S_j(\hat{t}_i) \\ &\quad -k^{-1} \textstyle\sum_q \lambda_q S_j\{S_q(\hat{t}_i)\} \\ &= (r_{i_1}^* + \cdots + r_{i_{n(i^*,j)}}^*)(1 - k^{-1})S_j(\hat{t}_i) \\ &\quad -k^{-1} \textstyle\sum_{q=1}^m \lambda_q \sum_{h=1}^m p(\Phi^*, i^*; h; j, q)S_h(\hat{t}_i) \\ &\quad -k^{-1}\lambda_j n(i^*, j)\hat{t}_i, \qquad\qquad \Phi^* = 1, 2, \cdots, g, \\ &= (r_{i_1}^* + \cdots + r_{i_{n(i^*,j)}}^*)(1 - k^{-1})S_j(\hat{t}_i) \\ &\quad -k^{-1} \textstyle\sum_{q=1}^m \lambda_q \sum_{h=1}^m p(\Phi^*, i^*; h; j, q)S_h(\hat{t}_i) \\ &\quad + k^{-1}\lambda_j n(i^*, j) \textstyle\sum_{h=1}^m S_h(\hat{t}_i), \qquad \Phi^* = 1, 2, \cdots, g, \end{aligned}$$

for, we assume that

$$\hat{t}_i + \textstyle\sum_{h=1}^m S_h(\hat{t}_i) = 0 \quad \text{for all } i.$$

These equations are similar to those of the ordinary PBIB design and can be solved similarly.

5. Methods of construction. Whenever necessary, we shall distinguish the parameters of a PBIB(TW) with asterisks. The $p(a, b; l; c, d)$ parameters are omitted from the following discussions, as they can be easily obtained once the groups into which the treatments fall and the association relation between the treatments are specified.

THEOREM 5.1. *The direct combination of n* PBIB *designs having parameters* v_i, b_i, r_i, k, λ_j, n_{ij}, $i = 1, 2, \cdots, n; j = 1, 2, \cdots, m$, *leads to a* PBIB(TW) *with parameters*

$$v_i^* = v_i, b^* = \textstyle\sum_{i=1}^m b_i, r_i^* = r_i, k^* = k,$$

$$\lambda_j^* = \lambda_j, \lambda_{m+1}^* = 0, n(i, j) = n_{ij},$$

$$n(i, m + 1) = \textstyle\sum_{l \neq i=1}^m v_l, i = 1, 2, \cdots, n; \qquad j = 1, 2, \cdots, m.$$

PROOF. The association relation between treatments of the derived design. We shall assume that the treatments of the design number i are $1, 2, \cdots, v_i$ so that in the derived design i.e. in the PBIB(TW), we denote them as $1_{(i)}$. $2_{(i)} \cdots, v_{i(i)}$. If $\theta_{(x)}$ is any treatment of the derived design, then its first m associates are the same as those in design number x and the treatments of designs other than x are its $(m + 1)$th associates.

GROUPS OF THE DERIVED DESIGN. Treatments having the same suffix (i) belong to group i. The $p(a, b; l; c, d)$ parameters can be easily obtained.

The designs obtained by this theorem are not connected and are, therefore, of theoretical interest only.

THEOREM 5.2. *If there exists a resolvable* PBIB *design with parameters* v, $b = nr$, *n any positive integer, r, k,* $\lambda_1 = 0, \lambda_2, n_1, n_2$ *then we can construct a* PBIB(TW) *with parameters*

$$v_1^* = v, v_2^* = r, b^* = b, r_1^* = r, r_2^* = n, k^* = k + 1, \lambda_1^* = 0,$$
$$\lambda_2^* = \lambda_2, \lambda_3^* = 1, n(1, 1) = n_1, n(1, 2) = n_2, n(1, 3) = r,$$
$$n(2, 1) = r - 1, n(2, 2) = 0, n(2, 3) = v.$$

PROOF. Number the r replicates of the PBIB from 1 to r. Let the treatments of the PBIB be $1, 2, \cdots, v$. If $(\theta_1, \cdots, \theta_r)$ is an additional set of r distinct treatments, augment each block of replicate i, $1 \leqq i \leqq r$, by θ_i.

GROUPS OF THE DERIVED DESIGN. All treatments of the original PBIB design belong to group 1 and θ's belong to group 2.

ASSOCIATION RELATION BETWEEN TREATMENTS OF THE DERIVED DESIGN. The first and second associates of a treatment belonging to group 1 are the same as those it had in the original PBIB and θ's are its third associates. The first associates of a treatment $\theta_i \varepsilon G(2)$ are θ_j, $j \neq i, j = 1, 2, \cdots, r$. θ_i has no second associates. Its third associates are the treatments of the original PBIB design.

THEOREM 5.3. *If there exists a resolvable* PBIB *design with parameters* $v, b = nr, r, k, \lambda_1 = 0, \lambda_2 = n, n_1, n_2$ *then we can construct a* PBIB(TW) *with parameters*

$$v_1^* = v, \quad v_2^* = rt, \quad b^* = b, \quad r_1^* = r, \quad r_2^* = n, \quad k^* = k + t,$$
$$\lambda_1^* = 0, \quad \lambda_2^* = n, \quad \lambda_3^* = 1, \quad n(1, 1) = n_1, \quad n(1, 2) = n_2,$$
$$n(1,3) = rt, \quad n(2,1) = (r-1)t, \quad n(2,2) = (t-1), \quad n(2,3) = v,$$

where t is any positive integer.

The procedure of construction is similar to that of Theorem 5.2. The difference is that instead of augmenting the blocks by a single treatment, here we augment each of them by a set of t distinct treatments and different replicates with different sets of treatments.

Let the new treatments attached to blocks of the ith replicate be $\theta_{i1}, \theta_{i2}, \cdots, \theta_{it}$, $i = 1, 2, \cdots, r$.

GROUPS OF TREATMENTS OF THE DERIVED DESIGN. All treatments of the original PBIB design belong to group 1 and θ's belong to group 2.

ASSOCIATION RELATION BETWEEN THE TREATMENTS OF THE DERIVED DESIGN. The first and second associates of a treatment belonging to group 1 are the same as those it had in the original PBIB design and θ's are its third associates. The first associates of a treatment $\theta_{ij} \varepsilon G(2)$ are θ_{lk}, $l \neq i, l = 1, 2, \cdots r; k = 1, 2, \cdots t$. Its second associates are θ_{if}, $f \neq j, f = 1, 2, \cdots, t$. The third associates of θ_{ij} are the treatments of the original PBIB design.

THE METHOD OF DIFFERENCES FOR CONSTRUCTING PBIB DESIGNS WITH TWO-WAY CLASSIFICATION OF TREATMENTS. Let there be m, $m \geqq 2$, groups of treatments, a typical treatment of group i being x_i, $x = 0, 1, 2, \cdots, (t - 1)$ modulo t. Then the difference modulo t between two elements having suffix, say i, is said to be a "pure difference" of type (i, i) and the difference modulo t between two elements having distinct suffixes, say i and j, is said to be a "mixed difference" of type (i, j) [2]. Now we prove the following theorem.

THEOREM 5.4. *Let there be two groups of treatments such that treatments of group i are x_i, $x = 0, 1, \cdots, (t-1)$ modulo t, $i = 1, 2$. Let it be possible to find q initial blocks $B_1, B_2, \cdots, B_q$ each containing k, $k < t$, distinct treatments such that*

(i) *there are r_i treatments belonging to group i, $i = 1, 2$;*

(ii) *the $k(k-1)q$ differences modulo t arising from the initial blocks satisfy the conditions:*

(a) *among the $(2t-1)$ distinct differences of types $(1, 1)$ and $(1, 2)$, $n(1, 1)$ of them occur λ_1 times and $n(1, 2) = (2t-1) - n(1, 1)$ occur λ_2 times each,*

(b) *among the $(2t-1)$ distinct differences of types $(2, 1)$ and $(2, 2)$, $n(2, 1)$ of them occur λ_1 times and $n(2, 2) = (2t-1) - n(2, 1)$ occur λ_2 times each,*

(c) *$n(i, 1)$ should be equal to t for $i = 1$ or 2 such that all pure differences of type (i, i) should be among these $n(i, 1) = t$ differences, the other difference being of type (i, j), $i \neq j$, $i, j = 1, 2$. If this difference of type (i, j) is say l, then $n(j, 1)$ should be equal to 1 and this difference say f, should be of type (j, i) and such that $l + f = 0$ modulo t. Then developing these blocks modulo t we get a* PBIB(TW).

PROOF. That we have $v_1 = v_2 = t$, $b = qt$ are obvious. Because of (i) each treatment of group i occurs r_i times in the design.

Now let us consider the association relation between treatments. Let a_1, a_2, $\cdots$, $a_{n(1,1)}$ and $c_1, c_2, \cdots, c_{n(1,2)}$ be the $n(1, 1)$ and $n(1, 2)$ differences of types $(1, 1)$ and $(1, 2)$ referred to in condition (ii)-(a). Then the first and second associates of a treatment, say $\theta \varepsilon G(1)$ are given respectively by $(\theta - a_i)$ modulo t, $i = 1, 2, \cdots, n(1, 1)$ and $(\theta - c_j)$ modulo t, $j = 1, 2 \cdots, n(1, 2)$, where we give the suffix i to the treatment $(\theta - a_i)$ if a_i is a difference of type $(1, i)$ and so for $(\theta - c_j)$. The first and second associates of a treatment, say $\Phi \varepsilon G(2)$ are obtained in a similar way. The complementary nature of the two differences arising from a pair of treatments retains the symmetry of the association relation. Condition (ii)-(c) ensures the constancy of the parameters $p(a, b; l; c, d)$ in the sense that the value of this parameter is the same for all treatments $\alpha \varepsilon G(a)$, $\beta \varepsilon G(b)$, $(\alpha, \beta) = l$. For example, if $n(1, 1) = t$ we have

$$p(1, 1; 1; i, j) = \begin{pmatrix} t-2 & 1 \\ 1 & t-2 \end{pmatrix}, \qquad p(1, 1; 2; i, j) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$p(2, 2; 1; i, j) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad p(2, 2; 2; i, j) = \begin{pmatrix} 0 & 1 \\ 1 & 2t-4 \end{pmatrix},$$

$$p(1, 2; 1; i, j) = \begin{pmatrix} 0 & t-1 \\ 0 & t-1 \end{pmatrix}, \qquad p(1, 2; 2; i, j) = \begin{pmatrix} 1 & t-1 \\ 0 & t-2 \end{pmatrix}.$$

Hence the theorem.

TABLE 5.1

Type of difference	The differences arising from the blocks
(1, 1)	1, 2, 3, 4, 1, 2, 3, 4
(1, 2)	0, 1, 2, 3, 4, 4
(2, 1)	0, 1, 2, 3, 4, 1
(2, 2)	1, 2, 3, 4

EXAMPLE. The following example is given to make the theorem clear. Let $t = 5$. Then consider the initial blocks

$$(0_1\ 1_1\ 3_1),\ (0_2\ 4_2\ 0_1),\ (0_2\ 2_2\ 4_1),\ (0_1\ 1_1\ 2_2).$$

We have $r_1 = 7$, $r_2 = 5$. The 24 differences generated by the treatments of the blocks are given in Table 5.1.

The $n(1, 1) = 5$ distinct differences occurring $\lambda_1 = 2$ times are 1, 2, 3, 4 of type (1, 1) and 4 of type (1, 2). The rest of the differences 0, 2, 3, 4 of type (2, 1) and 1, 2, 3, 4 of type (2, 2) occur $\lambda_2 = 1$ times. The association scheme and plan of the design are given below:

Association Scheme

Treatment	*1st associates*	*2nd associates*
0_1	$1_1\ 2_1\ 3_1\ 4_1\ 1_2$	$0_2\ 2_2\ 3_2\ 4_2$
1_1	$0_1\ 2_1\ 3_1\ 4_1\ 2_2$	$0_2\ 1_2\ 3_2\ 4_2$
2_1	$0_1\ 1_1\ 3_1\ 4_1\ 3_2$	$0_2\ 1_2\ 2_2\ 4_2$
3_1	$0_1\ 1_1\ 2_1\ 4_1\ 4_2$	$0_2\ 1_2\ 2_2\ 3_2$
4_1	$0_1\ 1_1\ 2_1\ 3_1\ 0_2$	$1_2\ 2_2\ 3_2\ 4_2$
0_2	4_1	$0_1\ 1_1\ 2_1\ 3_1\ 1_2\ 2_2\ 3_2\ 4_2$
1_2	0_1	$1_1\ 2_1\ 3_1\ 4_1\ 0_2\ 2_2\ 3_2\ 4_2$
2_2	1_1	$0_1\ 2_1\ 3_1\ 4_1\ 0_2\ 1_2\ 3_2\ 4_2$
3_2	2_1	$0_1\ 1_1\ 3_1\ 4_1\ 0_2\ 1_2\ 2_2\ 4_2$
4_2	3_1	$0_1\ 1_1\ 2_1\ 4_1\ 0_2\ 1_2\ 2_2\ 3_2$

Plan

$0_1\ 1_1\ 3_1$	$0_2\ 4_2\ 0_1$	$0_2\ 2_2\ 4_1$	$0_1\ 1_1\ 2_2$
$1_1\ 2_1\ 4_1$	$1_2\ 0_2\ 1_1$	$1_2\ 3_2\ 0_1$	$1_1\ 2_1\ 3_2$
$2_1\ 3_1\ 0_1$	$2_2\ 1_2\ 2_1$	$2_2\ 4_2\ 1_1$	$2_1\ 3_1\ 4_2$
$3_1\ 4_1\ 1_1$	$3_2\ 2_2\ 3_1$	$3_2\ 0_2\ 2_1$	$3_1\ 4_1\ 0_2$
$4_1\ 0_1\ 2_1$	$4_2\ 3_2\ 4_1$	$4_2\ 1_2\ 3_1$	$4_1\ 0_1\ 1_2$

The parameters of the above design are:

$v_1 = v_2 = 5$, $b = 20$, $r_1 = 7$, $r_2 = 5$, $k = 3$, $\lambda_1 = 2$, $\lambda_2 = 1$, $n(1,1) = 5$, $n(1,2) = 4$, $n(2,1) = 1$, $n(2,2) = 8$.

$$p(1, 1; 1; i, j) = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}, \quad p(1, 1; 2; i, j) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$p(2, 2; 1; i, j) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad p(2, 2; 2; i, j) = \begin{pmatrix} 0 & 1 \\ 1 & 6 \end{pmatrix},$$

$$p(1, 2; 1; i, j) = \begin{pmatrix} 0 & 4 \\ 0 & 4 \end{pmatrix}, \quad p(1, 2; 2; i, j) = \begin{pmatrix} 1 & 4 \\ 0 & 3 \end{pmatrix}.$$

THEOREM 5.5. *If there exists a* BIB *design with parameters* v, b, r, k, λ *then we can always construct a* PBIB(TW) *having parameters*

$$v^* = v + k, \quad b^* = b + \lambda, \quad r_1^* = \lambda, \quad r_2^* = r,$$

$$\lambda_1^* = 0, \quad \lambda_2^* = \lambda, \quad n(1, 1) = v, \quad n(1, 2) = k - 1,$$

$$n(2, 1) = k, \quad n(2, 2) = (v - 1).$$

PROOF. Let the treatments of the BIB design be $1, 2, \cdots, v$. Then, if $(\theta_1, \theta_2, \cdots, \theta_k)$ is an additional set of k distinct treatments, add the block $(\theta_1, \theta_2, \cdots, \theta_k)\lambda$ times to the BIB design. The resulting design is the required one.

GROUPS OF TREATMENTS. We consider $\theta_1, \theta_2, \cdots \theta_k$ as forming group 1 and treatments $1, 2, \cdots v$, forming group 2.

ASSOCIATION RELATION. The first associates of a treatment belonging to group i, $i = 1, 2$, are all those belonging to group j, $j \neq i = 1, 2$. The second associates of a treatment belonging to group i, $i = 1, 2$, are all the other treatments of the same group.

This theorem is a special case of Theorem 5.1 except that the 1st and 2nd associates are reversed.

THEOREM 5.6. *If from a* PBIB *design obtained by dualising the* BIB *design with parameters* [7]$v^* = rk - k + 1, b^* = (rk - k + 1)k/r^{-1}, r^* = k, k^* = r, \lambda^* = 1$ we omit the r blocks in which a particular treatment θ occurs, then the resulting design is always a PBIB(TW).

PROOF. The PBIB design so obtained has $k(r - 1) + 1$ blocks in it and has $\lambda_1 = 0, \lambda_2 = 1$ so that any two blocks have one treatment in common.

We shall denote the original PBIB design by D and the derived design obtained from it by omitting the r blocks as D^*.

GROUPS OF TREATMENTS OF D^*. Those treatments which occurred in the omitted set of blocks of D, except θ, constitute the first group of treatments and the rest of the treatments of D^* constitute the second group.

ASSOCIATION RELATION BETWEEN TREATMENTS OF D^*. The discussion about the associate classes of a treatment α in D^* shall be carried out in terms of those of α in D. Thus we shall be talking about the changes that have taken place in the associate classes of α in passing from D to D^*.

Two treatments which occur together in the same block of D are considered as 2nd associates and two treatments which do not occur together in D are considered as first associates. Hence the second associates of θ form group 1 and the first associates of it form group 2. Hence (in passing from D to D^*) to the 1st associates of a treatment belonging to group 1, $(k - 2)$ additional first associates are added which are taken away from its second associates. Also, θ is taken away from the 2nd associates of group 1 and from the 1st associates of group 2 treatments.

$$\therefore \quad \begin{aligned} n(1, 1) &= (k - r)(r - 1)(k - 1)r^{-1} + (k - 2), \\ n(1, 2) &= r(k - 1) - (k - 2) - 1, \\ n(2, 1) &= (k - r)(r - 1)(k - 1)r^{-1} - 1, \\ n(2, 2) &= r(k - 1). \end{aligned}$$

We shall now determine $p(a, b; l; c, d)$'s.

Let $\alpha \varepsilon G(1)$. Consider those group 1 treatments which were its 2nd associates in D, but are now its 1st associates in D^*. There are $(k - 2)$ of them. Let β be one such. This implies that α and β had occurred together in one of the omitted blocks so that the set of $(k - 2)$ newly acquired first associates of α and the set of $(k - 2)$ newly acquired first associates of β have $(k - 3)$ treatments in common. Hence as far as two such treatments are concerned, $p(1, 1; 1; 1, 1) = p_{11}^2 + (k - 3)$.

Now let γ be a first associate of α in D, $\gamma \varepsilon G(1)$, which means that γ is not a newly acquired 1st associate of α. Since $(\alpha, \gamma) = 1$, the pair (α, γ) never occur and also since $\gamma \varepsilon G(1)$, a block in which γ occurred has been deleted from D. Hence the newly acquired sets of 1st associates of α and γ have no treatment in common. But the newly acquired 1st associates of γ have $\{(k - 2) - (r - 1)\}$ treatments common with the 1st associates of α in D and so for those of α to γ. Hence as far as two such treatments are concerned $p(1, 1; 1; 1, 1) = p_{11}^1 + 2\{(k - 2) - (r - 1)\}$. For $p(1, 1; 1; 1, 1)$ to satisfy condition (3) of the definition we must, therefore, have

$$(5.1) \qquad \begin{aligned} p_{11}^2 + (k - 3) &= p_{11}^1 + 2\{(k - 2) - (r - 1)\} \\ &= p(1, 1; 1; 1, 1). \end{aligned}$$

Now let $\delta \varepsilon G(1)$ and $(\alpha, \delta) = 2$. This implies that a block involving δ has been omitted from D and that in that block α and δ did not occur together. But they have to occur together as $(\alpha, \delta) = 2$. Hence their newly acquired sets of 1st associates have no treatment in common; but the newly acquired set of first associates of δ has $\{(k - 2) - (r - 2)\}$ treatments common with the 1st associates of α in D and so for those of α to δ. Hence for two such treatments

$$(5.2) \qquad p(1, 1; 2; 1, 1) = p_{11}^2 + 2\{(k - 2) - (r - 2)\}.$$

By a similar line of arguments it follows that we must have

$$p_{22}^2 - (k - 3) - 1 = p_{22}^1 - 2(r - 1) - 1 = p(1, 1; 1; 2, 2),$$

$$p_{12}^2 = p_{12}^2 + (r - 1) - (r - 1) = p(1, 1; 1; 1, 2),$$

$$\begin{aligned} p(1, 1; 2; 2, 2) &= p_{22}^2 - 2(r - 2) - 1, \\ p(1, 1; 2; 1, 2) &= p_{12}^2 + (r - 2) - \{(k - 2) - (r - 2)\}, \\ p(1, 2; 2; 1, 1) &= p_{11}^2 + \{(k - 2) - (r - 1)\}, \\ p(1, 2; 1; 1, 1) &= p_{11}^1 + \{(k - 2) - r\} \\ p(1, 2; 2; 2, 2) &= p_{22}^2 - (r - 1), \end{aligned}$$

$$(5.3)\qquad \begin{aligned} p(1,2;1;2,2) &= p^1_{22} - r, \\ p(1,2;2;1,2) &= p^2_{12} + (r-1), \\ p(1,2;2;2,1) &= p^2_{12} - \{(k-2)-(r-1)\} - 1, \\ p(1,2;1;1,2) &= p^1_{12} + r, \\ p(1,2;1;2,1) &= p^1_{12} - \{(k-2)-r\} - 1, \\ p(2,2;1;i,j) &= \begin{pmatrix} p^1_{11}-1 & p^1_{12} \\ p^1_{12} & p^1_{22} \end{pmatrix}, \\ p(2,2;2;i,j) &= \begin{pmatrix} p^2_{11}-1 & p^2_{12} \\ p^2_{21} & p^2_{22} \end{pmatrix}. \end{aligned}$$

Shrikhande [7] has proved that for the PBIB design, i.e. D, mentioned in the theorem

$$p^1_{ij} = \begin{pmatrix} (k-r)^2 + 2(r-1) - k(k-1)r^{-1} & r(k-r-1) \\ r(k-r-1) & r^2 \end{pmatrix}$$

$$p^2_{ij} = \begin{pmatrix} (r-1)(k-r)(k-r-1)r^{-1} & (r-1)(k-r) \\ (r-1)(k-r) & (k-2)+(r-1)^2 \end{pmatrix}.$$

Substituting these values of p^i_{jk} in relations (5.1) and (5.3) we see that they are satisfied. In fact, we have in D^*

$$v_1 = n_2 = r(k-1), \qquad v_2 = n_1 = (k-r)(r-1)(k-1)r^{-1},$$
$$b = rk - 2r + 1, \qquad r_1 = (r-1), \qquad r_2 = r.$$

Hence the theorem.

EXAMPLES. The design (2.1) given in Section 2 was obtained from design number (1) of Shrikhande.

6. Some construction methods for Shah's intra- and inter-group partially balanced (IIPB) designs. Below we shall give two methods of construction of the IIPB [6] designs. In the discussion we shall follow the notations used by Shah.

THEOREM 6.1. *If there exists a Group Divisible* (GD) *design with parameters* $v, b, r, k, \lambda_1 = 0, \lambda_2, m, n$ *where* n *is even, then we can construct IIPB designs with parameters*

$$v_1^* = \tfrac{1}{2}nt, \quad v_2^* = (m-t)n, \quad b^* = b, \quad r_1^* = 2r, \quad r_2^* = r,$$
$$k^* = k, \quad \lambda_{11:1} = 0, \quad \lambda_{11:2} = 4\lambda_2, \quad \lambda_{12:1} = 2\lambda_2, \quad \lambda_{22:1} = 0,$$
$$\lambda_{22:2} = \lambda_2, \quad n_{11:1} = \tfrac{1}{2}n - 1, \quad n_{11:2} = \tfrac{1}{2}n(t-1), \quad n_{12:1} = (m-t)n,$$
$$n_{21:1} = \tfrac{1}{2}nt, \quad n_{22:1} = (n-1), \quad n_{22:2} = (m-t-1)n,$$

where t *is any positive integer such that* $1 < t < m$.

PROCEDURE. Let the treatments occurring in the ith row of the GD scheme be

$$(i-1)n, (i-1)n+1, \cdots, (i-1)n+(n-1), \quad i = 1, 2, \cdots, m.$$

Replace the treatments of the GD design,

$$(i-1)n+\tfrac{1}{2}n, \qquad (i-1)n+\tfrac{1}{2}n+1, \cdots, (i-1)n+(n-1),$$

respectively by the treatments:

$$(i-1)n, \qquad (i-1)n+1, \cdots, (i-1)\,n+\tfrac{1}{2}n-1$$

for t distinct values of i, say for $i = i_1, i_2, \cdots i_t$. Then the resulting design is the required one.

The association relation between the treatments of the resulting design and the groups into which the treatments are divided are discussed below.

GROUPS. The first group consists of treatments

$$(i-1)n, \qquad (i-1)n+1, \cdots, \qquad (i-1)n+\tfrac{1}{2}n-1, \qquad i = i_1, \quad i_2, \cdots, i_t.$$

The second group consists of treatments

$$(i-1)n, \qquad (i-1)n+1, \cdots, \qquad (i-1)n+(n-1), \qquad i \neq i_1, i_2, \cdots, i_t, \qquad i = 1, 2, \cdots, \quad m.$$

ASSOCIATION RELATION. For convenience, let us call a row of the GD scheme a subgroup. Let α and β be any two treatments. If $\alpha \varepsilon G(i)$, $\beta \varepsilon G(i)$ and if α and β belong to the same subgroup we define $(\alpha, \beta) = ii{:}1$, and if they belong to different subgroups we define $(\alpha, \beta) = ii{:}2$, $i = 1, 2$. If $\alpha \varepsilon G(1)$, $\beta \varepsilon G(2)$ or vice versa, $(\alpha, \beta) = 12{:}1$. Hence the theorem.

CONSTRUCTION OF IIPB DESIGNS FROM THE COMPLEMENTS OF PBIB(TW). Let N be the incidence matrix of a PBIB(TW) having parameters v_i, b, r_i, k, λ_j, $i = 1, 2, \cdots, g; j = 1, 2, \cdots, m$. Then we can partition N as:

$$N = \begin{bmatrix} N_1 \\ N_2 \\ \vdots \\ N_g \end{bmatrix}.$$

Where N_i is of order $v_i \times b$, and where each row total of N_i is r_i. If $N^* = (E_{vb} - N)$ and if we denote the (i, s)th matrix element of $N^*N^{*\prime}$ by d_{is} we have

$$N^*N^{*\prime} = (d_{is}) = (b - r_i - r_s)E_{v_iv_s} + N_iN_s', \qquad i, s = 1, 2, \cdots g.$$

Now, the elements of the matrix N_iN_s' give the number of times a treatment from the ith and another from the sth group occur together, and this is λ_l if they are lth associates. Hence if α and β are any two treatments of the PBIB(TW) and if $\alpha \varepsilon G(i)$, $\beta \varepsilon G(s)$, $(\alpha, \beta) = l$, then they occur together in the complementary design $(b - r_i - r_s + \lambda_l)$, $i, s = 1, 2, \cdots, g$, $l = 1, 2, \cdots, m$ times and each of them occurs $(b - r_i)$ and $(b - r_s)$ times respectively. Hence if we preserve the grouping of treatments as in the PBIB(TW), define $(\alpha, \beta) = is{:}l$ if $\alpha \varepsilon G(i)$, $\beta \varepsilon G(s)$ or vice versa and $(\alpha, \beta) = l$ in the PBIB(TW), and in addition if the parameters $q(ij{:}t, i_1j_1{:}t_1, i_2j_2{:}t_2)$ satisfy condition (iii)-(c) of Shah's definition, then the complement of a PBIB(TW) gives IIPB design. It can be easily seen that the complement of a PBIB(TW), the association scheme of which can be written in arrays forming a geometrical configuration, like GD scheme etc., [3] satisfies all the above conditions and thus gives IIPB design. Thus, the complements of the PBIB(TW)'s obtained by Theorem 5.5 are of Shah's IIPB type.

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