The Annals of Mathematical Statistics
1969, Vol. 40, No. 2, 653–657

A NOTE ON SEQUENTIAL MULTIPLE DECISION PROCEDURES[1]

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0. Introduction. We introduce a family of procedures for choosing one out of k decisions concerning the (unknown) mean of a normal distribution (with known variance). Sobel and Wald proposed in [7] a procedure for the case $k = 3$. Their procedure can be expressed as a composition of two SPRT's for testing simple hypotheses. We followed their way of reasoning, but applied it to Anderson's modification of the SPRT [1]. We show that Paulson's procedure [4] is of the form of the suggested procedures, but can be improved. More explicitly, the (sampling) continuation region of some of the suggested procedures are subsets of those of Paulson's. As a consequence, the number of observations required by any one of them is never greater than the sample size required by Paulson.

1. The problem. Let $a_1 < a_2 < a_3 < \cdots < a_{k-1}$ be real numbers. Denote $a_0 = -\infty$, $a_k = +\infty$.

Let X be a rv normally distributed with unit variance and unknown mean θ. We want to choose one of the k decisions

$$D_i: \theta \,\varepsilon\, (a_{i-1}, a_i], \qquad i = 1, 2, \cdots, k, \tag{1}$$

when the loss function for the decision D_i is defined as the indicator of the complement of the interval $(a_{i-1} - \frac{1}{2}\Delta, a_i + \frac{1}{2}\Delta)$, where Δ is a positive real number satisfying

$$\Delta \ll \min_{1 \leqq i \leqq k-2} (a_{i+1} - a_i). \tag{2}$$

(The interval $(a_i + \frac{1}{2}\Delta, a_{i+1} - \frac{1}{2}\Delta)$ will be called "nonindifference interval").

A "solution" to the problem is a sequential procedure δ satisfying for a preassigned number $\alpha \,\varepsilon\, (0, 1)$.

$$\sup_\theta E_\theta l(\delta(X), \theta) \leqq \alpha. \tag{3}$$

The present work deals with a special kind of solution, that can be described as a partition of the (n, s_n) plane into $k + 1$ sets: one sampling continuation set and k decision sets. The boundaries of these sets are broken lines.

This kind of procedure was treated extensively by Gordon Simons [5], [6]. The procedure presented in the next chapter is in some sense a special case of his general model.

Received 11 January 1968; revised 9 August 1968.

[1] This paper is a portion of the author's Master's thesis at the Hebrew University. Its publication was partially supported by Tel Aviv University and by the Air Force Office of Scientific Research, Office of Aerospace Research, U.S.A.F., under AFOSR Grant 1312-67.

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2. The procedure. Let $A > 0$, $B \geqq 0$ be two numbers. Their value is to be determined later.

Denote $S_m = \sum_{i=1}^{m} X_i$.

Define (for $i = 1, 2, \cdots, k-1$):

$$t_{0i}: \text{The first integer } j \text{ satisfying } S_j \leqq j(a_i + B) - A; \tag{4}$$

$$t_{1i}: \text{The first integer } j \text{ satisfying } S_j \geqq j(a_i - B) + A. \tag{5}$$

In (4) or (5), if no such integer exists, put $t_{ni} = \infty$ $(n = 0, 1)$

$$t_i = \min(t_{0i}, t_{1i}), \tag{6}$$

$$t = \max(t_1, t_2, \cdots, t_{k-1}). \tag{7}$$

REMARK. We shall later use t as the stopping time of a procedure. It is easy to verify that the t_i (and hence t), are stopping times, since for $B > 0$ they are bounded, and for $B = 0$, t_i is actually the stopping time used in SPRT for testing $H_0: \theta = a_i - c$ against $H_1: \theta = a_i + c$, where c is any positive number.

Let the vector

$$S = S(t_{ji}) = (P_1, P_2, \cdots, P_k) \tag{8}$$

be defined by

$$P_k = 0, \quad P_i = 0 \quad t_i = t_{0i}; \qquad = 1 \quad t_i = t_{1i} \qquad (\text{for } i = 1, 2, \cdots, k-1). \tag{9}$$

It can be shown that the P_i's form a nonincreasing sequence.

We shall use the vector S to define a solution in the following way:

(10) *If m is the first coordinate j of S at which $P_j = 0$, decide D_m.*

The Sobel-Wald procedure [7] is a special case of the last, for $k = 3$. They require $B = 0$, and find A to be approximately

$$A^{\text{S.W.}} = \Delta^{-1} \ln((1 - \alpha)\alpha^{-1}). \tag{11}$$

They represented their procedure as the composition of two tests of simple hypotheses: One testing $H_0: a_1 - \frac{1}{2}\Delta$ against $H_1: \theta = a_1 + \frac{1}{2}\Delta$, the other testing $H_0': \theta = a_2 - \frac{1}{2}\Delta$ against $H_1': \theta = a_2 + \frac{1}{2}\Delta$. They tested both by means of SPRT's. Their final decision is D_1 if both H_0 and H_0' are accepted, D_2 if H_1 and H_0' are accepted, D_3 if H_1 and H_1'. They prove that the acceptance of both H_0 and H_1' is impossible. The probability of deciding D_i can be expressed in terms of the operating characteristic of the SPRT's. (3) gives a system of inequalities that determine the boundaries of both SPRT's. It is intuitively clear that the same kind of treatment can be applied to $k > 3$. The possibility of doing it was pointed out by Lechner and Ginsburg in [2]. We carried it out in [3], where we also computed upper and lower bounds for the expected number of observations (as a function of θ). Anderson proposed in [1] a modification of the SPRT for testing $H_0:\theta = \theta_0$ against $H_1:\theta = \theta_1$, where θ is the (unknown) mean of a normal distribution with unit variance. Instead of a sampling region

$$C \leqq \sum_{i=1}^{n} (x_i - \tfrac{1}{2}(\theta_0 + \theta_1)) \leqq D,$$

he replaced C and D by linear functions of n. In Corollary 4.5 he shows that if the sampling region is

$$-c(1 - nN^{-1}) \leqq \sum_{i=1}^{n} (x_i - \tfrac{1}{2}(\theta_0 + \theta_1)) \leqq c(1 - nN^{-1}),$$

then the probability of accepting H_1, when the expectation of X is θ is approximately

$$\varphi[(\theta - \tfrac{1}{2}(\theta_0 + \theta_1))N^{\frac{1}{2}}(\tfrac{1}{4}\beta^2 N c^{-2} + 1)^{-\frac{1}{2}}] \tag{12}$$

where $\beta = 1.702$ and φ is the cumulative standard normal distribution.

We repeated the arguments of Sobel-Wald, but applied them to Anderson's results rather than those related to SPRT.

An advantage of using Anderson's tests is that they lead to bounded stopping times. Furthermore, we can fix a bound N and find values A, B defining a procedure that requires at most N observations, and for which

$$\sup_\theta E_\theta l(\delta(X), \theta) \approx \alpha. \tag{13}$$

NOTE. We must require $N > 4\Delta^{-2}Z^2_{(1-\alpha)}$. This number is the sample size n required by a nonsequential procedure that decides D_i if $\bar{x}_n \varepsilon [a_{i-1}, a_i)$.

The values A and B are

$$A_N = 1.702\, Z_{(1-\alpha)}[N(N\Delta^2 - 4\, Z^2_{(1-\alpha)})^{-1}]^{\frac{1}{2}}, \tag{14}$$
$$B_N = A_N N^{-1}.$$

As in Sobel-Wald, these values are approximations to the true ones, and (3) need not be satisfied. E. Paulson [4] proposed a solution of an apparently different kind, that satisfies (3). We shall see in the next section that his solution is of the form we are dealing with.

3. Paulson's procedure. Paulson defines two rv

$$U_r = \max_{1 \leqq i \leqq r} \{\bar{X}_i - \tfrac{1}{4}\Delta - 2\,(\log 2\alpha^{-1})(\Delta i)^{-1}\}, \tag{15}$$
$$V_r = \min_{1 \leqq i \leqq r} \{\bar{X}_i + \tfrac{1}{4}\Delta + 2\,(\log 2\alpha^{-1})(\Delta i)^{-1}\}.$$

He stops the sampling the first time that either $U_r \geqq V_r$ (and then he decides the interval containing $\bar{x}_r$), or the interval (U_r, V_r) does not intersect two nonindifference regions (in which case he decides the interval whose nonindifference part is intersected by (U_r, V_r). If (U_r, V_r) turns out to be a subset of some indifference region, he chooses one of the adjacent intervals by throwing a coin).

Actually, he got a family of procedures, but he required the bound of the stopping time to be minimal. This requirement led to (15). If this restriction is not necessary, we may multiply Δ (in (15)) by any $R\, \varepsilon\, [1, 2)$ in both the numerators and denominators. The bound for the stopping time will then be

(16) $$N = 8 \log 2\alpha^{-1} \cdot R^{-1}(2 - R)^{-1}\Delta^{-2}.$$

As we easily see, the minimum of N is attained at $R = 1$. We shall work with the modified U_r, V_r, maintaining the same notation.

THEOREM. *Let*

(17) $$A_N^P = 2\Delta^{-1}R^{-1} \log 2\alpha^{-1}, \qquad B_N^P = \tfrac{1}{4}\Delta(2 - R),$$

(*where N and R are related by* (16)). *t defined in* (7) *with A, B replaced by* A_N^P, B_N^P *stops at the first j satisfying* (*for some i*)

(18) $$\min_{m \leqq j} \{\bar{x}_m + \tfrac{1}{4}\Delta R + 2\Delta^{-1}R^{-1}m^{-1} \ln 2\alpha^{-1}\} \leqq a_i + \tfrac{1}{2}\Delta,$$
$$\max_{m \leqq j} \{\bar{x}_m - \tfrac{1}{4}\Delta R - 2\Delta^{-1}R^{-1}m^{-1} \ln 2\alpha^{-1}\} \geqq a_{i-1} - \tfrac{1}{2}\Delta.$$

PROOF. Denote $t_0 = t_{1,0} = 1$, $t_k = t_{0,k} = 1$. t is the first j satisfying (for some i) $t_i = t_{0,i}$; $t_{i-1} = t_{1,i-1}$; $\max(t_{1,i-1}; t_{0,i}) = j$. That is, if $t = j$, there exist an i ($i = 1, 2, \cdots, k$) and two integers j_1, j_2 with $\max(j_1, j_2) = j$ such that: j_1 is the first integer satisfying

(19) $$S_{j_1} \leqq j_1(a_i + \tfrac{1}{4}\Delta(2 - R)) - 2\Delta^{-1}R^{-1} \log 2\alpha^{-1};$$

j_2 is the first integer satisfying

(20) $$S_{j_2} \geqq j_2(a_{i-1} - \tfrac{1}{4}\Delta(2 - R)) + 2\Delta^{-1}R^{-1} \log 2\alpha^{-1}.$$

(19) and (20) can be written as

(21) $$\bar{x}_{j_1} + \tfrac{1}{4}\Delta R + 2\Delta^{-1}R^{-1}j_1^{-1} \log 2\alpha^{-1} \leqq a_i + \tfrac{1}{2}\Delta,$$
$$\bar{x}_{j_2} - \tfrac{1}{4}\Delta R - 2\Delta^{-1}R^{-1}j_2^{-1} \log 2\alpha^{-1} \geqq a_{i-1} - \tfrac{1}{2}\Delta.$$

Hence t is the first integer j for which there exists an i satisfying (18). Q.E.D.

The decisions made by (10) applied to A_N^P, B_N^P and by Paulson are almost the same. The only situation in which the decision may be different is the case when (U_r, V_r) is a subset of an indifference region. (10) tells exactly what to do, and Paulson randomizes the solution. The difference is immaterial.

TABLE

A comparison between $A^{\text{S.W.}}$, A_∞ *and* A_∞^P *as defined in (22) for different values of* α

	α						
	.1	.05	.04	.03	.02	.01	.001
$\Delta A^{\text{S.W.}} = \ln(1 - \alpha)\alpha^{-1}$	2.1972	2.9444	3.1780	3.4758	3.8918	4.5951	6.9067
$\Delta A_\infty = 1.702 Z_{(1-\alpha)}$	2.18	2.8	2.98	3.2	3.49	3.965	5.26
$\Delta A_\infty^P = \ln 2\alpha^{-1}$	2.9957	3.6889	3.9120	4.195	4.6052	5.2983	7.5998

The preceding table gives a comparison between

$$(22) \qquad A_\infty^P = \lim_{N\to\infty} A_N^P, \qquad A_\infty = \lim_{N\to\infty} A_N \text{ and } A^{\text{S.W.}}.^3$$

$A_N^P - A_N$ can be shown to exceed $A_\infty^P - A_\infty$, in the range of α's of the table. Furthermore, it can also be shown that for each R there exists an N such that the sampling continuation region of the procedure defined by A_N, B_N is a subset of the sampling continuation region of the Paulson procedure defined by R. This shows that the family of procedures determined by A_N, B_N is uniformly (in θ) better than the family of Paulson procedures, in the sense that it requires the same or fewer observations and hence decreases the expected sample size. Naturally, we pay in probability of error for this gain, but if (13) is a good approximation, it doesn't matter very much, because we don't require the probability of error to be small, we just require the satisfaction (or quasi-satisfaction) of (3).

As pointed out in Section 2, the Sobel-Wald procedure can be easily generalized to any $k > 3$. The values of $A^{\text{S.W.}}$ shown in the table are not restricted to $k = 3$.

Acknowledgment. This work was done under the guidance of Dr. J. Yahav. The subject was suggested by Dr. E. Samuel. I am deeply indebted to both for their advice and encouragement. My thanks are also due to Prof. P. Bickel, for his valuable comments and suggestions. I also wish to thank the referee, whose comments helped to clarify the presentation of the subject.

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[3] We would expect A_∞ and $A^{\text{S.W.}}$ to be equal, since Anderson's procedure becomes an SPRT as the truncation point $N \to \infty$. But both are only approximations of the true value, obtained in different ways. Anderson obtains (12) via a function that approximates the normal cumulative distribution. This element is extraneous to Wald's work.