The Annals of Mathematical Statistics
1969, Vol. 40, No. 2, 697–701

PROPERTIES OF POWER FUNCTIONS OF SOME TESTS CONCERNING DISPERSION MATRICES OF MULTIVARIATE NORMAL DISTRIBUTIONS[1]

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1. Introduction. The following testing problems relating the parameters of a p-variate normal distribution $N_p(\mu, \Sigma)$ are considered.

(a) Hypothesis: $\Sigma = \Sigma_0$, a known positive definite matrix.
Alternative: $\Sigma \neq \Sigma_0$.

(b) Hypothesis: $\Sigma = \sigma^2 I_p$, where σ^2 is unknown and I_p stands for the $p \times p$ identity matrix.
Alternative: $\Sigma \neq \sigma^2 I_p$.

(c) Hypothesis: $\Sigma = \Sigma_0$, a known pd matrix, $\mu = \mu_0$, a known vector.
Alternative: $\Sigma \neq \Sigma_0$ and/or $\mu \neq \mu_0$.

It has been shown that the likelihood ratio test is unbiased for the problems (b) and (c) but not for the problem (a); however, the "modified" likelihood ratio test for the problem (a) is not only unbiased but its power function has the usual monotonicity property. The above result for the problem (b) was first found by Gleser [4] using a different method; Cohen [2] obtained similar results for a more general version of (b).

It is further shown that the likelihood ratio test for testing the equality of the covariance matrices of two p-variate normal distributions is biased when the sample sizes are unequal.

All the above results are known to be true when $p = 1$. For simplicity, only the canonical forms of the above problems are considered.

2. Test of the hypothesis $\Sigma = I_p$. The following lemmas will be used to prove the main result. These lemmas are stated without proofs (which are fairly easy).

LEMMA 2.1. *For $S > 0$, $r > 0$, the region*

$$S^r \operatorname{etr}(-S/2) \geqq k$$

is equivalent to the region $s_1 \leqq S \leqq s_2$ where

$$s_1^r \operatorname{etr}(-s_1/2) = s_2^r \operatorname{etr}(-s_2/2) = k.$$

LEMMA 2.2. *Let S be a random variable such that the distribution of S/σ^2 is χ^2 with m degrees of freedom. For $r > 0$, let*

$$\beta(\sigma^2) = P[S^r \operatorname{etr}(-S/2) \geqq k; \sigma^2].$$

Then

$$d\beta(\sigma^2)/d(\sigma^2) \gtreqless 0$$

Received 13 May 1968.

[1] Prepared with the support of the National Science Foundation Grant GP-3813.

according as

$$\sigma^2 \lesseqqgtr 2r/m.$$

COROLLARY 2.2.1. *If in Lemma* 2.2, $2r = m$ *then* $\beta(\sigma^2)$ *monotonically decreases as* $|\sigma^2 - 1|$ *increases.*

The critical regions of the likelihood ratio test and the modified likelihood ratio test for the hypothesis $\Sigma = I_p$ against $\Sigma \neq I_p$ based on a random sample of size N from $N_p(\mu, \Sigma)$ are given by

$$\omega: |S|^{N/2} \operatorname{etr}(-S/2) \leqq k,$$

and

$$\omega^*: |S|^{n/2} \operatorname{etr}(-S/2) \leqq k^*,$$

respectively, where $n = N - 1$ and $S = [S_{ij}]$ is distributed according to the Wishart distribution $W(\Sigma; n, p)$.

THEOREM 2.1. (i) *The likelihood ratio test for the problem* (a) *is biased.* (ii) *The power function of the modified likelihood ratio test for the problem* (a) *involves the parameters only through the characteristic roots of* Σ *and the power increases as the absolute deviation of each such characteristic root from* 1 *increases.*

PROOF. (i) It is known that the probability of ω depends on the parameters only through the characteristic roots of Σ. So, without any loss of generality, we shall take Σ to be a diagonal matrix

$$\Delta = \operatorname{diag}\{\delta_1, \delta_2, \cdots, \delta_p\}.$$

Note that

$$|S|^m \operatorname{etr}(-S/2) = |S|^m[\textstyle\prod_{k=1}^p S_{kk}^m]^{-1} \prod_{k=1}^m [S_{kk}^m \operatorname{etr}(-S_{kk}/2)].$$

Moreover, $|S|^m/\prod_{k=1}^p S_{kk}^m$, $S_{kk}^m \operatorname{etr}(-S_{kk}/2)$, $(k = 1, \cdots, p)$ are mutually independent. The distribution of $|S|^m/\prod_{k=1}^p S_{kk}^m$ is free from δ's and S_{kk} is distributed as $\chi^2_{(n)} \cdot \delta_k$.

It follows from Lemma 2.2 that there exists a constant δ_p^* such that $1 < \delta_p^* < Nn^{-1}$ and

$$P[S_{pp}^{N/2} \operatorname{etr}(-S_{pp}/2) \geqq C_p;\ \delta_p = 1] < P[S_{pp}^{N/2} \operatorname{etr}(-S_{pp}/2) \geqq C_p;\ \delta_p = \delta_p^*]$$

irrespective of the value of C_p chosen. The desired result will follow if we evaluate the probability integral over ω first with respect to S_{pp} fixing the other variates $S_{11}, \cdots, S_{p-1,p-1}$ and $|S|^{N/2}/\prod_{k=1}^p S_{kk}^{N/2}$.

(ii) This follows from the facts noted in the proof of (i) and Corollary 2.2.1.

3. Unbiasedness of the likelihood ratio test for the sphericity.

THEOREM 3.1. *For testing the hypothesis* $H: \Sigma = \sigma^2 I_p$ *against the alternatives* $K: \Sigma \neq \sigma^2 I_p$ *the likelihood ratio test based on a random sample of size N from* $N_p(\mu, \Sigma)$ *is unbiased.*

PROOF. The critical region of the likelihood ratio test is given by

$$\omega: |S|[\operatorname{tr}(S)]^{-p} \leqq k$$

where $S \cap W(\Sigma; n, p)$. Since the power of the above test involves the parameters only through the characteristic root of Σ, we shall assume Σ to be a diagonal matrix. Note that

$$|S|[\text{tr}\,(S)]^{-p} = [|S|/\textstyle\prod_{k=1}^{p} S_{kk}][\prod_{k=1}^{p} S_{kk}/(\text{tr}\,(S))^{p}].$$

The factors in the right-hand side of the above are independently distributed and the distribution of the first factor is free from any parameter. The unbiasedness of the likelihood ratio test now follows from the result of Brown [1].

4. Unbiasedness of the likelihood ratio test for $\mu = 0$ and $\Sigma = I_p$.

THEOREM 4.1. *The likelihood ratio test for the hypothesis $H: \mu = 0$, $\Sigma = I_p$ against the alternatives K: not H, based on a random sample of size N from $N_p(\mu, \Sigma)$ is unbiased.*

PROOF. The critical region of the likelihood ratio test is given by

$$\omega: |S|^{N/2} \text{etr}\,[-(S + N\bar{X}\bar{X}')/2] \leqq k$$

where $\bar{X}$ is the sample mean vector and S/N is the sample covariance matrix. We shall show the unbiasedness by using the following two inequalities.

(i) $P(\bar{\omega} \mid \mu = 0, \Sigma = I_p) > P(\bar{\omega} \mid \mu = 0, \Sigma)$ for any pd matrix $\Sigma \neq I_p$.

(ii) $P(\bar{\omega} \mid \mu = 0, \Sigma) > P(\bar{\omega} \mid \mu, \Sigma)$ for any $\mu \neq 0$.

To prove the inequality (i) we shall assume, without loss of generality, that Σ is diagonal. Let

$$A = [a_{ij}] = S + N\bar{X}\bar{X}'.$$

Note that

$$|S|^{N/2} \text{etr}\,(-A/2) = [(|S|/|A|)^{N/2}][|A|^{N/2} \text{etr}\,(-A/2)].$$

When $\mu = 0$, the distribution of the first factor in above is free from any parameter and it is independent of the second factor. The inequality (i) now follows from Lemma 2.2 and the fact that $A \cap W(\Sigma; N, p)$. The second inequality follows from the results of Das Gupta, Anderson and Mudholkar [3]

5. Equality of the covariance matrices of two p-variate normal distributions.

THEOREM 5.1. *The likelihood ratio test for the hypothesis $H: \Sigma_1 = \Sigma_2$ against the alternatives $K: \Sigma_1 \neq \Sigma_2$ based on random samples of sizes N_1 and N_2 ($N_1 \neq N_2$) from $N_p(\mu_1, \Sigma_1)$ and $N_p(\mu_2, \Sigma_2)$, respectively, is biased.*

First we prove the following lemma.

LEMMA 5.1. *Let Y be a random variable such that δY ($\delta > 0$) is distributed according to the F-distribution with degrees of freedom $n_1 - 1$ and $n_2 - 1$ ($n_1 < n_2$). Let*

$$\beta(\delta) = P[Y^{n_1}/(1 + Y)^{n_1+n_2} \geqq k \mid \delta].$$

Then there exists a constant $\lambda(<1)$ independent of k such that

$$\beta(\delta) > \beta(1)$$

for all δ lying between λ and 1.

NOTE. If $n_1 > n_2$, the same result holds with $\lambda > 1$.

PROOF. The region

$$Y^{n_1}/(1 + Y)^{n_1+n_2} \geqq k$$

is equivalent to $y_2 \leqq Y \leqq y_1$ where

$$(y_1/y_2)^{n_1} = [(1 + y_1)/(1 + y_2)]^{n_1+n_2}.$$

Then

$$\beta(\delta) = B \int_{\delta y_2}^{\delta y_1} y^{(n_1-3)/2}/(1 + y)^{(n_1+n_2)/2-1}\, dy$$

where B is a numerical constant. Differentiating with respect to δ, we get

$$\beta'(\delta) = B\delta^{(n_1-3)/2}[y_1^{(n_1-1)/2}/(1 + \delta y_1)^{(n_1+n_2)/2-1} - y_2^{(n_1-1)/2}/(1 + \delta y_2)^{(n_1+n_2)/2-1}].$$

Thus $\beta'(\delta) \gtreqless 0$ according as

$$(y_1/y_2)^{(n_1-1)/2} \gtreqless ((1 + \delta y_1)/(1 + \delta y_2))^{(n_1+n_2)/2-1}$$

i.e.

$$((1 + y_1)/(1 + y_2))^{\lambda} \gtreqless ((1 + \delta y_1)/(1 + \delta y_2))$$

where

$$\lambda = (n_1 + n_2)n_1^{-1}(n_1 - 1)(n_1 + n_2 - 2)^{-1}.$$

Note that $\lambda < 1$. It follows now that there exist δ_0 such that

$$\beta'(\delta) \gtreqless 0 \qquad \text{according as} \qquad \delta \lesseqgtr \delta_0,$$

where $\delta_0 < 1$. Since $\log(1 + \lambda x) - \lambda \log(1 + x)$ is an increasing function of x, we find $\beta'(\lambda) < 0$. Thus $\delta_0 \leqq \lambda < 1$. Therefore

$$\beta(\delta) > \beta(1)$$

for all δ lying between λ and 1.

PROOF OF THEOREM 5.1. The critical region of the likelihood ratio test is given by

$$|S_1|^{N_1}|S_2|^{N_2}/|S_1 + S_2|^{N_1+N_2} \leqq k,$$

where S_1/N_1 and S_2/N_2 are the maximum likelihood estimates of Σ_1 and Σ_2, respectively. For considering the power of this test we may assume, without any loss of generality, that $\Sigma_1 = \Gamma$, a diagonal matrix, and $\Sigma_2 = I_p$. In particular, let

$$\Gamma = \operatorname{diag}\{\delta, 1, 1, \cdots, 1\}.$$

Let $S_1 = [S_{ij}^{(1)}]$, $S_2 = [S_{ij}^{(2)}]$. Then

$$|S_1|^{N_1}|S_2|^{N_2}/|S_1 + S_2|^{N_1+N_2} = [(S_{11}^{(1)})^{N_1}(S_{11}^{(2)})^{N_2}/(S_{11}^{(1)} + S_{11}^{(2)})^{N_1+N_2}]\cdot Z.$$

It is easy to see that under the above structure of Γ the distribution of Z is free from all the parameters and it is independent of the first factor in the above. The desired result now follows from Lemma 5.1.

REMARKS. For $p = 1$, the result in Section 5 was shown by Brown [1]. My technique does not show that the likelihood ratio test for $\Sigma_1 = \Sigma_2$ is unbiased when $N_1 = N_2$. Using a modification of Pitman's [5] technique, Sugiura and Nagao [6] prove this result. They also show the unbiasedness part in Theorem 2.1 (ii) and Theorems 3.1 and 4.1 using methods quite different from the methods of this paper.

REFERENCES

[1] BROWN, G. W. (1939). On the power of the L_1 test for the equality of several variances. *Ann. Math. Statist.* **10** 119–128.

[2] COHEN, A. (1967). A note on the unbiasedness of the likelihood ratio tests for some normal covariance matrices. (Unpublished)

[3] DAS GUPTA, S, ANDERSON, T. W. and MUDHOLKAR, G. S. (1964). Monotonicity of the power functions of some tests of the multivariate linear hypothesis. *Ann. Math. Statist.* **35** 200–205.

[4] GLESER, L. J. (1966). A note on the sphericity test. *Ann. Math. Statist.* **37** 464–467.

[5] PITMAN, E. J. G. (1939). Tests of hypothesis concerning location and scale parameters. *Biometrika* **31** 200–215.

[6] SUGIURA, N. and NAGAO, H. (1968). Unbiasedness of some test criteria for the equality of one or two covariance matrices. *Ann. Math. Statist.* **39** 1686–1692.