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THE EXISTENCE OF CERTAIN STOPPING TIMES ON BROWNIAN MOTION[1]

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1. Introduction. Let $(\Omega, \mathcal{B}_t, t \geqq 0, P)$ be the space of continuous sample paths of standard Brownian motion starting at zero. Let I be the unit interval with Borel sets, $\mathcal{L}$, and uniform measure λ. In [8] Skorokhod showed that if X is a random variable (rv) with $\sigma^2(X) < \infty$ and $E(X) = 0$ then there is a stopping time τ defined on $(\Omega \times I, \mathcal{B}_t \times \mathcal{L}, P \times \lambda)$ such that X_τ and X are equal in law, where $X_\tau(\omega, s) = \omega(\tau(\omega, s))$ for $(\omega, s) \in \Omega \times I$. It is the purpose here to show that for such a rv X there is a stopping time τ defined on $(\Omega, \mathcal{B}_t, P)$ such that X_τ and X are equal in law, where $X_\tau(\omega) = \omega(\tau(\omega))$. A second method of defining stopping times τ directly on $(\Omega, \mathcal{B}_t, P)$ such that X and X_τ are equal in law has recently been given by Dubins [3].

2. Construction of the stopping time.

2.1 Theorem. *If X is a* rv *with $\sigma^2(X) < \infty$ and $E(X) = 0$ then there is a stopping time τ defined on Ω such that $\mathcal{L}(X) = \mathcal{L}(X_\tau)$ and $E(\tau) = \sigma^2(X)$.*

Two lemmas will be established before proving this theorem.

2.2 Lemma. *If X is a* rv *taking only a finite number of values and satisfying $E(X) = 0$, then there is a stopping time τ defined on Ω such that $\mathcal{L}(X) = \mathcal{L}(X_\tau)$ and $\sigma^2(X) = E(\tau)$.*

Proof. If $X \equiv 0$ then $\tau \equiv 0$ suffices. Assume therefore that $P[X = 0] < 1$. Set $p_i = P[X = x_i] > 0$, where $x_1 < x_2 < \cdots < x_n$ are the possible values for X and $\sum_{i=1}^n p_i = 1$. Clearly $x_1 < 0 < x_n$ since $E(X) = 0$. Let $\beta = (b_1, \cdots, b_n)$ be an n-tuple of reals with $0 = b_1 = b_n$ and $0 \leqq b_i \leqq +\infty$ for all i. Define a stopping time τ_β as follows:

$$\tau_\beta(\omega) = \inf\{t;\ \omega(t) = x_i \text{ and } t \geqq b_i \text{ for some } i\}. \tag{2.1}$$

Then $X_\beta \equiv X_{\tau_\beta}$ is a rv such that $\sum_i P[X_\beta = x_i] = 1$. In [7] it is shown that if τ is any stopping time and $E(\tau) < \infty$ then $\sigma^2(X_\tau) = E(\tau)$ and $E(X_\tau) = 0$. Thus $E(X_\beta) = 0$ and $E(\tau_\beta) = \sigma^2(X_\beta)$ because τ_β is bounded by the first exit time of $[x_1, x_n]$ which has finite mean. Define A_n by

$$A_n = \{(p_1, \cdots, p_n);\ p_i \geqq 0,\ \textstyle\sum_{i=1}^n p_i = 1,\ \sum_{i=1}^n p_i x_i = 0\}. \tag{2.2}$$

Geometrically this is a convex set in R^n with a finite number of extreme points.

Define B_n by

$$B_n = \{(b_1, \cdots, b_n);\ b_1 = b_n = 0 \text{ and } 0 \leqq b_i \leqq +\infty\}. \tag{2.3}$$

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This is also a convex set with a finite number of extreme points. By the previous discussion, each $\beta \varepsilon B_n$ yields a stopping time whose associated rv has a distribution which is a point in A_n. Thus we have a well defined mapping $f_n : B_n \to A_n$. Let A_n be topologized by the Euclidian metric and B_n by the metric d defined by

$$(2.4) \qquad d(\beta, \beta') = [\sum_{i=1}^{n} (b_i(b_i + 1)^{-1} - b_i'(1 + b_i')^{-1})^2]^{\frac{1}{2}},$$

where $\beta = (b_1, \cdots, b_n)$ and $\beta' = (b_1', \cdots, b_n')$. With these topologies both A_n and B_n are homeomorphic to the unit ball in R^{n-1}. Thus to show that f_n is onto it is sufficient to show that it is continuous, that it carries the boundary of B_n (∂B_n) onto the boundary of A_n (∂A_n), and that its restriction to the interior of B_n is one-to-one. Continuity of f_n is clear. For the rest we proceed by induction on n. If $n = 2$ each space has only one point. Assume $n > 2$. The induction hypothesis implies that $f_n : \partial B_n \to \partial A_n$ is onto. Let $\beta = (b_1, \cdots, b_n)$ and $\bar{\beta} = (\bar{b}_1, \cdots, \bar{b}_n)$ be points in the interior of B_n and assume $\mathfrak{L}(X_\beta) = \mathfrak{L}(X_{\bar{\beta}})$. Let $K = \{x_i ; \bar{b}_i < b_i\}$ and observe that $P[X_{\bar{\beta}} \varepsilon K] > P[X_\beta \varepsilon K]$ if K is not empty. Thus K is empty and $\bar{\beta} \geqq \beta$. By symmetry $\bar{\beta} = \beta$. Lemma 2.2 is proved.

Before extending this result to non-discrete rv's we introduce some concepts which will be used in the proof of the extension.

2.3 Definition. A barrier is a subset B of $[0, +\infty] \times [-\infty, +\infty]$ satisfying

(1) B is closed,

(2) $(+\infty, x) \varepsilon B$ for all x,

(3) $(t, \pm\infty) \varepsilon B$ for all $t \varepsilon [0, +\infty]$,

(4) if $(t, x) \varepsilon B$ then $(s, x) \varepsilon B$ whenever $s > t$.

Map the closed half plane H homeomorphically to a bounded rectangle F by $(t, x) \to (t/(1 + t), x/(1 + |x|))$. Let F have the ordinary Euclidian metric ρ and H the corresponding induced metric r. Define a metric, also denoted by r, on $\mathfrak{C}$, the space of closed subsets of H by:

$$(2.5) \qquad r(C, D) = \max (\sup_{x \varepsilon C} r(x, D), \sup_{y \varepsilon D} r(y, C)).$$

Under r, $\mathfrak{C}$ is a separable compact metric space and $\mathfrak{B}$, the space of all barriers is closed in $\mathfrak{C}$ and hence compact. For $B \varepsilon \mathfrak{B}$ define τ_B by

$$\tau_B(\omega) = \inf \{t; (t, \omega(t)) \varepsilon B\}.$$

2.4 Lemma. *If B is a barrier with corresponding stopping time τ and $E(\tau) < \infty$, then for any $\epsilon > 0$ there exists $\delta > 0$ such that if $\bar{B} \varepsilon \mathfrak{B}$ and $\bar{\tau}$ is the corresponding stopping time and $r(B, \bar{B}) < \delta$ then $P[\bar{\tau} > \tau + \epsilon] < \epsilon$.*

Proof. Choose $\eta > 0$ so small that

$$P[\sup_{\eta < t < \epsilon} \omega(t) > \eta \text{ and } \inf_{\eta < t < \epsilon} \omega(t) < -\eta] > 1 - \epsilon/3.$$

Choose $T > 3E(\tau)/\epsilon$ then $P[\tau \geqq T] < \epsilon/3$. Choose M and δ so that if $(t, x) \varepsilon B$ and $t \leqq T$ and $|x| \leqq M$ and $r(B, \bar{B}) < \delta$ then $\rho((t, x), \bar{B}) < \eta$ where ρ is the Euclidian metric. Each of these choices can clearly be made. Let $A = \{\omega \varepsilon \Omega; \omega$ satisfies (i), (ii), (iii) below$\}$:

(i) $\sup_{\tau+\eta<t<\tau+\epsilon} [\omega(t) - \omega(\tau)] > \eta$ and $\inf_{\tau+\eta<t<\tau+\epsilon} [\omega(t) - \omega(\tau)] < -\eta$,
(ii) $\tau(\omega) < T$,
(iii) $|\omega(\tau(\omega))| < M$.

From the definition of A and the choice of δ we see that if $\omega \varepsilon A$ then $\bar{\tau}(\omega) < \tau(\omega) + \epsilon$. From the choice of η, T, M and the strong Markov property we see that $P[A] > 1 - \epsilon$. Lemma 2.4 is proved.

Proof of Theorem 2.1. Let $\{X_n, n \geqq 1\}$ be a sequence of rv taking only a finite number of values such that (a) $E(X_n) = 0$ for all n, (b) $\sigma^2(X_n) \leqq \sigma^2(X)$, and (c) $\mathcal{L}(X_n) \to \mathcal{L}(X)$. By Lemma 2.2 we know that there are barriers B_n with corresponding stopping times τ_n such that $\mathcal{L}(\omega(\tau_n)) = \mathcal{L}(X_n)$ and $E(\tau_n) = \sigma^2(X_n)$. From compactness we may without loss of generality assume $\{B_n\}$ is a convergent sequence. Let B be the limit of $\{B_n\}$ and τ the corresponding stopping time. Since $\{E(\tau_n)\}$ is uniformly bounded we may conclude from Lemma 2.4 that for any $\epsilon > 0$, $P[\tau > \tau_n + \epsilon] \to 0$ as $n \to \infty$. Thus $E(\tau) < \infty$. We may now conclude from Lemma 2.4 that $P[|\tau - \tau_n| > \epsilon] \to 0$ as $n \to \infty$. Without loss of generality we assume $\tau_n \to \tau$ a.s. as $n \to \infty$. Thus $\mathcal{L}(X) = \lim \mathcal{L}(\omega(\tau_n)) = \mathcal{L}(\omega(\tau))$. The finiteness of $E(\tau)$ implies by [7] that $E(\tau) = \sigma^2(X)$. Theorem 2.1 is proved.

It can be shown that the barrier described in Theorem 2.1 corresponding to a normal random variable with mean zero and variance v is $\{(t, x); t \geqq v\}$. The barriers which correspond to the uniform distribution on $[-1, 1]$ or the centered negative exponential distribution, $P[X < a] = 1 - e^{-1-a}$ $(a > -1)$, are unknown.

2.5 Theorem. *If X is a* rv *and $E(X) = 0$ and $E(X^2) < \infty$ and τ is the corresponding stopping time of Theorem* 2.1, *then $E|X|^{2n} < \infty$ if and only if $E(\tau^n) < \infty$.*

Proof. It is known that for each n there exist n constants $a_1, \cdots, a_n$ such that the process $V_t = X_t^{2n} + a_1 t X_t^{2n-2} + \cdots + a_n t^n$ is a martingale defined on $(\Omega, P, \mathcal{B}_t\, t > 0)$. (In ([1], Section IV) these polynomials are given as $(t/2)^n H_{2n}(x/(2t)^{\frac{1}{2}})$ where H_n is the nth Hermite polynomial and it should read $(t)^{n/2} H_n(x/t^{\frac{1}{2}})$.) Assume $E(\tau^n) < \infty$. Define $\tau_t(\omega) = \min(\tau(\omega), t)$. Since $\{V_s; s \leqq t\}$ is a uniformly integrable martingale, it follows from Chapter VII, Theorem 11.8 of [2] that $E(V_{\tau_t}) = 0$ or

$$0 = E(X_{\tau_t}^{2n}) + a_1 E(\tau_t X_{\tau_t}^{2n-2}) + \cdots + a_n E(\tau_t^{\,n}). \tag{2.7}$$

Applying Holder's inequality to this, we have

$$E(X_{\tau_t}^{2n}) < |a_1| E^{1/n}(\tau_t^{\,n}) E^{(n-1)/n}(X_{\tau_t}^{2n}) + |a_2| E^{2/n}(\tau_t^{\,n}) E^{(n-2)/n}(X_{\tau_t}^{2n}) + \cdots + |a_n| E(\tau_t^{\,n}).$$

Hence there is a K_n such that

$$E(X_{\tau_t}^{2n}) < K_n E(\tau_t^{\,n}) \leqq K_n E(\tau^n).$$

Since $\{X_{\tau_t}^{2n}, t \geqq 0\}$ is a non-negative submartingale and $\lim_{t\to\infty} X_{\tau_t}^{2n} = X_\tau^{2n}$, we have (see [2], pp. 324–325) $\lim_{t\to\infty} E(X_{\tau_t}^{2n}) = E(X_\tau^{2n})$. Thus $E(X_\tau^{2n}) \leqq K_n E(\tau^n)$.

To prove the converse, observe that for any stopping time τ we obtain from (2.7)

$$E(\tau_t^n) < |a_n|^{-1}[|a_1|E^{1/n}(\tau_t^n)E^{(n-1)/n}(X_{\tau_t}^{2n}) + \cdots + E(X_{\tau_t}^{2n})].$$

As before there exists k_n such that $E(\tau_t^n) < k_n E(X_{\tau_t}^{2n})$ thus

$$E(\tau^n) = \lim_{t\to\infty} E(\tau_t^n) \leqq k_n \lim_{t\to\infty} E(X_{\tau_t}^{2n}) = k_n E(X_\tau^{2n}).$$

3. Applications. The stopping times of [3] or [8] or Theorem 2.1 can be used to represent sequences of processes, e.g. random walks, converging a.s. to Brownian motion. It is also possible to represent the empirical processes so as to obtain a.s. convergence of their sample paths to the sample paths of a Brownian bridge. These representations are contained in [6] and will appear under the title, "Construction of Almost Surely Convergent Random Processes."

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