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SOME FIRST PASSAGE PROBLEMS FOR $S_n/n^{\frac{1}{2}}$ [1]

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1. Introduction and summary. Let $x_1, x_2, \cdots$ be independent random variables with mean 0 and variance 1. Let $s_n = x_1 + \cdots + x_n$, $n \geqq 1$, and for each $c > 0$ define

$$\tau_1 = \tau_1(c) = \text{first } n \geqq 1 \text{ for which } s_n > cn^{\frac{1}{2}}$$
$$= \infty \quad \text{if} \quad s_n \leqq cn^{\frac{1}{2}} \quad \text{for all} \quad n,$$

and

$$\tau_2 = \tau_2(c) = \text{first } n \geqq 1 \text{ for which } |s_n| > cn^{\frac{1}{2}}$$
$$= \infty \quad \text{if} \quad |s_n| \leqq cn^{\frac{1}{2}} \quad \text{for all} \quad n.$$

The stopping times τ_1 and τ_2 have received considerable attention recently (see, for example, [1], [2], [3], [6], and [8]), and naturally there has arisen the question as to whether $P\{\tau_k < \infty\} = 1$, $k = 1, 2$. One contribution of this note is

(1) THEOREM. *If $s_n/n^{\frac{1}{2}}$ does not tend in probability to* 0, *then for each $c > 0$, $P\{\tau_2 < \infty\} = 1$.*

We show by examples that (1) is no longer true with τ_2 replaced by τ_1. The final section contains two remarks bearing on the converse to (1).

2. Proof of (1). By hypothesis there exists a subsequence (n') of positive integers along which

$$P\{|S_{n'}/(n')^{\frac{1}{2}}| > \epsilon\} > \epsilon. \tag{2}$$

According to the Helly-Bray lemma there exists a further subsequence (n'') and a distribution function F for which

$$P\{s_{n''}/(n'')^{\frac{1}{2}} \leqq x\} \rightarrow F(x) \quad \text{as} \quad n'' \rightarrow \infty$$

at all continuity points x of F. From the fact that $E\{(s_n/n^{\frac{1}{2}})^2\} \equiv 1$ it follows that

(3) $(s_n/n^{\frac{1}{2}})$ is stochastically bounded, and hence F has total variation 1,

and also that

$$F \quad \text{has mean} \quad 0. \tag{4}$$

By a well-known characterization of limits in law of the row sums of triangular

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arrays, F is infinitely divisible. Hence

$$F(-y) + (1 - F(y)) > 0 \tag{5}$$

for every real y, for if this were not the case we could conclude from the definition of infinite divisibility that F is degenerate, contradicting either (2) or (4).

(5) implies that for each positive c,

$$\begin{aligned}(6)\quad P\{\tau_2(c) < \infty\} &\geqq P\{\limsup |s_n|/n^{\frac{1}{2}} > 2c\} \\ &= \lim_{N\to\infty} P\{\sup_{n\geqq N} |s_n|/n^{\frac{1}{2}} > 2c\} \geqq \limsup P\{|s_n|/n^{\frac{1}{2}} > 2c\} > 0.\end{aligned}$$

In view of the Kolmogorov 0-1 law, the second term of (6) is either 0 or 1; hence $P\{\tau_2(v) < \infty\} = 1$.

3. Two counterexamples in the one-sided case. The following example shows that (1) is no longer true with τ_2 replaced by τ_1.

Let G be any infinitely divisible distribution function with mean 0, variance 1, and $G(c_0) = 1$ for some $\infty > c_0 > 0$. Assume $k(n)$ is a sequence of positive integers for which

$$k(n)/(k(1) + \cdots + k(n)) \to 1 \quad \text{as} \quad n \to \infty, \tag{7}$$

and set $\nu(n) = k(1) + \cdots + k(n)$ $(\nu(0) = k(0) = 0)$. Now suppose y_{nk}, $n = 1, 2, \cdots, k = 1, 2, \cdots, k(n)$, is an array of mutually independent random variables with the property that for $n = 1, 2, \cdots, y_{n1}, y_{n2}, \cdots, y_{nk(n)}$ have a common distribution, and

$$P\{y_{n1} + \cdots + y_{nk(n)} \leqq x\} = G(x) \qquad \text{for all real} \quad x. \tag{8}$$

Finally, define the sequence $x_1, x_2, \cdots$ of independent random variables with mean 0 and variance 1 by

$$x_{\nu(n-1)+k} = (k(n))^{\frac{1}{2}} y_{nk}, \qquad n = 1, 2, \cdots, \quad k = 1, 2, \cdots, k(n).$$

By (8), $P\{y_{nk} \leqq c_0/k(n)\} = 1$, and hence

$$P\{(k(n))^{\frac{1}{2}}(y_{n1} + \cdots + y_{nk})/k^{\frac{1}{2}} \leqq (k/k(n))^{\frac{1}{2}} c_0\} = 1. \tag{9}$$

We write

$$\begin{aligned}(10)\quad & s_{\nu(n-1)+k}/(\nu(n-1) + k)^{\frac{1}{2}} \\ &= [\nu(n-1)/(\nu(n-1) + k)]^{\frac{1}{2}}[s_{\nu(n-1)}/(\nu(n-1))^{\frac{1}{2}}] \\ &\qquad + [k/(\nu(n-1) + k)]^{\frac{1}{2}}[(k(n))^{\frac{1}{2}}(y_{n1} + \cdots + y_{nk})/k^{\frac{1}{2}}]\end{aligned}$$

for $n = 1, 2, \cdots, k = 1, 2, \cdots, k(n)$, where $s_0/0$ is taken to be 0. Putting $k = k(n)$, it is easy to conclude from (7) and (8) that

$$P\{s_{\nu(n)}/(\nu(n))^{\frac{1}{2}} \leqq x\} \to G(x)$$

at continuity points x of G. According to (9) and (10),

$$s_{\nu(n-1)+k}/(\nu(n-1)+k)^{\frac{1}{2}} \leqq (s_{\nu(n-1)}/(\nu(n-1))^{\frac{1}{2}})^{+} + c_0 .$$

Choose a sequence (c_n) of positive constants decreasing to c_0 as $n \to \infty$ satisfying

$$\sum_{n=1}^{\infty} P\{s_{\nu(n)}/(\nu(n))^{\frac{1}{2}} > c_n\} < \infty. \tag{11}$$

Then for each fixed $\epsilon > 0$ and $m = 1, 2, \cdots$, letting $n^* = n^*(m)$ denote the largest integer $n - 1$ for which $\nu(n-1) \leqq m$, it follows from (11) that

$$\begin{aligned} P\{\sup_{j \geqq m} s_j/j^{\frac{1}{2}} > 2c_0 + \epsilon\} &\leqq \sum_{n=n^*}^{\infty} P\{\textstyle\bigcup_{k=1}^{k(n)} \{s_{\nu(n-1)+k}/(\nu(n-1)+k)^{\frac{1}{2}} \\ &\qquad > 2c_0 + \epsilon\}\} \\ &\leqq \sum_{n=n^*}^{\infty} P\{s_{\nu(n-1)}/(\nu(n-1))^{\frac{1}{2}} > c_0 + \epsilon\} \to 0 \\ &\qquad \text{as} \quad m \to \infty. \end{aligned}$$

Thus

$$P\{\limsup s_n/n^{\frac{1}{2}} \leqq 2c_0\} = 1,$$

and it follows that $P\{\tau_1(c) < \infty\} < 1$ for each $c > 2c_0$.

The point of the preceding example is that for any infinitely divisible distribution F with mean 0 and variance 1 there exist a sequence $x_1, x_2, \cdots$ of independent random variables having mean 0 and variance 1 and a subsequence $(\nu(n))$ of positive integers such that $(\nu(n))^{-\frac{1}{2}} s_{\nu(n)}$ converges in law to F as $n \to \infty$. It seems natural to inquire whether in our problem, that is, whether under the additional constraint that $F(c_0) = 1$ for some $c_0 > 0$, it is possible to take $\nu(n) \equiv n$. The answer in general is no, since a minimal additional requirement of F is that it belongs to the class L (see [5], p. 145 or [4], p. 554 for a definition of the class L of distribution functions).

Now let F be any distribution function of the class L having mean 0 and variance 1, and let φ denote the characteristic function of F. For any $n = 1, 2, \cdots$ let $\psi_n(t) = \varphi(n^{\frac{1}{2}}t)/\varphi((n-1)^{\frac{1}{2}}t)$. A consequence of [4], p. 554 (see also [5], p. 152) is that each ψ_n is a characteristic function, and because $\psi_n(t)\varphi((n-1)^{\frac{1}{2}}t) = \varphi(n^{\frac{1}{2}}t)$, the distribution of which ψ_n is the characteristic function has mean 0 and variance 1. Also,

$$\varphi(n^{\frac{1}{2}}t) = \prod_{k=1}^{n} \psi_k(t).$$

So if $X_1, X_2, \cdots$ are independent with X_k having the characteristic function ψ_k, then $n^{-\frac{1}{2}}(\sum_1^n X_k)$ has distribution function F for every $n = 1, 2, \cdots$. It remains to exhibit a member of the class L having mean 0, variance 1 and $F(c_0) = 1$ for some $c_0 > 0$ (or what is more convenient $F(c_0) = 0$ for some $c_0 < 0$). The following example is mentioned by Gnedenko and Kolmogorov ([5], p. 152) in a different context; we follow their notation. Suppose that F is the infinitely divisible distribution with finite variance and (in indicator function notation) Kolmogorov canonical measure $K(u) = u^2 I_{[0<u<1]}$, and also $\gamma = 0$. It is easy to see that F has mean 0 and variance 1. Moreover,

$$\log \varphi(t) = 2\int_0^1 \{e^{itu} - u - itu\} u^{-1}\, du,$$

which it is convenient to rewrite as

$$(12) \qquad \int_0^\infty [(e^{itu} - 1)/u] I_{[0<u<1]}\, du - 2it.$$

It follows from (12) and a theorem of Lévy and Baxter and Shapiro ([4], p. 539) that F is supported by the interval $[-2, +\infty)$. Consequently, if $x_n = -X_n$ $(n = 1, 2, \cdots)$, then $P\{s_n/n^{\frac{1}{2}} \leqq 2\} = 1$ $(n = 1, 2, \cdots)$ and $\tau_1(c) = +\infty$ a.s. for every $c \geqq 2$.

4. Remarks and acknowledgment. (a) The condition of the theorem (1) is not necessary. There are examples for which $s_n/n^{\frac{1}{2}}$ tends in probability to 0, but $P\{\tau_1(c) < \infty\} = 1$ for all c. One is as follows. Let y_n and z_n, $n = 1, 2, \cdots$, be a family of independent random variables with the following properties: the y_n are iid $N(0, 1)$; $E\{z_n\} \equiv 0$, $\text{Var}\{z_n\} \equiv 1$, $\sum P\{z_n \neq 0\} < \infty$. Suppose $\psi(n) \uparrow \infty$, $n - \psi(n) \uparrow \infty$ and that

$$\psi(n)/n \to 0 \quad \text{and} \quad \psi(n) \log\log \psi(n)/n \to \infty \qquad \text{as} \quad n \to \infty.$$

Put

$$s_n = \sum_{j=1}^{\psi(n)} y_j + \sum_{k=1}^{n-\psi(n)} z_k .$$

It is easy to infer from Borel-Cantelli and the law of the iterated logarithm that the required two conditions are indeed fulfilled by this s_n.

(b) A partial converse to (1) is this: if $s_n/n^{\frac{1}{2}} \to 0$ a.s., then there exists a constant $d \geqq 0$ with the property that for any $c < d$ there exists an integer $N(c)$ for which $P\{\tau_2(c) < N(c)\} = 1$; for any $c > d$, $\Pr\{\tau_2(c) = \infty\} > 0$. We conjecture, but have been unable to prove in complete generality, that $\tau_2(d)$ cannot be finite with probability one and unbounded.

(c) Our proof of (1) uses the independence of the x's in a striking way, first to conclude that F is infinitely divisible, and second to invoke the Kolmogorov 0–1 law. Our use of the 0–1 law can be replaced by a lengthier argument which is valid under more general conditions. We do not know, however, if (1) remains true for, say, martingale differences x_k for which $E\{x_k^2\} < \infty$, $k \geqq 1$, and $\liminf n^{-1} \sum_{k=1}^n E\{x_k^2 \mid x_1, \cdots, x_{k-1}\} \geqq \epsilon > 0$.

(d) Straightforward but tedious calculations show that for the first example of Section 3

$$P(s_n/n^{\frac{1}{2}} > c_0) = 0 \qquad \text{for each} \quad n,$$

from which it follows that $P\{\tau_1(c) < \infty\} = 0$ for any $c \geqq c_0$.

(e) We thank Charles Stein for a suggestion which led to the first example of Section 3.

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