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THE ASYMPTOTIC BEHAVIOR OF A CERTAIN MARKOV CHAIN

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1. Introduction. This note is concerned with the limiting behavior of a Markov chain which arose as a model for a peculiar kind of growth process. Its precise definition is as follows: let $S_1 = 1$, and

$$S_{N+1} = S_N + X_N\,(N + 1 - S_N), \qquad N = 1, 2, \cdots, \tag{1.1}$$

where $\{X_N\}$ is a sequence of independent random variables satisfying $0 \leqq X_N \leqq 1$. A problem that we pose, but do not solve, is to find necessary and sufficient conditions on the sequence $\{X_N\}$ to ensure that S_N/N converges a.s. to 1. A sufficient condition for this is presented, and also some related facts concerning the moments of S_N. It should be noted that the Markov chain S_N does not have temporally stationary transition probabilities and that the state space may be uncountable.

2. Convergence results.

THEOREM 1. *If there exists an $\alpha > 0$ such that $E(X_i) \geqq \alpha$ for all i, then* $S_N/N \to 1$, a.s.

PROOF. Since $S_N = (1 - X_{N-1})\,S_{N-1} + N\cdot X_{N-1}$ and $E(1 - X_N) \leqq 1 - \alpha < 1$, for all N,

$$\begin{aligned} E(N - S_N) &= E\{(1 - X_{N-1})(N - S_{N-1})\} \\ &= E(1 - X_{N-1})\cdot E(N - 1 - S_{N-1}) + E(1 - X_{N-1}) \\ &\leqq (1 - \alpha)\,E(N - 1 - S_{N-1}) + (1 - \alpha). \end{aligned}$$

By induction then $E(N - S_N) \leqq \sum_1^{N-1} (1 - \alpha)^i < K < \infty$. Hence $E(N - S_N) < K$, for all N. Similarly,

$$\begin{aligned} E\{(N - S_N)^2\} &= E\{(1 - X_{N-1})^2\}E\{(N - S_{N-1})^2\} \\ &= E\{(1 - X_{N-1})^2\}E\{(N - 1 - S_{N-1})^2\} + E\{(1 - X_{N-1})^2\} \\ &\quad + 2E\{(1 - X_{N-1})^2\}E\{N - 1 - S_{N-1}\} \\ &\leqq (1 - \alpha)E\{N - 1 - S_{N-1})^2\} + K' \end{aligned}$$

where K' does not depend on N. Again by induction we have $E\{(N - S_N)^2\} \leqq K''$, for all N. Now for $\epsilon > 0$,

$$P(|S_N/N - 1| > \epsilon) = P(|S_N - N|N^{-1} > \epsilon) \leqq E\{(N - S_N)^2\}/N^2\epsilon^2 \leqq K''/N^2\epsilon^2.$$

Thus $\sum_1^\infty P(|S_N/N - 1| > \epsilon) < \infty$ which implies that $S_N/N \to 1$, a.s.

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THEOREM 2. *The sequence* $\{\text{Var } S_N\}$ *is bounded. Moreover, if* $E(X_N) = \alpha$, *and* $E(X_N^2) = \alpha_2$, *for all* N, *where* $\alpha > 0$ *and* α_2 *are constants then* $\lim_{N\to\infty} \text{Var } S_N = \alpha^{-2}\cdot(\alpha_2 - \alpha^2)(2\alpha - \alpha_2)^{-1}$.

PROOF. The first statement follows from the proof of the last theorem: $\text{Var } S_N = \text{Var}(N - S_N) \leqq E\{(N - S_N)^2\} \leqq K''$. Now suppose $E(X_N) = \alpha$, $E(X_N^2) = \alpha_2$, and $\text{Var } S_N = \sigma_N^2$, for all N. Then by squaring both sides of (1.1), taking expectations, and performing some routine algebraic reductions we derive the identity

$$\sigma_{N+1}^2 = \beta\sigma_N^2 + r_N + \gamma, \quad N = 1, 2, \cdots, \quad \sigma_1^2 = 0, \tag{2.1}$$

where $\beta = 1 - 2\alpha + \alpha_2$, $\gamma = \alpha_2/\alpha^2 - 1$, and $r_N \to 0$ as $N \to \infty$. Since $0 < \beta < 1$, it is clear that the solution to the difference equation (2.1) converges to a limit L satisfying

$$L = \beta L + \gamma,$$

so that $L = \gamma/1 - \beta = \alpha^{-2}\cdot(\alpha_2 - \alpha^2)(2\alpha - \alpha_2)^{-1}$. (To derive (2.1) one should first note that $E(S_N) = N + 1 - \alpha^{-1} + \alpha^{-1}(1 - \alpha)^N$.)

3. Remarks.

(i) Theorem 1 may fail to hold if, for example, $E(X_N) \to 0$. In fact, if $E(X_N) = 1/N + 1$, then $E(S_N/N) \to \frac{1}{2}$ so that $S_N/N \nrightarrow 1$.

(ii) For the case of an arbitrary sequence $\{X_N\}$, $0 \leqq X_N \leqq 1$, it is not known whether $\lim_{N\to\infty} E(N - S_N)$ exists.

(iii) All of these questions pertain to the stability of the solutions to a linear stochastic difference equation, for various conditions on the (random) coefficients. It would be of interest to have some general results along these lines.

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