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ON MINIMUM VARIANCE UNBIASED ESTIMATION OF RELIABILITY

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1. Introduction. The purpose of the present paper is to present a simple method of deriving the unique minimum variance unbiased estimate (MVUE) of the reliability function associated with a life distribution. The widely applicable life distributions are the normal, one- and two-parameter exponential, gamma and Weibull distributions. A good deal of work has been done on this problem. Barton [1] estimated the probability that a normal variable will take a value between two points X_1 and X_2 in its range. Pugh [7] and Laurent [6] obtained the MVUE of reliability under the one-parameter and two-parameter exponential life distributions respectively. Tate [8] considered the two-parameter exponential, gamma and Weibull distributions and obtained MVUE's of some functions of the parameters. Recently, Basu [2] put forward a method of deriving MVUE of reliability and verified these estimates. The methods used by most of these authors consist of finding conditional distribution of a component of sample-observations given the sufficient statistics. In our present method, we find a statistic which is stochastically independent of the complete sufficient statistics and whose distribution can be very easily obtained. The MVUE is based on this distribution.

2. A general theorem for finding the MVUE. Let $\mathbf{X} = (X_1, X_2, \cdots, X_n)$ be a random sample of size n from a distribution function $F(x; \boldsymbol{\theta})$, where $\boldsymbol{\theta}$ denotes a vector of parameters. Let $\hat{\boldsymbol{\theta}}$ be a complete sufficient statistic for $\boldsymbol{\theta}$. Let $\psi(t, \boldsymbol{\theta})$ be a parametric function, t being a real number, and let

$$U(\mathbf{X}) = \lambda \quad \text{if } Z(\mathbf{X}) \geqq t, \qquad = 0 \quad \text{otherwise}, \tag{2.1}$$

for $Z(\mathbf{X})$ a function of $\mathbf{X}$ and λ any real number, be an unbiased estimate of $\psi(t, \boldsymbol{\theta})$. We introduce the following:

THEOREM 1. *If there exists a function* $V(Z, \hat{\boldsymbol{\theta}})$ *such that*

(i) *it is stochastically independent of* $\hat{\boldsymbol{\theta}}$,

(ii) *it is a strictly increasing function of* Z *for fixed* $\hat{\boldsymbol{\theta}}$, *and*

(iii) *its distribution function* $H(x)$ *is such that*

$$H(x) = 0 \quad \text{for } x < a, \qquad = 1 \quad \text{for } x > b.$$

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[1] This work was done when the author was a U.G.C. Scholar.

a and b ($a < b$) *being either finite or infinite, then the unique* MVUE *of* $\psi(t, \boldsymbol{\theta})$ *is given by*

$$\begin{aligned} \psi^*(t, \hat{\boldsymbol{\theta}}) &= \lambda && \text{if } V(t, \hat{\boldsymbol{\theta}}) < a, \\ &= \lambda \textstyle\int_{V(t,\hat{\boldsymbol{\theta}})}^{b} dH(x) && \text{if } a \leqq V(t, \hat{\boldsymbol{\theta}}) \leqq b, \\ &= 0 && \text{if } V(t, \hat{\boldsymbol{\theta}}) > b. \end{aligned} \tag{2.2}$$

Proof. Since $U(\mathbf{X})$ is an unbiased estimate of $\psi(t, \boldsymbol{\theta})$ and $\hat{\boldsymbol{\theta}}$ is a complete sufficient statistic for $\boldsymbol{\theta}$, therefore, by the Rao-Blackwell-Lehmann-Scheffé theorem, the unique MVUE of $\psi(t, \boldsymbol{\theta})$ is

$$\begin{aligned} \psi^*(t, \hat{\boldsymbol{\theta}}) &= E[U(\mathbf{X}) \mid \hat{\boldsymbol{\theta}}] \\ &= \lambda \operatorname{Prob}[Z(\mathbf{X}) \geqq t \mid \hat{\boldsymbol{\theta}}] \\ &= \lambda \operatorname{Prob}[V(Z, \hat{\boldsymbol{\theta}}) \geqq V(t, \hat{\boldsymbol{\theta}}) \mid \hat{\boldsymbol{\theta}}] \quad \text{by (ii)} \\ &= \lambda \operatorname{Prob}[V(Z, \hat{\boldsymbol{\theta}}) \geqq V(t, \hat{\boldsymbol{\theta}})] \quad \text{by (i)}. \end{aligned}$$

Hence by (iii) we get (2.2).

The steps in the application of this theorem are the selections of suitable functions Z and V. Function V satisfying condition (i) can be obtained by making use of the following

Theorem 2. *If V is any statistic (not a function of the sufficient statistic $\hat{\boldsymbol{\theta}}$ alone) such that its distribution does not depend upon $\boldsymbol{\theta}$, then V is stochastically independent of $\hat{\boldsymbol{\theta}}$.*

The proof of this theorem is given by Hogg and Craig [5] in Section 8.9.

3. Applications. The unique MVUE's of reliability and its powers under various life distributions are obtained in this section by applying Theorem 1 of the preceding section.

Reliability at 'mission time' t is defined by

$$R(t, \boldsymbol{\theta}) = \operatorname{Prob}[X \geqq t \mid \boldsymbol{\theta}] = \textstyle\int_t^\infty dF(x; \boldsymbol{\theta})$$

where $F(x; \boldsymbol{\theta})$ is the underlying life distribution.

The following lemma will be frequently used to find the density function of V.

Lemma 1. *If $X_1, X_2, \cdots, X_n$ are independently and identically distributed with the density function*

$$f(x) = \theta^{-p}\Gamma^{-1}(p)e^{-x/\theta}x^{p-1}, \qquad x \geqq 0;\ p, \theta > 0,$$

then the random variable

$$\xi = X_i/(X_i + \textstyle\sum_{j \neq i}^{m} X_j),$$

where $\sum^m$ denotes the sum of m terms ($0 < m < n$), has the beta density function

$$f(x) = \beta^{-1}(p, mp)x^{p-1}(1 - x)^{mp-1} \tag{3.1}$$

for $0 \leqq x \leqq 1$ and 0 otherwise.

The proof of this lemma is simple; hence it is omitted. The density function (3.1) will be denoted by $B(p, mp)$.

TABLE 1

MVUE's *of reliability function*

Life distribution	$\hat{\theta}$	V	Density function of V	$R^*(t, \hat{\theta})$
Normal: $N(\theta, 1)$ $(2\pi)^{-\frac{1}{2}} \exp[-(x-\theta)^2/2]$	$\bar{X}$	$Z - \bar{X}$	$N(0, (n-1)/n)$	$G[(n/(n-1))^{\frac{1}{2}}(t-\bar{X})]$
Normal: $N(\theta_1, \theta_2)$ $(2\pi\theta_2)^{-\frac{1}{2}} \exp[-(x-\theta_1)^2/2\theta_2]$	$(\bar{X}, s^2)$	$(Z-\bar{X})/s$	$[1 - x^2/(n-1)]^{(n-4)/2}/[(n-1)^{\frac{1}{2}}\beta(\frac{1}{2}, (n-2)/2)]$ for $\lvert x \rvert < (n-1)^{\frac{1}{2}}$ See Cramér [3] Sec. 29.4	$\begin{cases} \frac{1}{2}[1 + I_{\tau^2}(\frac{1}{2}, (n-2)/2)] & \text{if } \tau \leqq 0 \\ \frac{1}{2}[1 - I_{\tau^2}(\frac{1}{2}, (n-2)/2)] & \text{if } \tau \geqq 0 \end{cases}$ where $\tau = (t-\bar{X})/s(n-1)^{\frac{1}{2}}$
Gamma: $\theta^{-p}\Gamma^{-1}(p)e^{-x/\theta}x^{p-1}$ p known	S_1	Z/S_1	$B(p, (n-1)p)$	$1 - I_{t/S_1}(p, (n-1)p)$
Exponential: $(1/\theta)e^{-x/\theta}$	S_1	Z/S_1	$B(1, (n-1))$	$(1 - t/S_1)^{n-1} \cdot \delta(S_1 - t)$
Weibull: $(p/\theta)x^{p-1} \exp[-x^p/\theta]$ p known	S_p	Z^p/S_p	$B(1, n-1)$	$(1 - t^p/S_p)^{n-1} \cdot \delta(S_p - t^p)$

In all the above examples, $\lambda = 1$ and $Z = X_1$ are used in $U(\mathbf{X})$.

$$\bar{X} = \sum_1^n X_i/n, \qquad s^2 = \sum_1^n (X_i - \bar{X})^2/n, \qquad S_p = \sum_1^n X_i^p.$$

$$G(w) = (2\pi)^{-\frac{1}{2}} \int_w^\infty \exp[-x^2/2]\, dx. \quad \text{For} \quad a, b > 0,\ I_w(a, b) = \begin{cases} 0 & \text{if } w < 0 \\ \beta^{-1}(a, b) \int_0^w x^{a-1}(1-x)^{b-1}\, dx & \text{if } 0 \leqq w \leqq 1 \\ 1 & \text{if } w > 1 \end{cases}$$

TABLE 2

MVUE's *of kth power* ($k \leqq n$) *of reliability function associated with two parameter exponential distribution using a censored sample*

Case	$\hat{\theta}$	$\psi(t, \boldsymbol{\theta})$	λ	Z	V	Density function of V	$\psi^*(t, \hat{\theta})$
θ_1 known	S'	$R^k(t, \theta_2)$	1	$(Y_1 - (n-k)\theta_1)/k$	$\frac{k(Z-\theta_1)}{S' - n\theta_1}$	$B(1, r-1)$	$\begin{cases} 1, & t < \theta_1 \\ [1 - k(t-\theta_1)/(S' - n\theta_1)]^{r-1}, & \theta_1 \leqq t \leqq [S' - (n-k)\theta_1]/k \\ 0, & t > [S' - (n-k)\theta_1]/k \end{cases}$
θ_2 known	$X_{(1)}$	$R^k(t, \theta_1) - (k/n)R^n(t, \theta_1)$	$1 - k/n$	$(Y_2/k) + (Y_1/n)$	$k(Z - X_{(1)})$	$(1/\theta_2)e^{-x/\theta_2}$	$\begin{cases} 1 - k/n, & t \leqq X_{(1)} \\ (1-k/n)\exp[-k(t - X_{(1)})/\theta_2], & t \geqq X_{(1)} \end{cases}$
		$(k/n)R^n(t, \theta_1)$	k/n	Y_1/n			†$\begin{cases} k/n, & t \leqq X_{(1)} \\ 0, & t > X_{(1)} \end{cases}$
		$*R^k(t, \theta_1)$					$\begin{cases} 1, & t \leqq X_{(1)} \\ (1-k/n)\exp[-k(t - X_{(1)})/\theta_2], & t > X_{(1)} \end{cases}$
θ_1, θ_2 unknown	$(X_{(1)}, S')$	$R^k(t, \boldsymbol{\theta}) - (k/n)R^n(t, \boldsymbol{\theta})$	$1 - k/n$	$(Y_2/k) + (Y_1/n)$	$\frac{k(Z - X_{(1)})}{S' - nX_{(1)}}$	$B(1, r-2)$	$\begin{cases} 1 - k/n, & t < X_{(1)} \\ (1-k/n)[1 - k(t - X_{(1)})/(S' - nX_{(1)})]^{r-2}, & X_{(1)} \leqq t \leqq [S' - (n-k)X_{(1)}]/k \\ 0, & t > [S' - (n-k)X_{(1)}]/k \end{cases}$
		$(k/n)R^n(t, \boldsymbol{\theta})$	k/n	Y_1/n			†$\begin{cases} k/n, & t \leqq X_{(1)} \\ 0, & t > X_{(1)} \end{cases}$
		$*R^k(t, \boldsymbol{\theta})$					$\begin{cases} 1, & t < X_{(1)} \\ (1-k/n)[1 - k(t - X_{(1)})/(S' - nX_{(1)})]^{r-2}, & X_{(1)} \leqq t \leqq [S' - (n-k)X_{(1)}]/k \\ 0, & t > [S' - (n-k)X_{(1)}]/k \end{cases}$

* By addition.

† The unbiased estimate $U(X)$, being a function of the sufficient statistic, is itself the MVUE.

$$S' = \sum_{i=1}^{r} X_{(i)} + (n-r)X_{(r)}$$

The steps in the derivation of the MVUE's of reliability function associated with the normal, gamma, one-parameter exponential and Weibull distributions are given in Table 1. Similar steps for obtaining the MVUE's of powers of reliability function, associated with the two-parameter exponential distribution

$$f(x) = (1/\theta_2) \exp[-(x - \theta_1)/\theta_2]\delta(x - \theta_1), \quad \theta_1 \geqq 0, \theta_2 > 0,$$

where

$$\delta(w) = 1 \quad \text{if } w \geqq 0$$
$$= 0 \quad \text{otherwise}$$

using only r ($r \leqq n$) smallest observations (censored sample) are shown in Table 2. In order to apply Theorem 1 effectively to this distribution, the following transformation (Epstein and Sobel [4]) from X's to Y's is made:

$$Y_i = (n - i + 1)(X_{(i)} - X_{(i-1)}), \quad X_{(0)} = 0, \quad i = 1, 2, \cdots, r.$$

Here $X_{(i)}$ is the ith smallest observation. The Y's are then mutually independent. The density function of Y_1 is

$$f(x) = (1/\theta_2) \exp[-(x - n\theta_1)/\theta_2], \qquad x \geqq n\theta_1,$$

and that of Y_i, ($i = 2, 3, \cdots, r$), is

$$f(x) = (1/\theta_2) \exp[-x/\theta_2], \qquad x \geqq 0.$$

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