The Annals of Mathematical Statistics
1969, Vol. 40, No. 1, 51–65

FURTHER REMARKS ON TOPOLOGY AND CONVERGENCE IN SOME ORDERED FAMILIES OF DISTRIBUTIONS

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0. Introduction. In [3] three different types of order relations between distribution functions on the real line were discussed:

(1) $F_1 \leqq F_2$ iff $F_2(t) \leqq F_1(t)$ for all t;
(2) $F_1 \leqq F_2$ iff both
(2′) $F_2(t)/F_1(t)$ is nondecreasing in t, and
(2″) $(1 - F_2(t))/(1 - F_1(t))$ is nondecreasing in t;
(3) $F_1 \leqq F_2$ iff $F_1(t) = 0$ implies $F_2(t) = 0$ and $[F_2(t'') - F_2(t')]/[F_1(t'') - F_1(t')]$ is nondecreasing in both variables t' and t'' whenever $F_1(t') < F_1(t'')$.

It was shown that the order relations (1), (2), (3) are of increasing stringency.

The study of families which are ordered (1) or (2) or (3) is justified by the prominent role which monotone likelihood ratio families play in statistical theory and the fact that monotonicity of likelihood ratios is equivalent to order (3).

Let two distance functions d', d'' be defined by

$$d'(F_1, F_2) = \sup\{|F_1(t) - F_2(t)|: -\infty < t < \infty\},$$

$$d''(F_1, F_2) = \sup\{|P_1(B) - P_2(B)|: B \varepsilon \mathfrak{B}\},$$

where $\mathfrak{B}$ is the Borel-algebra over the real line and P_i the p-measures pertaining to F_i.

In [3] it was shown that $d'(F_1, F_2) \leqq d''(F_1, F_2) \leqq 2(d'(F_1, F_2))^{\frac{1}{2}}$ if $F_1 \leqq F_2$ or $F_2 \leqq F_1$ in the sense of order (2).

Hence the topologies pertaining to these two metrics are equivalent in this case. In view of Lemma (3.1) this immediately implies that the strong and the uniform topology are identical for families which are ordered (2) (Proposition 1.1). It is the purpose of the first part of this paper to extend this result to other topologies. It will be shown that for any family which is ordered (2), the weak and strong neighborhood systems of any nondegenerate distribution are identical (Theorem 1.3). Furthermore it will be shown that for any family of distributions which is ordered (1), the weak topology is equivalent to the induced interval topology. The section concludes with some propositions on connected families.

In the second part of this paper it will be shown that for a dominated family of distributions which is ordered (2), there exists a system of densities such that strong convergence of measures implies convergence of the densities everywhere

Received 22 December 1967; revised 20 May 1968.

with the possible exception of the two boundary points of the convex support of the limit measure (Theorem 2.1). Moreover, for monotone likelihood ratio families by an appropriate choice of densities we may achieve that the likelihood ratios are nondecreasing everywhere (Theorem 2.10). Finally it is shown that for a family of mutually absolutely continuous distributions which is ordered (2), having continuous densities with respect to a nonatomic σ-finite measure which is positive on each interval, strong convergence implies pointwise convergence of the continuous densities (Theorem 2.12).

The basic concepts of [3] will be assumed to be known. We remark that for any pair of functions $f/\mathbb{R}$ and $g/\mathbb{R}$ we consider the quotient $f(t)/g(t)$ as undefined if $f(t) = g(t) = 0$. We put $f(t)/g(t) = \infty$ if $f(t) > 0$ and $g(t) = 0$.

1. Strong, weak and order topologies. Let F be a distribution function and P the pertaining p-measure: $F(t) = P(-\infty, t]$. In the following it will be convenient to use also the function $F^*(t) = P[t, \infty)$. The properties of F^* are dual to the properties of F. Using F and F^*, the properties specifying "order (2)" can be brought into the following convenient form:

$$F_1 \leqq F_2 \text{ iff } t' < t'' \text{ implies} \begin{cases} F_2(t')F_1(t'') \leqq F_1(t')F_2(t'') & (2') \\ F_2^*(t')F_1^*(t'') \leqq F_1^*(t')F_2^*(t'') & (2''). \end{cases}$$

(That the second line is equivalent to $(2'')$ as defined above follows easily from continuity considerations.)

Let $\mathfrak{Q}$ be the family of *all* distribution functions. The order relation (k) defines a partial order on $\mathfrak{Q}$. Furthermore, let $\mathfrak{F} \subset \mathfrak{Q}$ be a linearly ordered (k) subset, where k is one of the numbers 1, 2, 3.

We define two strong topologies:

Let $\mathfrak{I}'$ be the topology generated in $\mathfrak{F}$ by the subbase $\{\{F \varepsilon \mathfrak{F} : a < F(t) < b\}, \{F \varepsilon \mathfrak{F} : c < F^*(t) < d\} : -\infty < t < \infty,\ a, b, c, d \varepsilon \mathbb{R}\}$ and $\mathfrak{I}''$ the topology generated by the subbase

$$\{\{F \varepsilon \mathfrak{F} : P(B) < a\} : B \varepsilon \mathfrak{B}, 0 < a \leqq 1\}.$$

Let furthermore $\mathfrak{I}_{d'}$ and $\mathfrak{I}_{d''}$ be the topologies pertaining to the metrics d' and d'', respectively.

From these definitions together with Lemma 3.1. we obtain for arbitrary families $\mathfrak{F}$

$$\mathfrak{I}_{d'} = \mathfrak{I}' \subset \mathfrak{I}'' \subset \mathfrak{I}_{d''}.$$

For families which are ordered (2) we have $\mathfrak{I}_{d'} = \mathfrak{I}_{d''}$ ([3], p. 1220, Theorem 1). This proves the following

(1.1) PROPOSITION. *If a family of distribution functions is ordered* (2), *we have* $\mathfrak{I}_{d'} = \mathfrak{I}' = \mathfrak{I}'' = \mathfrak{I}_{d''}$.

Now we define the weak topology $\mathfrak{I}_W$ on $\mathfrak{F}$ by the following local subbase for $F_0 \varepsilon \mathfrak{F}$:

$$\{\{F \varepsilon \mathfrak{F} : |F(t) - F_0(t)| < \epsilon\} : \epsilon > 0, t \varepsilon \mathbb{R} \text{ with } P_0\{t\} = 0\}.$$

An equivalent subbase may be defined using F^* instead of F. We remark that $\mathfrak{J}_W$ is metrizable by the Lévy-metric

$$(1.2)\qquad \rho(F_1, F_2) = \inf\{h \varepsilon \mathbb{R}: F_1^*(t+h) - h \leqq F_2^*(t) \leqq F_1^*(t-h) + h \text{ for all } t \varepsilon \mathbb{R}\}.$$

(1.3) REMARK. If $\mathfrak{F}$ is ordered (1), ρ is monotone in the following sense:

$$F_1 \leqq F_2 \leqq F_3 \text{ implies } \rho(F_1, F_2) \leqq \rho(F_1, F_3) \quad\text{and}\quad \rho(F_2, F_3) \leqq \rho(F_1, F_3).$$

As any of the subbase elements belongs to $\mathfrak{J}'$, we have $\mathfrak{J}_W \subset \mathfrak{J}'$.

(1.4) THEOREM. *If a family of distribution $\mathfrak{F}$ is ordered (2) and $F_0 \varepsilon \mathfrak{F}$ is nondegenerate, we have $\mathfrak{J}_W(F_0) = \mathfrak{J}'(F_0)$ where $\mathfrak{J}_W(F_0)$ and $\mathfrak{J}'(F_0)$ are the neighborhood systems of F_0 with respect to $\mathfrak{J}_W$ and $\mathfrak{J}'$, respectively.*

PROOF. As $\mathfrak{J}_W \subset \mathfrak{J}'$ we have $\mathfrak{J}_W(F_0) \subset \mathfrak{J}'(F_0)$. In order to prove the inverse conclusion, it suffices to show that to each

$$U \varepsilon \{\{F \varepsilon \mathfrak{F}: |F(t) - F_0(t)| < a\}, \{F \varepsilon \mathfrak{F}: |F^*(t) - F_0^*(t)| < b\}: t \varepsilon \mathbb{R}, a, b \varepsilon \mathbb{R}_+\}$$

(the local subbase of F_0 with respect to $\mathfrak{J}'$) there exists $V \varepsilon \mathfrak{J}_W$ such that $F_0 \varepsilon V \subset U$.

If $P_0\{t_0\} = 0$, this is trivial, for then $U \varepsilon \mathfrak{J}_W$. If $P_0\{t_0\} > 0$, we also have $F_0(t_0) > 0$. Let

$$U = \{F \varepsilon \mathfrak{F}: |F(t_0) - F_0(t_0)| < a\}.$$

(i) If there exists $s < t_0$ with $F_0(s) > 0$, there also exists $s_1 < t_0$ with $P_0\{s_1\} = 0$ and $F_0(s_1) > 0$. Let

$$V_1 = \{F \varepsilon \mathfrak{F}: |F(s_1) - F_0(s_1)| < a \cdot F_0(s_1)/F_0(t_0)\}.$$

We have $V_1 \varepsilon \mathfrak{J}_W$. It remains to show that $V_1 \subset U$.

If $F \varepsilon V_1$, $F > F_0$, order (2′) implies

$$1 \geqq F(t_0)/F_0(t_0) \geqq F(s_1)/F_0(s_1) > 1 - a/F_0(t_0)$$

and therefore $F_0(t_0) \geqq F(t_0) > F_0(t_0) - a$, i.e. $F \varepsilon U$.

If $F \varepsilon V_1$, $F < F_0$, order (2′) implies

$$1 \leqq F(t_0)/F_0(t_0) \leqq F(s_1)/F_0(s_1) < 1 + a/F_0(t_0)$$

and therefore $F_0(t_0) \leqq F(t_0) < F_0(t_0) + a$, i.e. $F \varepsilon U$.

(ii) If $F_0(s) = 0$ for all $s < t_0$, we have $F_0(t_0) < 1$ (for otherwise F_0 were degenerate). As F_0 is right continuous, there exists $s_2 > t_0$, $P_0\{s_2\} = 0$, such that $F_0(s_2) < 1$ which implies $F_0^*(s_2) > 0$. Let

$$V_2 = \{F \varepsilon \mathfrak{F}: |F^*(s_2) - F_0^*(s_2)| < \tfrac{1}{2}a \cdot F_0^*(s_2)\}.$$

We have $V_2 \varepsilon \mathfrak{Z}_W$. It remains to show that $V_2 \subset U$. If $F \varepsilon V_2$, $F > F_0$, order (2″) implies for each $t \varepsilon (t_0, s_2)$

$$F^*(t)/F_0^*(t) \leqq F^*(s_2)/F_0^*(s_2) < 1 + a/2$$

$$= 1 + a/2 \cdot F_0^*(t_0) < 1 + a/2 \cdot F_0^*(t).$$

Hence $F^*(t) - F_0^*(t) = F_0(t-0) - F(t-0) < a/2$ for all $t \varepsilon (t_0, s_2)$ and therefore $F_0(t_0) - F(t_0) \leqq a/2$, i.e. $F \varepsilon U$.

If $F \varepsilon V_2$, $F < F_0$, order (2″) implies for each $t \varepsilon (t_0, s_2)$

$$F^*(t)/F_0^*(t) \geqq F^*(s_2)/F_0^*(s_2) > 1 - a/2$$

$$= 1 - a/2 \cdot F_0^*(t_0) > 1 - a/2 \cdot F_0^*(t).$$

Hence $F_0^*(t) - F^*(t) < a/2$ for all $t \varepsilon (t_0, s_2)$ which implies $F(t_0) - F_0(t_0) \leqq a/2$. Therefore $F \varepsilon U$. For $\tilde{U} = \{F \varepsilon \mathfrak{F} : |F^*(t_0) - F_0^*(t_0)| < a\}$ the assertion follows similarly.

Remark. The following example shows that the assumption of F_0 being nondegenerate cannot be dispensed with: For $n \varepsilon \mathbb{N}$, let

$$F_n(t) = 0, \quad t \leqq 0,$$
$$= t^n, \quad 0 < t < 1,$$
$$= 1, \quad t \geqq 1.$$

Let furthermore

$$F_0(t) = 0, \quad t < 1,$$
$$= 1, \quad t \geqq 1.$$

The family $\mathfrak{F} = \{F_1, F_2, \cdots, F_0\}$ is ordered (2) by $F_1 < F_2 < \cdots < F_0$. We have $\{F \varepsilon \mathfrak{F} : |F^*(1) - F_0^*(1)| < \frac{1}{2}\} = \{F_0\} \varepsilon \mathfrak{Z}'(F_0)$, but $\{F_0\} \not\varepsilon \mathfrak{Z}_W(F_0)$, for otherwise $\{F \varepsilon \mathfrak{F} : \rho(F, F_0) < \epsilon\} = \{F_0\}$ for some $\epsilon > 0$, which contradicts the fact that $(F_n)_{n \varepsilon N}$ converges to F_0 pointwise and hence also with respect to the metric ρ.

In the following we shall give two theorems on the existence of infima and suprema for ordered families of distribution.

(1.5) Proposition. *Let $\mathfrak{F}$ be a family of distribution functions which is ordered* (1). *If there exists a distribution function F_1 (F_2) such that $F < F_1$ ($F > F_2$) for all $F \varepsilon \mathfrak{F}$, then there exists a distribution function G_1 (G_2), which is the $\mathfrak{Z}_W$-supremum ($\mathfrak{Z}_W$-infimum) of $\mathfrak{F}$.*

Proof. For all $t \varepsilon \mathbb{R}$, let $G_1(t) = \inf \{F(t) : F \varepsilon \mathfrak{F}\}$. G_1 is the infimum of nondecreasing and upper semicontinuous functions and therefore nondecreasing and upper semicontinuous itself.

We have for all $t \varepsilon \mathbb{R}$ and all $F \varepsilon \mathfrak{F}$:

$$F_1(t) \leqq G_1(t) \leqq F(t).$$

Hence $\lim_{t\to-\infty} G_1(t) = 0$, $\lim_{t\to+\infty} G_1(t) = 1$. This implies the existence of some p-measure, say P_1, such that $G_1(t) = P_1(-\infty, t]$ for all $t \varepsilon \mathbb{R}$.

To establish that G_1 belongs to the $\mathfrak{J}_W$-closure of $\mathfrak{F}$, it suffices to show that for any $\epsilon > 0$, any $n \varepsilon \mathbb{N}$ and any $t_i \varepsilon \mathbb{R}$, $i = 1, \cdots, n$, there exists $F \varepsilon \mathfrak{F}$ such that $|F(t_i) - G_1(t_i)| < \epsilon$ for $i = 1, \cdots, n$. By definition of G_1, to any $i = 1, \cdots, n$, there exists $F_i \varepsilon \mathfrak{F}$ such that $0 \leqq F_i(t_i) - G_1(t_i) < \epsilon$. The assertion now holds with $F = \max\{F_1, \cdots, F_n\}$.

The proof for the existence of G_2 runs similarly, starting from the definition $G_2(t) = \inf\{F^*(t):F \varepsilon \mathfrak{F}\}$.

If the family is ordered (2), the assertion of Proposition (1.5) can be sharpened as follows:

(1.6) PROPOSITION. *Let $\mathfrak{F}$ be a set of distribution functions which is ordered* $(2')[(2'')]$. *Then we have either* $\inf\{F(t):F \varepsilon \mathfrak{F}\} = 0$ $[\sup\{F(t): F \varepsilon \mathfrak{F}\} = 1]$ *for all $t \varepsilon \mathbb{R}$ or there exists a distribution function G_1 $[G_2]$ which is the $\mathfrak{J}_W$-supremum ($\mathfrak{J}_W$-infimum) of $\mathfrak{F}$.*

PROOF. For all $t \varepsilon \mathbb{R}$, let $G_1(t) := \inf\{F(t):F \varepsilon \mathfrak{F}\}$. The proof is the same as that for Proposition (1.5) with the following exception: We have to show that the existence of $t_1 \varepsilon \mathbb{R}$ with $\inf\{F(t_1):F \varepsilon \mathfrak{F}\} > 0$ implies $\lim_{t\to\infty} G_1(t) = 1$. Let $\epsilon > 0$ and choose $F_\epsilon \varepsilon \mathfrak{F}$ such that $(1 - \epsilon/3)F_\epsilon(t_1) \leqq G_1(t_1)$. Furthermore, we choose $t_\epsilon > t_1$ such that $F_\epsilon(t_\epsilon) \geqq 1 - \epsilon/3$ and $G_\epsilon \varepsilon \mathfrak{F}$ such that $F_\epsilon \leqq G_\epsilon$ and $(1 - \epsilon/3)G_\epsilon(t_\epsilon) \leqq G_1(t_\epsilon)$. Then

$$1 - \epsilon/3 \leqq G_1(t_1)/F_\epsilon(t_1) \leqq G_\epsilon(t_1)/F_\epsilon(t_1) \leqq G_\epsilon(t_\epsilon)/F_\epsilon(t_\epsilon)$$
$$\leqq [G_1(t_\epsilon)/(1 - \epsilon/3)]/(1 - \epsilon/3).$$

Hence $1 - \epsilon \leqq (1 - \epsilon/3)^3 \leqq G_1(t_\epsilon)$. As $\epsilon > 0$ was arbitrary, this completes the proof.

The proof for the second assertion proceeds correspondingly.

In addition to the topologies studied so far we shall consider the interval topology, say $\mathfrak{J}_I$, induced in $\mathfrak{F}$. This topology is defined by the subbase

(1.7) $$\{\{F \varepsilon \mathfrak{F}:F > F_1\}, F \varepsilon \mathfrak{F}:F < F_2\}: F_i \varepsilon \mathfrak{Q}, \quad i = 1, 2\}.$$

We remark that this topology is in general finer than the interval topology of $\mathfrak{F}$ (which is defined with the additional restriction $F_i \varepsilon \mathfrak{F}$, $i = 1, 2$.)

(1.8) PROPOSITION: $\mathfrak{J}_I = \mathfrak{J}_W$ *for any family of distributions which is ordered* (1).

PROOF. $\mathfrak{J}_I \subset \mathfrak{J}_W$: We have to show that for any $F_0 \varepsilon \mathfrak{F}$, $F_1 \varepsilon \mathfrak{Q}$ there exists $U \varepsilon \mathfrak{J}_W$ with $F_0 \varepsilon U \subset \{F \varepsilon \mathfrak{F}:F < F_1\}$. As ρ is monotone (see Remark (1.3)), this holds true with $U = \{F \varepsilon \mathfrak{F}:\rho(F_0, F) < \rho(F_0, F_1)\}$.

$\mathfrak{J}_W \subset \mathfrak{J}_I$: We have to show that $F_0 \varepsilon U \varepsilon \mathfrak{J}_W$ implies the existence of $V \varepsilon \mathfrak{J}_I$ such that $F_0 \varepsilon V \subset U$. Let

$$\mathfrak{F}_1 = \{F \varepsilon \mathfrak{F}:F \not\varepsilon U, F \leqq F_0\}, \qquad \mathfrak{F}_2 = \{F \varepsilon \mathfrak{F}:F \not\varepsilon U, F \geqq F_0\}.$$

If $\mathfrak{F}_1 = \mathfrak{F}_2 = \varnothing$, the assertion holds with $V = \mathfrak{F}$. If $\mathfrak{F}_1 \neq \varnothing$, by Proposition (1.5) there exists a distribution function G_1 which is $\mathfrak{J}_W$-infimum of $\mathfrak{F}_1$.

(1.9) PROPOSITION. *If a family of distributions is ordered* (1) *and connected with respect to a locally connected topology* $\mathfrak{J} \supset \mathfrak{J}_I$, *then* $\mathfrak{J} = \mathfrak{J}_I$.

PROOF. To any $U \varepsilon \mathfrak{J}$ with $F_0 \varepsilon U$ there exists a connected set $U_0 \varepsilon \mathfrak{J}$ with $F_0 \varepsilon U_0 \subset U$. Hence it suffices to show that to any connected set $U_0 \varepsilon \mathfrak{J}$ with $F_0 \varepsilon U_0$ there exists $V \varepsilon \mathfrak{J}_I$ such that $F_0 \varepsilon V \subset U_0$.

If $\mathfrak{F} = \{F_0\}$, the assertion holds with $V = \mathfrak{F}$. Without loss of generality we may assume $\mathfrak{F}_1 = \{F \varepsilon \mathfrak{F}: F < F_0\} \neq \emptyset$. If $\mathfrak{F}_1 \cap U_0 = \emptyset$, then $\mathfrak{F} = \mathfrak{F}_1 + \bar{\mathfrak{F}}_1$ is a partition into two disjoint $\mathfrak{J}$-open sets: $\mathfrak{F}_1 \varepsilon \mathfrak{J}_I \subset \mathfrak{J}$. $\bar{\mathfrak{F}}_1 \varepsilon \mathfrak{J}$, for to any element of $\bar{\mathfrak{F}}_1$ there exists a $\mathfrak{J}$-neighborhood contained in $\bar{\mathfrak{F}}_1$: For F_0 this is the neighborhood U_0, for any $F > F_0$ the neighborhood $\{F \varepsilon \mathfrak{F}: F > F_0\} \varepsilon \mathfrak{J}_I \subset \mathfrak{J}$. Hence $\mathfrak{F}_1 \neq \emptyset$ implies the existence of $F_1 \varepsilon \mathfrak{F}_1 \cap U_0$. If $\mathfrak{F}_2 = \{F \varepsilon \mathfrak{F}: F > F_0\} = \emptyset$, the assertion holds with $V = \{F \varepsilon \mathfrak{F}: F > F_1\}$. As $V = \{F \varepsilon \mathfrak{F}: F_1 < F \leqq F_0\}$ and F_1, $F_0 \varepsilon U_0$, the local connectedness of U_0 implies $V \subset U_0$. If $\mathfrak{F}_2 \neq \emptyset$, there exists $F_2 \varepsilon U_0 \cap \mathfrak{F}_2$ and the assertion holds with $V = \{F \varepsilon \mathfrak{F}: F_1 < F < F_2\}$.

(1.10) REMARK. If ρ is a 'monotone' metric such as ρ and d' (see (1.3) and [3], Lemma 6, p. 1223), any connected family of distributions is also locally connected. Furthermore, monotony of ρ implies $\mathfrak{J}_\rho \supset \mathfrak{J}_I$.

(1.11) PROPOSITION. *If a family of p-measures is ordered* (2) *and* $\mathfrak{J}'$*-connected, then it is dominated.*

PROOF. We shall show that any $\mathfrak{J}'$-connected family $\mathfrak{P}$ of p-measures contains at most a countable number of degenerate p-measures. The assertion then follows from [3], p. 1220, Theorem 3. Assume that P_0 is degenerate: $P_0\{t_0\} = 1$. If $\mathfrak{P} = \{P_0\}$ the assertion is trivial. Hence we may assume without loss of generality that $\{P \varepsilon \mathfrak{P}: P < P_0\} \neq \emptyset$. For all $P < P_0$, we have $P(-\infty, t_0) > 0$ (for $P(-\infty, t_0) = 0$ together with $P(t_0, \infty) \leqq P_0(t_0, \infty) = 0$ implies $P\{t_0\} = 1$ whence $P = P_0$). If there exists a monotone sequence $P_n \varepsilon \mathfrak{P}$ $\mathfrak{J}'$-converging to P_0 the relation $P_n(-\infty, t_0) \downarrow P_0(-\infty, t_0)$ implies the existence of $n_0 \varepsilon N$ such that $0 < P_{n_0}(-\infty, t_0) < 1$. Therefore the convex support of n_0 is a nondegenerate interval containing t_0 (as $\mathfrak{P}$ is ordered, no degenerate p-measure of $\mathfrak{P}$ can have its support in the interior of this interval). If $\mathfrak{P}$ is connected, this is the only possibility.

Hence, t_0 belongs to a nondegenerate interval whose interior does not contain the support of a degenerate element of $\mathfrak{P}$. This implies that the number of degenerate elements of $\mathfrak{P}$ is countable.

2. Convergence and lifting of densities. It is well known (see e.g. [2], p. 352) that a.e.-convergence of densities implies convergence in the first mean and therefore $\mathfrak{J}_{d''}$-convergence of the pertaining p-measures. The converse is not true in general. For sequences of p-measures which are ordered (2), however, $\mathfrak{J}_{d''}$-convergence implies a.e.-convergence of the densities. Even more can be asserted: It is possible to choose a coherent system of densities in such a way that $\mathfrak{J}_{d''}$-convergence implies convergence of densities everywhere except on the boundary of the convex support of the limit measure. If the measures are mutually absolutely continuous convergence of densities without any restriction can be obtained. For monotone likelihood ratio families the densities may be chosen such that, in addition, the likelihood ratios are monotone whenever defined. This is the content of the following Theorems 2.1 and 2.10 and the pertaining Corollaries.

Furthermore, it is shown that for an ordered (2) family of mutually absolutely continuous distributions having continuous densities with respect to a nonatomic σ-finite measure which is positive on each interval, $\mathfrak{J}_{d''}$-convergence implies pointwise convergence of the continuous densities (Theorem 2.12).

We remark that it is impossible in general to describe a.e.-convergence of densities by a topology. Each sequence of densities converging in measure to a density contains a subsequence converging a.e. to the same limit. Hence whenever convergence a.e. can be described by a topology, it coincides with convergence in measure. This coincidence does, however, not hold true in general.

For any p-measure $P/\mathfrak{B}$ we define $P^- = \{t \in \mathbb{R}: F(t) = 0\}$ and $P^+ = \{t \in \mathbb{R}: F^*(t) = 0\}$. The set $P^* = \overline{P^-} \cap \overline{P^+}$ is the convex support as defined in [3], p. 1219, (12). If $\mathfrak{P}$ is ordered (1), $P < Q$ implies $P^- \subset Q^-$ and $P^+ \supset Q^+$. We denote

$$t_P^- = \inf P^* \qquad \text{and} \qquad t_P^+ = \sup P^*.$$

(We shall write t_ν^-, t_ν^+ instead of $t_{P_\nu}^-$, $t_{P_\nu}^+$, if no confusion is possible). The convex support P^* is the interval between t_P^- and t_P^+. The boundary points belong to P^* iff they are of positive probability.

If $\mathfrak{P}/\mathfrak{B}$ is an ordered family of p-measures endowed with a topology $\mathfrak{J}$, we say that a sequence $(P_n)_{n \varepsilon N} \subset \mathfrak{P}$ $\mathfrak{J}$-converges from below (above) to $P_0 \varepsilon \mathfrak{P}$, if $(P_n)_{n \varepsilon N}$ $\mathfrak{J}$-converges to P_0 and $P_n \leqq P_0$ ($P_n \geqq P_0$) for all $n \varepsilon N$.

(2.1) THEOREM. *Let $\mathfrak{P}/\mathfrak{B}$ be a family of p-measures which is ordered* (2) *and dominated by a σ-finite measure $\mu/\mathfrak{B}$. Then it is possible to determine for each $P \varepsilon \mathfrak{P}$ a μ-density p such that*

(i) *$p(t) = 0$ for $t \varepsilon P^- \cup P^+$,*

(ii) *for any sequence $(P_n)_{n \varepsilon N} \subset \mathfrak{P}$ $\mathfrak{J}_{d''}$-converging from below (above) to $P_0 \varepsilon \mathfrak{P}$ we have $\lim_{n \varepsilon N} p_n(t) = p_0(t)$ for all $t \neq t_0^-$ ($t \neq t_0^+$) and for all $t \varepsilon \mathbb{R}$ if $\mu\{t_0^-\} > 0$ ($\mu\{t_0^+\} > 0$).*

If the members of $\mathfrak{P}$ are mutually absolutely continuous, (ii) *holds for all $t \varepsilon \mathbb{R}$.*

(2.2) COROLLARY. *If P_0 is nondegenerate, Theorem* (2.1) *is even valid under the weaker assumption that $(P_n)_{n \varepsilon N}$ $\mathfrak{J}_w$-converges from below (above) to P_0.*

Corollary (2.2) follows immediately from Theorem (2.1) together with Theorem (1.4) and Proposition (1.1).

PROOF. A. As $\mathfrak{P}/\mathfrak{B}$ is dominated and $\mathfrak{B}$ is separable, there exists a countable subset $\mathfrak{Q} = \{Q_1, Q_2, \cdots\} \subset \mathfrak{P}$ which is dense in $\mathfrak{P}$ with respect to $\mathfrak{J}_{d''}$ (see e.g. Lehmann (1959), p. 352). For each Q_n, $n \varepsilon N$, we determine a finite μ-density q_n such that $q_n(t) = 0$ for $t \varepsilon Q_n^- \cup Q_n^+$. By Lemma 3.3 $Q_m < Q_n$ implies

$$G_n(t)/G_m(t) \leqq q_n(t)/q_m(t) \leqq G_n^*(t)/G_m^*(t) \tag{2.3}$$

for μ-a.a. $t \varepsilon \mathbb{R}$ for which the expressions are defined. ($G_n/\mathbb{R}$ is the distribution function pertaining to Q_n, $n \varepsilon N$.)

The set for which (2.3) is violated for some pair m, $n \varepsilon N$ is the countable union of μ-null sets and hence a μ-null set itself. Hence we may put $q_n(t) = 0$ for all t of this set and all $n \varepsilon \mathbb{N}$. In other words: Without loss of generality we may assume that (2.3) holds for all $t \varepsilon \mathbb{R}$ for which the occurring expressions are defined.

For any $P \varepsilon \mathfrak{P}$ we define

$$\Delta_P^- = P^- - \bigcup \{Q^-: Q \varepsilon \mathfrak{P} \text{ and } Q < P\} \quad \text{and}$$
$$\Delta_P^+ = P^+ - \bigcup \{Q^+: Q \varepsilon \mathfrak{P} \text{ and } Q > P\}.$$

There are three possibilities: (a) $\Delta_P^* = \emptyset$, (b) $\Delta_P^* = \{t_P^*\}$, (c) Δ_P^* is a nondegenerate interval, $* = -, +$. For $P \neq Q$ we have $\Delta_P^* \cap \Delta_Q^* = \emptyset$. Hence the set of all $P \varepsilon \mathfrak{P}$ for which Δ_P^- or Δ_P^+ is a nondegenerate interval is at most countable.

For $* = -, +$ let $\mathfrak{P}^*$ be the set of all elements $P \varepsilon \mathfrak{P}$ for which there exists an infinite subset $\mathbb{N}_P^* \subset \mathbb{N}$ such that $(Q_n)_{n \varepsilon \mathbb{N}_P^*}$ d''-converges to P from

$$\left.\begin{matrix}\text{below}\\ \text{above}\end{matrix}\right\} \text{ for } \begin{cases} * = - \\ * = + \end{cases} \text{ and such that } \begin{cases} Q_n < P \\ Q_n > P \end{cases} \text{ for all } n \varepsilon \mathbb{N}_P^*.$$

For $P \varepsilon \mathfrak{P}^*$, let $p^*(t) = \lim_{n \varepsilon \mathbb{N}_P^*} q_n(t)$. According to Lemma (3.6) $p^*(t)$ is defined for all $t \varepsilon \mathbb{R}$ and is a μ-density of P. This implies $p^*(t) = 0$ for μ — a.a. $t \varepsilon \Delta_P^*$. According to Corollary 3.7 the limit $p^*(t)$ is independent of the approximating sequence $(Q_n)_{n \varepsilon \mathbb{N}_P^*}$ as long as $\begin{cases} Q_n < P \\ Q_n > P \end{cases}$ for all $n \varepsilon \mathbb{N}_P^*$.

For $P \varepsilon \mathfrak{P}^*$, $* = -, +$, let

$$A_P^* = \{t \varepsilon \Delta_P^*: p^*(t) > 0\}.$$

Let A^* be the union of all A_P^*, $P \varepsilon \mathfrak{P}^*$, for which Δ_P^* is a nondegenerate interval. A^* is the countable union of μ-null sets and hence a μ-null set itself.

As $\mathfrak{Q}$ is dense in $\mathfrak{P}$ with respect to $\mathfrak{J}_{d''}$, we have $\mathfrak{P} = \mathfrak{P}^- \cup \mathfrak{P}^+ \cup \mathfrak{Q}$. For $Q_n \varepsilon \mathfrak{Q}$ we define

$$(2.4) \qquad \hat{q}_n(t) = 0, \qquad t \varepsilon A^- \cup A^+,$$
$$= q_n(t), \qquad \text{elsewhere.}$$

For $P \varepsilon \mathfrak{P} - \mathfrak{Q}$, we have $P \varepsilon \mathfrak{P}^*$ for $* = -$ and/or $+$. We define

$$(2.5) \qquad \hat{p}(t) = 0, \qquad t \varepsilon P^- \cup P^+,$$
$$= \lim_{n \varepsilon \mathbb{N}_P^*} \hat{q}_n(t), \qquad \text{elsewhere.}$$

We remark that on account of Lemma (3.6), definition (2.5) is unique also for $P \varepsilon (\mathfrak{P}^- \cap \mathfrak{P}^+) - \mathfrak{Q}$.

B. We shall show that for $P \varepsilon \mathfrak{P}^*$, $* = -, +$, we have

$$(2.6) \quad \hat{p}(t) = \lim_{n \varepsilon \mathbb{N}_P^*} \hat{q}_n(t) \quad \text{for all } t \neq t_P^* \text{ and for all } t \text{ if } \mu\{t_P^*\} > 0.$$

We shall give a proof for "−" only.

(a) $t \varepsilon P^+$: Order (2) implies $Q_n^+ \supset P^+$ for all $n \varepsilon N_P^-$ whence $\hat{q}_n(t) = \hat{p}(t) = 0$ for all $n \varepsilon N_P^-$.

(b) $t \varepsilon P^*$: If $P \varepsilon \mathfrak{P} - \mathfrak{Q}$ the assertion follows immediately from definition (2.5). Therefore, let $P \varepsilon \mathfrak{Q}$. As $t \varepsilon P^*$ implies $F_P(t) > 0$ and $F_P^*(t) > 0$ and hence $G_n(t) > 0$ and $G_n^*(t) > 0$ for all $n \varepsilon N_P^-$ with $n \geqq n(t)$, say, we have by (2.3)

$$F_P(t)/G_n(t) \leqq \hat{p}(t)/\hat{q}_n(t) \leqq F_P^*(t)/G_n^*(t) \tag{2.7}$$

for all $n \varepsilon N_P^-$, $n \geqq n(t)$ for which $\hat{p}(t)/\hat{q}_n(t)$ is defined. As $0 < F_P(t)/G_n(t) < \infty$ and $0 < F_P^*(t)/G_n^*(t) < \infty$ for all $n \varepsilon N_P^-$, $n \geqq n(t)$, $\hat{p}(t) = 0$ implies $\hat{q}_n(t) = 0$ and $\hat{p}(t) > 0$ implies $\hat{q}_n(t) > 0$ for all $n \varepsilon N_P^-$, $n \geqq n(t)$. Hence (2.7) implies for $n \to \infty$: $\hat{p}(t) = \lim_{n \varepsilon N_P^-} \hat{q}_n(t)$.

(c) $t < t_P^-$: If $t \varepsilon \bigcup\{Q^-: Q \varepsilon \mathfrak{P}$ and $Q < P\}$ we have $t \varepsilon Q_n^-$ and therefore $\hat{q}_n(t) = 0$ for all sufficiently large $n \varepsilon N_P^-$ (see [3], p. 1223, Lemma 6, and p. 1220, Theorem 1). As $\hat{p}(t) = 0$, this implies the assertion.

If $t < t_P^-$ and $t \not\varepsilon \bigcup\{Q^-: Q \varepsilon \mathfrak{P}$ and $Q < P\}$, we have $t \varepsilon \Delta_P^-$ and Δ_P^- is a nondegenerate interval. If $t \varepsilon A^- \cup A^+$ the assertion is trivial. If $t \not\varepsilon A^- \cup A^+$ we have $\lim_{n \varepsilon N_P^-} q_n(t) = 0$. As $q_n(t) = \hat{q}_n(t)$ for all $n \varepsilon N_P^-$ according to (2.4), this implies the assertion.

(a), (b) and (c) together establish assertion (2.6) for $t \neq t_P^-$. If $\mu\{t_P^-\} > 0$ we obtain furthermore that $\lim_{n \varepsilon N_P^-} \hat{q}_n(t_P^-) = \hat{p}(t_P^-)$ from the fact that

$$\lim_{n \varepsilon N_P^-} \hat{q}_n(t)$$

is a μ-density of P (see Lemma 3.6).

C. Now we shall show that the system of densities defined by (2.4) and (2.5) fulfills assumptions (i) and (ii) of Lemma (3.6).

While (i) is trivial, (ii) may be proved by distinguishing a number of cases. We shall give a proof for the first inequality and for $P \varepsilon \mathfrak{P}^-$, $Q \varepsilon \mathfrak{P}^+$, $P < Q$, only.

If $t \varepsilon A^- \cup A^+$, the assertion is void. Let now $t \not\varepsilon A^- \cup A^+$. If $F_P(t) = 0$, we have $F_Q(t) = 0$ and the assertion is void either. (F_P and F_Q denote the distribution functions pertaining to P and Q, respectively.) Let now $F_P(t) > 0$. If $\hat{p}(t) = 0$, the assertion is void or obvious. Therefore we may assume $\hat{p}(t) > 0$ which implies $t \not\varepsilon P^- \cup P^+$. As $P^+ \supset Q^+$, we also have $t \not\varepsilon Q^+$.

For each $n \varepsilon N_Q^+$ we have $G_n(t)/F_P(t) \leqq F_Q(t)/F_P(t)$. As $(Q_n)_{n \varepsilon N_Q^+}$ d''-converges to Q, $\lim_{n \varepsilon N_Q^+} (G_n(t)/F_P(t)) = F_Q(t)/F_P(t)$. Therefore to an arbitrary $\epsilon > 0$ there exists $n(\epsilon) \varepsilon N_Q^+$ such that

$$F_Q(t)/F_P(t) \leqq G_n(t)/F_P(t) + \epsilon/2 \text{ for all } n \varepsilon N_Q^+ \text{ with } n \geqq n(\epsilon). \tag{2.8}$$

Let $n \geqq n(\epsilon)$, $n \varepsilon N_Q^+$ be fixed. As $\lim_{m \varepsilon N_P^-} G_m(t) = F_P(t)$ and $F_P(t) > 0$, we may assume $G_m(t) > 0$ for all $m \varepsilon N_P^-$. For all $m \varepsilon N_P^-$ we have $G_n(t)/G_m(t) \leqq G_n(t)/F_P(t)$ and $\lim_{m \varepsilon N_P^-} (G_n(t)/G_m(t)) = G_n(t)/F_P(t)$. Therefore there exists $m(\epsilon, n) \varepsilon N_P^-$ such that

$$G_n(t)/F_P(t) \leqq G_n(t)/G_m(t) + \epsilon/2 \text{ for all } m \varepsilon N_P^- \text{ with } m \geqq m(\epsilon, n). \tag{2.9}$$

As for each $m \varepsilon \mathbb{N}_P^-$ and each $n \varepsilon \mathbb{N}_Q^+$ we have $Q_m < Q_n$, (2.8) and (2.9) together with (2.3) imply

$$F_Q(t)/F_P(t) \leqq G_n(t)/G_m(t) + \epsilon \leqq \hat{q}_n(t)/\hat{q}_m(t) + \epsilon$$

for all $n \varepsilon \mathbb{N}_Q^+$ with $n \geqq n(\epsilon)$ and all $m \varepsilon \mathbb{N}_P^-$ with $m \geqq m(\epsilon, n)$. According to (2.6) we obtain for $m \to \infty$ and $n \to \infty$

$$F_Q(t)/F_P(t) \leqq \hat{q}(t)/\hat{p}(t) + \epsilon.$$

As $\epsilon > 0$ was arbitrary, this implies the assertion.

D. Finally we shall show that for any sequence $(P_m)_{m\varepsilon N} \subset \mathfrak{P}\ d''$-converging from below to $P_0 \varepsilon \mathfrak{P}$, we have $\lim_{m\varepsilon N} \hat{p}_m(t) = \hat{p}_0(t)$ for all $t \neq t_0^-$ and for all t if $\mu\{t_0^-\} > 0$. (The proof for sequences converging from above runs similarly.)

As it suffices to consider those elements P_m for which $P_m < P_0$, we may assume without loss of generality that $P_m < P_0$ for all $m \varepsilon \mathbb{N}$. Furthermore $(P_m)_{m\varepsilon N}$ contains an increasing subsequence, determined by the subset $\mathbb{N}_0 \subset \mathbb{N}$, say. As $\lim_{m\varepsilon N} \hat{p}_m(t) = \lim_{m\varepsilon N_0} \hat{p}_m(t)$ by Lemma 3.6, we may assume that $(P_m)_{m\varepsilon N}$ is increasing without loss of generality. Now we define a sequence $(Q_{n(m)})_{m\varepsilon N} \subset \mathfrak{Q}$ as follows: We have $P_m < P_{m+1} < P_0$. If $P_{m+1} \varepsilon \mathfrak{P}^-$, there exists $Q \varepsilon \mathfrak{Q}$ such that $P_m < Q < P_{m+1}$. If $P_{m+1} \varepsilon \mathfrak{P}^+$, there exists $Q \varepsilon \mathfrak{Q}$ such that $P_{m+1} < Q < P_0$. Hence there always exists $Q_{n(m)} \varepsilon \mathfrak{Q}$ such that $P_m < Q_{n(m)} < P_0$. As $(P_m)_{m\varepsilon N}$ and $(Q_{n(m)})_{m\varepsilon N}$ are both d''-converging to P_0 from below and $P_m < P_0$, $Q_{n(m)} < P_0$ for all $m \varepsilon \mathbb{N}$, we have $\lim_{m\varepsilon N} \hat{p}_m(t) = \lim_{m\varepsilon N} \hat{q}_{n(m)}(t)$ by Lemma (3.6). As by (2.6) $\lim_{m\varepsilon N} \hat{q}_{n(m)} = \hat{p}_0(t)$ for all $t \neq t_0^-$, and for all t if $\mu\{t_0^-\} > 0$, this implies the assertion.

If the members of $\mathfrak{P}$ are mutually absolutely continuous, we have $P^- = Q^-$ and $P^+ = Q^+$ for all $P, Q \varepsilon \mathfrak{P}$. This implies that $\lim_{n\varepsilon N} \hat{p}_n(t) = \hat{p}_0(t)$ holds without any restriction: If, for instance, $\mu\{t_0^-\} = 0$, we have $t_0^- \varepsilon P_0^- = P_n^-$ for all $n \varepsilon \mathbb{N}$ and therefore $\hat{p}_0(t_0^-) = 0 = \hat{p}_n(t_0^-)$.

(2.10) THEOREM. *Let $\mathfrak{P}/\mathfrak{B}$ be a monotone likelihood ratio family of p-measures which is dominated by a σ-finite measure $\mu/\mathfrak{B}$. Then it is possible to determine for each $P \varepsilon \mathfrak{P}$ a density p such that*

(i) *$p(t) = 0$ for $t \varepsilon P^- \cup P^+$,*

(ii) *for any sequence $(P_n)_{n\varepsilon N} \subset \mathfrak{P}\ d''$-converging from below (above) to $P_0 \varepsilon \mathfrak{P}$, we have* $\lim_{n\varepsilon N} p_n(t) = p_0(t)$ *for all $t \neq t_0^-$ ($t \neq t_0^+$) and for all $t \varepsilon \mathbb{R}$, if $\mu\{t_0^-\} > 0$ ($\mu\{t_0^+\} > 0$),*

(iii) *if $P < Q$, $q(t)/p(t)$ is a nondecreasing function on the set of all $t \varepsilon \mathbb{R}$ for which this ratio is defined.*

If the members of $\mathfrak{P}$ are mutually absolutely continuous, (ii) *holds for all $t \varepsilon \mathbb{R}$.*

(2.11) COROLLARY. *If P_0 is nondegenerate, assertion* (ii) *of Theorem* 2.9 *holds even under the weaker assumption that $(P_n)_{n\varepsilon N}$ $\mathfrak{J}_w$-converges from below (above) to P_0.*

We remark that Theorem (2.10) generalizes the lemma in [4].

PROOF. For each $P \varepsilon \mathfrak{P}$ we choose a μ-density p such that the properties (i) and (ii) of Theorem (2.1) are fulfilled. Let $\mathfrak{Q} = \{Q_1, Q_2, \cdots\}$ be a countable subset of $\mathfrak{P}$ which is dense in $\mathfrak{P}$ with respect to $\mathfrak{J}_{d''}$. As $\mathfrak{P}$ has monotone likelihood ratios (i.e. $\mathfrak{P}$ is ordered (3)), to any pair Q_m, $Q_n \varepsilon \mathfrak{Q}$ with $Q_m < Q_n$ there exists a nondecreasing function $H_{m,n}:\mathbb{R} \to [0, \infty]$ such that $q_n(t)/q_m(t) = H_{m,n}(t)$ μ-a.e. if $q_n(t)/q_m(t)$ is defined.

Let

$$B_{m,n} = \{t \varepsilon \mathbb{R}: q_n(t)/q_m(t) \text{ is defined and } q_n(t)/q_m(t) \neq H_{m,n}(t)\}.$$

Then $B = \bigcup\{B_{m,n} : m, n \varepsilon N, Q_m < Q_n\}$ is a μ-null set and we may put $p(t) = 0$ for all $t \varepsilon B$, $P \varepsilon \mathfrak{P}$. The modified densities (which again will be denoted by p), still have the properties (i) and (ii) of Theorem (2.1).

Let $(Q_n')_{n\varepsilon N}$ and $(Q_n'')_{n\varepsilon N}$ be subsequences of $\mathfrak{Q}$ d''-converging to P and Q, respectively. If $P < Q$, we have $Q_n' < Q_n''$ for all sufficiently large n. Hence monotonicity of $q_n''(t)/q_n'(t)$ (for sufficiently large n) and property (ii) of Theorem (2.1) together imply the monotonicity of $q(t)/p(t)$ on $\mathbb{R} - \{t_P^-, t_P^+, t_Q^-, t_Q^+\}$ whenever the ratio is defined. An easy discussion shows that the monotonicity assertion holds on all of $\mathbb{R}$.

REMARK. That $\lim_{n\varepsilon N} p_n(t) = p_0(t)$ for all $t \varepsilon \mathbb{R}$ cannot be achieved in general may be seen by an analysis of the following trivial example:

Let $\mathfrak{P} = \{P_\vartheta : \vartheta \varepsilon [0, 1]\}$, where P_ϑ is defined by its density with respect to the Lebesgue-measure:

$$p_\vartheta(t) = 0, \quad \text{if} \quad t \leqq \vartheta \text{ or } t \geqq 1 + \vartheta,$$
$$= 1, \quad \text{if} \quad \vartheta < t < 1 + \vartheta.$$

In the situations met within statistics, dominated families of p-measures are usually given by their densities. One among the possible families of densities is usually distinguished: The author never saw the family of all normal distributions with variance 1 given by densities other than $(2\pi)^{-\frac{1}{2}} \exp [-\frac{1}{2}(x - \mu)^2]$, $-\infty < \mu < \infty$. Hence the question usually is whether for a *given* family d''-convergence implies pointwise convergence. The general result of Theorem (2.1) that there always exists a family of densities for which d''-convergence implies pointwise convergence (everywhere with the possible exception of t_P^-, t_P^+) does not help to answer this question. Hence it might be useful to have the following theorem available.

(2.12) THEOREM. *Let $\mathfrak{P}/\mathfrak{B}$ be a family of mutually absolutely continuous p-measures which is ordered (2) and dominated by a σ-finite measure $\mu/\mathfrak{B}$, with the following properties:*

(i) $\mu\{a\} = 0$ *for all* $a \varepsilon \mathbb{R}$,

(ii) $\mu(a, b) > 0$ *for all* $a, b \varepsilon \mathbb{R}$ *with* $a < b$.

If for each $P \varepsilon \mathfrak{P}$ there exists a continuous μ-density, then d''-convergence implies convergence of the continuous densities everywhere.

PROOF. For $P \varepsilon \mathfrak{P}$ let p be a continuous μ-density. By continuity and assumption (ii) $p(t) = 0$ for $\mu -$ a.a. $t \varepsilon P^- \cup P^+$ implies $p(t) = 0$ for all $t \varepsilon P^- \cup P^+$. Furthermore, order (2) implies for $P < Q$ (see Lemma (3.3.)):

$$F_Q(t)/F_P(t) \leqq q(t)/p(t) \leqq F_Q^*(t)/F_P^*(t) \tag{2.13}$$

for μ-a.a. $t \varepsilon \mathbb{R}$ for which these expressions are defined. We shall show that (2.13) holds for all $t \varepsilon \mathbb{R}$, for which these expressions are defined. Assume that $F_Q(t_0)/F_P(t_0) > q(t_0)/p(t_0)$ for some $t_0 \varepsilon \mathbb{R}$ (for which both expressions are defined). Assumption (i) implies continuity of F_P and F_Q. Hence there exists an open interval, say (a, b), such that $F_Q(t)/F_P(t) > q(t)/p(t)$ for all $t \varepsilon (a, b)$. Because of assumption (ii), this contradicts (2.13).

If $(P_n)_{n\varepsilon N} \subset \mathfrak{P}$ is a sequence d''-converging to $P_0 \varepsilon \mathfrak{P}$ from below, we shall show that $\lim_{n\varepsilon N} p_n(t) = p_0(t)$ for all $t \varepsilon \mathbb{R}$. Without loss of generality we may assume that $P_n < P_0$ for all $n \varepsilon N$. As the elements of $\mathfrak{P}$ are mutually absolutely continuous, we have $P_0^- = P_n^-$ and $P_0^+ = P_n^+$ for all $n \varepsilon N$ and therefore $p_0(t) = p_n(t) = 0$ for all $t \varepsilon P_0^- \cup P_0^+$ and all $n \varepsilon N$. For $t \varepsilon P_0^*$, $\lim_{n\varepsilon N} p_n(t) = p_0(t)$ is a consequence of (2.13). (This can be shown in exactly the same manner as in the proof of Theorem (2.1), Part B, (b).)

If $(P_n)_{n\varepsilon N} \subset \mathfrak{P}$ is a sequence d''-converging to P_0 from above, it similarly follows that $\lim_{n\varepsilon N} p_n(t) = p_0(t)$ for all $t \varepsilon \mathbb{R}$. Hence for any sequence $(P_n)_{n\varepsilon N}$ in $\mathfrak{P}$ d''-converging to P_0 we have $\lim_{n\varepsilon N} p_n(t) = p_0(t)$ for all $t \varepsilon \mathbb{R}$.

3. A few lemmata.

(3.1) LEMMA. $\mathfrak{I}_{d'} = \mathfrak{I}'$ *for any family of distribution functions.*

PROOF. The relation $\mathfrak{I}' \subset \mathfrak{I}_{d'}$ is an immediate consequence of the definitions and the fact that $d'(F_1, F_2) = d'(F_1^*, F_2^*)$. It remains to show that to each $F_0 \varepsilon \mathfrak{F}$ and each $V = \{F \varepsilon \mathfrak{F}: d'(F, F_0) < \epsilon\}$ there exists $U \varepsilon \mathfrak{I}'$ such that

$$F \varepsilon U \subset V. \tag{3.2}$$

Let $a_0 = -\infty$ and define $a_{k+1} = \sup \{a \varepsilon \mathbb{R}: P_0(a_k, a) < \epsilon/3\}$, $k = 0, 1, 2, \cdots$. We have $P_0(a_k, a_{k+1}) \leqq \epsilon/3$ and $P_0(a_k, a_{k+1}] \geqq \epsilon/3$. As $P_0(a_0, a_k] = \sum_{i=1}^{k} P_0(a_{i-1}, a_i] \geqq k\cdot\epsilon/3$ there exists $K \varepsilon N$ such that $a_{K+1} = \infty$.

We shall show that (3.2) holds with

$$U = \{F \varepsilon \mathfrak{F}: |F(a_k) - F_0(a_k)| < \epsilon/3 \text{ and } |F^*(a_k) - F_0^*(a_k)| < \epsilon/3 \text{ for } k = 1, \cdots, K\}.$$

For any $t \varepsilon \mathbb{R}$ there exists $k \varepsilon \{0, \cdots, K\}$ such that $a_k \leqq t < a_{k+1}$. If $F_0(t) \leqq F(t)$, $F \varepsilon U$ implies $0 \leqq F(t) - F_0(t) \leqq P(-\infty, a_{k+1}) - P_0(-\infty, a_k] = F_0^*(a_{k+1}) - F^*(a_{k+1}) + P_0(a_k, a_{k+1}) \leqq 2\cdot\epsilon/3$. If $F(t) < F_0(t)$ and $F \varepsilon U$, we similarly obtain $0 \leqq F_0(t) - F(t) \leqq 2\cdot\epsilon/3$. Hence $F \varepsilon U$ implies $|F(t) - F_0(t)| \leqq 2\cdot\epsilon/3$ for all $t \varepsilon \mathbb{R}$ whence $d'(F, F_0) \leqq 2\cdot\epsilon/3 < \epsilon$.

(3.3) LEMMA. *Let $P/\mathfrak{B}$ and $Q/\mathfrak{B}$ be p-measures with $P < Q$ in the sense of order (2). Let p and q be densities of these measures with respect to a dominating σ-finite measure μ. Then we have*

$$F_Q(t)/F_P(t) \leqq q(t)/p(t) \leqq F_Q^*(t)/F_P^*(t)$$

for μ-a.a. $t \varepsilon \mathbb{R}$ for which these expressions are defined.

PROOF. We shall give a proof for the first inequality only. Without loss of generality we may assume that $\mu(a, b) > 0$ for all $a, b \varepsilon \mathbb{R}$ with $a < b$. (For otherwise μ may be substituted by a dominating measure with this property, say ν. If the ν-densities have the asserted property ν-a.e., the μ-densities have the asserted property μ-a.e.). Let p and q be μ-densities of P and Q, respectively. Then we have

$$\text{(3.4)} \qquad \begin{aligned} \lim_{h\to 0} P[t, t+h)/\mu[t, t+h) &= p(t) \quad \mu - \text{a.e.},\\ \lim_{h\to 0} Q[t, t+h)/\mu[t, t+h) &= q(t) \quad \mu - \text{a.e.} \end{aligned}$$

From [3], p. 1221, Lemma 3 we obtain

(3.5) $F_Q(t)P[t, t+h) \leqq F_P(t)Q[t, t+h)$ for all $t \varepsilon \mathbb{R}$ and all $h > 0$.

(Lemma 3 immediately implies (3.5) for the case $P[t, t+h) + Q[t, t+h) > 0$. If $P[t, t+h) = Q[t, t+h) = 0$, the assertion is trivial.)

As $\mu[t, t+h) > 0$ for all $t \varepsilon \mathbb{R}$ and all $h > 0$, (3.5) together with (3.4) implies

$$F_Q(t)p(t) \leqq F_P(t)q(t) \quad \mu \text{ - a.e.}$$

(3.6) LEMMA. *Let $\mathfrak{P}/\mathfrak{B}$ be a family of p-measures which is ordered* (2) *and dominated by a σ-finite measure $\mu/\mathfrak{B}$. Let $\mathfrak{P}_0 \subset \mathfrak{P}$ be a subfamily such that it is possible to determine for each $P \varepsilon \mathfrak{P}_0$ a μ-density p with the following properties*:

(i) $p(t) = 0$ *for all* $t \varepsilon P^- \cup P^+$,

(ii) $P < Q$, $P, Q \varepsilon \mathfrak{P}_0$, *implies*

$$F_Q(t)/F_P(t) \leqq q(t)/p(t) \leqq F_Q^*(t)/F_P^*(t)$$

for all $t \varepsilon \mathbb{R}$ for which these expressions are defined.

Let $(P_n')_{n\varepsilon N}$ and $(P_n'')_{n\varepsilon N}$ be subsequences of $\mathfrak{P}_0$ which d''-converge to $P_0 \varepsilon \mathfrak{P}$ from below and above, respectively, such that $P_n' < P_0$ and $P_n'' > P_0$ for all $n \varepsilon N$. Then the corresponding sequences of densities, $(p_n')_{n\varepsilon N}$ and $(p_n'')_{n\varepsilon N}$ converge for all $t \varepsilon \mathbb{R}$ and both limit functions are densities of P_0. Furthermore $\lim_{n\varepsilon N} p_n'(t) = \lim_{n\varepsilon N} p_n''(t)$ for all $t \varepsilon P_0^*$.

(3.7) COROLLARY. *The limit of the sequence of densities is the same for all sequences $(P_n)_{n\varepsilon N}$ of $\mathfrak{P}_0$ d''-converging to P_0 from below (above) with $P_n < P_0 (P_n > P_0)$ for all $n \varepsilon N$.*

PROOF. (1) Convergence will be proved for $(p_n')_{n\varepsilon N}$ only, as the proof for $(p_n'')_{n\varepsilon N}$ is dual. For reasons of notational convenience, we shall omit the dash in this and the following section (2).

(a) $t \varepsilon P_0^- \cup P_0^*$: In this case, $F_0^*(t) > 0$ and therefore $F_n^*(t) > 0$ for all $n \geqq n(t)$, say. Therefore we also have $F_n^*(\tau) > 0$ for all $n \geqq n(t)$ and all $\tau \leqq t$.

According to [3], p. 1220, Theorem 1, and p. 1223, Lemma 6, to any $m \varepsilon N$ there exists $N(m) \geqq m$ such that $P_m < P_n < P_0$ for all $n \geqq N(m)$. Hence $m \geqq n(t)$, $n \geqq N(m)$, $\tau \leqq t$ together with assumption (ii) implies

$$\text{(3.8)} \qquad p_n(\tau)/p_m(\tau) \leqq F_n^*(\tau)/F_m^*(\tau) \leqq F_n^*(t)/F_m^*(t)$$

whenever $p_n(\tau)/p_m(\tau)$ is defined. From (3.8) we obtain for $n \to \infty$

(3.9) $$F_m^*(t) \limsup_{n\varepsilon N} p_n(\tau) \leqq F_0^*(t)p_m(\tau).$$

(If $p_m(\tau) = 0$, (3.8) implies $p_n(\tau) = 0$ for all $n \geqq N(m)$ and (3.9) holds trivially.)

For $m \to \infty$ and $\tau = t$, (3.9) implies

$$F_0^*(t) \limsup_{n\varepsilon N} p_n(t) \leqq F_0^*(t) \liminf_{n\varepsilon N} p_n(t).$$

As $F_0^*(t) > 0$, this implies the existence of $\lim_{n\varepsilon N} p_n(t)$.

(b) $t \,\varepsilon\, P_0^+$: As $P_n < P_0$ for all $n \,\varepsilon\, N$, we have $P_0^+ \subset P_n^+$ and therefore according to assumption (i) $p_n(t) = 0$ for all $n \,\varepsilon\, N$. Hence $\lim_{n\varepsilon N} p_n(t)$ exists.

(2) Now we shall show that $\lim_{n\varepsilon N} p_n$ is a density of P_0.

(a) $t \,\varepsilon\, P_0^- \cup P_0^*$: (3.9) implies

$$F_m^*(t) \int_{(-\infty,t]} \lim_{n\varepsilon N} p_n(\tau)\mu(d\tau) \leqq F_0^*(t)F_m(t) \leqq F_0^*(t).$$

For $m \to \infty$ we obtain from $F_0^*(t) > 0$:

$$\int_{(-\infty,t]} \lim_{n\varepsilon N} p_n(\tau)\mu(d\tau) \leqq 1.$$

This relation holds for all $t \,\varepsilon\, P_0^- \cup P_0^*$ whence

(3.10) $$\int_{P_0^- \cup P_0^*} \lim_{n\varepsilon N} p_n(\tau)\mu(d\tau) \leqq 1.$$

On the other hand, if p_0 is any density of P_0, we have from Lemma (3.3) for μ-a.a. $t \,\varepsilon\, P_0^- \cup P_0^*$ and all sufficiently large n:

$$p_0(t)/p_n(t) \leqq F_0^*(t)/F_n^*(t), \qquad \text{or} \qquad p_0(t) = p_n(t) = 0.$$

In both cases we obtain

$$p_0(t) \leqq \lim_{n\varepsilon N} p_n(t) \text{ for } \mu\text{-a.a. } t \,\varepsilon\, P_0^- \cup P_0^*,$$

and therefore

$$1 = \int_{P_0^- \cup P_0^*} p_0(\tau)\mu(d\tau) \leqq \int_{P_0^- \cup P_0^*} \lim_{n\varepsilon N} p_n(\tau)\mu(d\tau).$$

Together with (3.10) this implies

$$\lim_{n\varepsilon N} p_n(t) = p_0(t) \text{ for } \mu\text{-a.a. } t \,\varepsilon\, P_0^- \cup P_0^*.$$

(b) $t \,\varepsilon\, P_0^+$: The relation $P_n^+ \supset P_0^+$ implies $p_n(t) = 0$ for all $n \,\varepsilon\, N$. As $p_0(t) = 0$ for μ-a.a. $t \,\varepsilon\, P_0^+$, we therefore have

$$\lim_{n\varepsilon N} p_n(t) = p_0(t) \text{ for } \mu\text{-a.a. } t \,\varepsilon\, P_0^+.$$

(3) Finally we shall show that $\lim_{n\varepsilon N} p_n'(t) = \lim_{n\varepsilon N} p_n''(t)$ for all $t \,\varepsilon\, P_0^*$. If $t \,\varepsilon\, P_0^*$ we have $F_0(t) > 0$ and $F_0^*(t) > 0$. Hence for all sufficiently large m, n we have $F_m'(t) \cdot F_n''(t) \cdot F_m'^*(t) \cdot F_n''^*(t) > 0$ and therefore by assumption (ii)

(3.11) $$F_n''(t)/F_m'(t) \leqq p_n''(t)/p_m'(t) \leqq F_n''^*(t)/F_m'^*(t)$$

whenever $p_n''(t)/p_m'(t)$ is defined. If $p_n''(t) > 0$ and $p_m'(t) > 0$ for all sufficiently large n and m, we may take the limit for $n \to \infty$ and $m \to \infty$ in (3.11). Using the result of (1) we immediately obtain the assertion. If $p_n''(t) = 0$ or $p_m'(t) = 0$ for infinitely many n or m, respectively, $F_m'(t) \cdot F_n''(t) \cdot F_m'^*(t) \cdot F_n''^*(t) > 0$ for all sufficiently large m and n together with (3.11) imply $p_n''(t) = p_m'(t) = 0$ for almost all n and m and the assertion is void.

Acknowledgment. The author wishes to thank Mr. W. Pierlo for the careful and critical study of the manuscript. The discussions with him helped much to improve the manuscript. He also discovered an error in an earlier proof of Theorem (1.4).

REFERENCES

[1] Gnedenko, B. V. and Kolmogorov, A. N. (1954). *Limit Distributions for Sums of Independent Random Variables*. Addison-Wesley, Cambridge.

[2] Lehmann, E. L. (1959). *Testing Statistical Hypotheses*. Wiley, New York.

[3] Pfanzagl, J. (1964). On the topological structure of some ordered families of distributions. *Ann. Math. Statist.* **35** 1216–1228.

[4] Pfanzagl, J. (1967). A technical lemma for monotone likelihood ratio families. *Ann. Math. Statist.* **38** 611–612.