The Annals of Mathematical Statistics
1969, Vol. 40, No. 2, 610–618

ANALYSIS OF COVARIANCE BASED ON GENERAL RANK SCORES[1]

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0. Summary. The purpose of the present investigation is to develop a class of rank order tests for the equality of treatment-effects in the presence of a set of concomitant variates for the analysis of covariance (ANOCA) model relating to completely randomized layouts. The proposed procedures are shown to be conditionally distribution-free and to have some desirable large sample properties.

1. Introduction. Let $\mathbf{Z}^{(p+1)\times 1} = (X^1, \mathbf{X}^{1\times p})'$ be a $(p + 1)$-stochastic vector, where X_0 (Scalar) is the primary variate and $\mathbf{X} = (X_1, \cdots, X_p)$ is a concomitant stochastic vector, $p \geqq 0$. Let $\mathbf{Z}_\alpha^{(k)} = (X_{0\alpha}^{(k)}, \mathbf{X}_\alpha^{(k)})'$, $\alpha = 1, \cdots, n_k$, be n_k independent and identically distributed (vector valued) random variables (iidrv) having a continuous cumulative distribution function (cdf) $G_k(\mathbf{z})$, $(\mathbf{z} \varepsilon R^{p+1}, k = 1, \cdots, c$, where all the $c(\geqq 2)$ samples are assumed to be mutually independent. The marginal (joint) cdf of $\mathbf{X}_\alpha^{(k)}$ is denoted by $F_k^{(1)}(\mathbf{x})$, $\mathbf{x} \varepsilon R^p$, and the conditional cdf of $X_{0\alpha}^{(k)}$ given $\mathbf{X}_\alpha^{(k)} = \mathbf{x}$ is denoted by $F_k^{(2)}(x_0 \mid \mathbf{x})$, $k = 1, \cdots, c$. The basic assumption of the paper is that the distribution of the concomitant variate is not affected by the application of the treatments, i.e.

$$F_k^{(1)}(\mathbf{x}) \equiv F^{(1)}(\mathbf{x}), \qquad k = 1, \cdots, c, \quad \mathbf{x} \varepsilon R^p. \tag{1.1}$$

Such an assumption is often found justified in practice [cf. Scheffé (1959), Chapter 6, where the corresponding parametric theory is thoroughly studied]. Consider then the following ANOCA model

$$F_k^{(2)}(y \mid \mathbf{x}) \equiv F^{(2)}(y - \tau_k \mid \mathbf{x}), \qquad k = 1, \cdots, c, \tag{1.2}$$

where $\boldsymbol{\tau} = (\tau_1, \cdots, \tau_c)$ is the treatment-effect vector. The null hypothesis of no treatment effect states that $\boldsymbol{\tau} = \mathbf{0}$, that is

$$H_0: F_k^{(2)}(y \mid \mathbf{x}) \equiv F^{(2)}(y \mid \mathbf{x}) \quad \text{for all} \quad k = 1, \cdots, c \quad \text{which implies that } \boldsymbol{\tau} = \mathbf{0} \text{ under (1.2)}. \tag{1.3}$$

The classical ANOCA test in the parametric case [cf. Scheffé (1959), Chapter 6] is based upon the linearity of regression of X_0 on $\mathbf{X}$ and the normality of the cdf's $F_k^{(2)}$, $k = 1, \cdots, c$. In the nonparametric set up, though a test for H_0 has recently been studied by Quade (1967), no unified theory for general rank order tests has yet been developed. The present paper aims in this direction. This has

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Received 4 April 1968.

[1] Work supported by (i) National Institutes of Health, Public Health Service, Grant GM-12868 (ii) The Office of Naval Research Contract Nonr-285(38).

been made possible by establishing a correspondence of the ANOCA problem with the multivariate multisample location problem and then utilizing the results of Puri and Sen (1966) on the ANOCA problem. The proposed tests are shown to be conditionally distribution-free, and they also include as a particular case, the test proposed by Quade (1967).[2] Various asymptotic properties of the proposed tests are studied and compared to the corresponding parametric (variance-ratio) test.

2. Preliminary notions. Let us denote the sample point by

$$(2.1)\quad \mathcal{C}_N = (\mathbf{Z}_1^{(1)}, \cdots, \mathbf{Z}_{n_1}^{(1)}, \cdots, \mathbf{Z}_{n_c}^{(c)});\quad \mathbf{Z}_\alpha^{(k)} = (X_{0\alpha}^{(k)}, X_{1\alpha}^{(k)}, \cdots, X_{p\alpha}^{(k)})',$$

$\alpha = 1, \cdots, n_k$, $k = 1, \cdots, c$, where $N = n_1 + \cdots + n_c$. Ranking the N elements in each row of $\mathcal{C}_N$ in increasing order of magnitude, we get a $(p + 1) \times N$ matrix

$$(2.2)\qquad \mathbf{R}_N = \begin{bmatrix} R_{01}^{(1)} & \cdots & R_{0n_1}^{(1)} & \cdots & R_{0n_c}^{(c)} \\ R_{11}^{(1)} & \cdots & R_{1n_1}^{(1)} & \cdots & R_{1n_c}^{(c)} \\ \vdots & & \vdots & & \vdots \\ R_{p1}^{(1)} & \cdots & R_{pn_1}^{(1)} & \cdots & R_{pn_c}^{(c)} \end{bmatrix},$$

where by virtue of the assumed continuity of G_k, $k = 1, \cdots, c$, the possibility of ties is neglected, in probability. Thus, each row of $\mathbf{R}_N$ is a permutation of the numbers $1, \cdots, N$. We replace the ranks $1, \cdots, N$ in the ith row of $\mathbf{R}_N$ by a set of general scores $\{E_{N,\alpha}^{(i)}, \alpha = 1, \cdots, N\}$, which is a set of N real (known) constants, for $i = 0, 1, \cdots, p$. This leads to the following rank-score matrix

$$(2.3)\qquad \mathbf{E}_N = \begin{pmatrix} E_{N,R_{01}^{(1)}}^{(0)} & \cdots & E_{N,R_{0n_1}^{(1)}}^{(0)} & \cdots & E_{N,R_{0n_c}^{(c)}}^{(0)} \\ \vdots & & \vdots & & \vdots \\ E_{N,R_{p1}^{(1)}}^{(p)} & \cdots & E_{N,R_{pn_1}^{(1)}}^{(p)} & \cdots & E_{N,R_{pn_c}^{(c)}}^{(p)} \end{pmatrix}.$$

Consider now the $(p + 1)c$ random variables

$$(2.4)\quad T_{N,i}^{(k)} = (1/n_k) \textstyle\sum_{\alpha=1}^{n_k} E_{N,R_{i\alpha}^{(k)}}^{(i)} \qquad \text{for} \quad i = 0, \cdots, p,\ k = 1, \cdots, c.$$

(Suitable conditions will be imposed on the scores $E_{N,\alpha}^{(i)}$'s at a latter stage). Define also

$$(2.5)\quad \bar{E}_N^{(i)} = (1/N) \textstyle\sum_{\alpha=1}^{N} E_{N,\alpha}^{(i)}, \qquad \bar{\mathbf{E}}_N = (\bar{E}_N^{(0)}, \bar{E}_N^{(1)}, \cdots, \bar{E}_N^{(p)}).$$

The proposed test statistic is based on $T_{N,i}^{(k)} - \bar{E}_N^{(i)}$, $i = 0, 1, \cdots, p$, $k = 1$ $\cdots, c$, and the rationality of the test procedure is explained in the next section.

3. The permutation-invariance principle and the test statistic. The (stochastic) matrix $\mathbf{E}_N$ can have only $(N!)^{p+1}$ possible realizations (constituting a set $\mathcal{E}_N$),

[2] Note that whereas Quade (1967) did not consider the exact test but proposed instead the asymptotically (unconditionally) distribution-free test, this paper deals with the exact as well as the asymptotic tests.

as each row of $\mathbf{E}_N$ can have only $N!$ possible realizations (over the permutations of the ranks). By a finite number of exchanges of its columns, the matrix $\mathbf{E}_N$ can always be converted into another matrix $\mathbf{E}_N^*$, whose first row has the scores $E_{N,1}^{(0)}, \cdots, E_{N,N}^{(0)}$, in natural order. Any two matrices, say $\mathbf{E}_N$ and $\mathbf{E}_N^0$ are said to be permutationally equivalent if both can be converted to a common $\mathbf{E}_N^*$ by only interchanging their columns. Thus, for each $\mathbf{E}_N^*(\epsilon \mathcal{E}_N)$, there exists a set of $N!$ possible $\mathbf{E}_N$ (to be denoted by $S(\mathbf{E}_N^*)$), all whose elements are permutationally equivalent to $\mathbf{E}_N^*$, and there are $(N!)^p$ such elements $\mathbf{E}_N^*$ of $\mathcal{E}_N$. Because of the association pattern of the components of the vectors $\mathbf{Z}_\alpha^{(k)}$, the probability distribution of $\mathbf{E}_N$ over $\mathcal{E}_N$, will, in general, depend on the unknown cdf's, even under H_0 in (1.3). However, under (1.3), $G_1(\mathbf{z}) \equiv \cdots \equiv G_c(\mathbf{z}) = G(\mathbf{z})$, and hence, $\mathbf{Z}_1^{(1)}, \cdots, \mathbf{Z}_{n_c}^{(c)}$ are all iidrv. Hence, the conditional distribution of $\mathbf{E}_N$ over the set $S(\mathbf{E}_N^*)$ will be uniform under (1.3), whatever be the common (unknown) G, that is

$$P\{\mathbf{E}_N = \mathbf{e}_N \mid S(\mathbf{E}_N^*)\} = 1/N! \quad \text{whatever be} \quad \mathbf{e}_N \,\varepsilon\, S(\mathbf{E}_N^*) \tag{3.1}$$

and for all continuous G. Consequently, any test depending explicitly on $\mathbf{E}_N$ and based on the permutational probability measure $\mathcal{P}_N$ in (3.1) will be conditionally distribution-free, and hence, unconditionally too, will be a similar test. [For details, refer to Puri and Sen (1966)]. To formulate the test statistic, we define $\mathbf{V}_N = ((v_{ij}))_{i,j=0,1,\cdots,p}$, where

$$v_{ij} = N^{-1} \sum_{k=1}^{c} \sum_{\alpha=1}^{n_k} E_{N,\alpha i}^{(k)} E_{N,\alpha j}^{(k)} - \bar{E}_N^{(i)} \bar{E}_N^{(j)}, \quad i, j = 0, \cdots, p, \tag{3.2}$$

where $E_{N,\alpha i}^{(k)}$ is the value of $E_{N,s}^{(i)}$ corresponding to the rank $s = R_{i\alpha}^{(k)}$. Then, it is easy to verify that

$$E(T_{N,i}^{(k)} - \bar{E}_N^{(i)} \mid \mathcal{P}_N) = 0, \qquad i = 0, \cdots, p,\ k = 1, \cdots, c, \tag{3.3}$$

$$E\{(T_{N,i}^{(k)} - \bar{E}_N^{(i)})(T_{N,j}^{(q)} - \bar{E}_N^{(j)}) \mid \mathcal{P}_N\} = (\delta_{kq} N - n_k) v_{ij} / n_k (N - 1), \tag{3.4}$$

for $i, j = 0, 1, \cdots, p$; $k, q = 1, \cdots, c$. It will be seen later on that under the permutational measure $\mathcal{P}_N$, the random variables $N^{\frac{1}{2}}\{T_{N,i}^{(k)} - \bar{E}_N^{(i)}, i = 0, \cdots, p;$ $k = 1, \cdots, c\}$ have asymptotically a multivariate normal distribution, with null means and covariances specified by (3.4). Hence, to utilize the information contained in the concomitant variates, we fit regression lines of $T_{N,0}^{(k)}$ on $T_{N,1}^{(k)}, \cdots, T_{N,p}^{(k)}$ for each $k = 1, \cdots, c$ and work with the residuals about these regression lines. Thus, it follows from (3.3) that the adjusted rank-scores for the c samples are

$$\begin{aligned} T_{N,k}^* &= T_{N,0}^{(k)} - \bar{E}_N^{(0)} \\ &\quad - (\text{the fitted value of } T_{N,0}^{(k)} - \bar{E}_N^{(0)} \text{ on } T_{N,1}^{(k)}, \cdots, T_{N,p}^{(k)}) \\ &= \sum_{i=0}^{p} (V_{N,i0}/V_{N,00})(T_{N,i}^{(k)} - \bar{E}_N^{(i)}), \qquad k = 1, \cdots, c, \end{aligned} \tag{3.5}$$

where $V_{N,ij}$ is the cofactor of v_{ij} in $\mathbf{V}_N$; $i, j = 0, 1, \cdots, p$. A quadratic form in

$(T^*_{N,1}, \cdots, T^*_{N,c})$ with the generalized inverse of their covariance matrix as its discriminant leads to the following test statistic

$$(3.6) \qquad \mathfrak{L}_N = \{V_{N,00}/|V_N|\} \textstyle\sum_{k=1}^{c} n_k(T^*_{N,k})^2; \qquad |V_N| = \det \mathbf{V}_N.$$

It may be noted that in the particular case when $E^{(i)}_{N,\alpha} = \alpha/(N+1)$, $\alpha = 1, \cdots, N$; $i = 0, 1, \cdots, p$, the statistic $\mathfrak{L}_N$ in (3.5) is a strictly monotonic function of the test statistic $VR(\lambda)$, proposed by Quade (1967). Further, it may be remarked that in his procedure, Quade considered the residuals of the individual ranks of $X^{(k)}_{0\alpha}$ on the coordinatewise ranks of $X^{(k)}_{1\alpha}, \cdots, X^{(k)}_{p\alpha}$, $\alpha = 1, \cdots, n_k$, $k = 1, \cdots, c$, which may be replaced by the comparatively simpler (but identical) procedure in (3.4) for the residuals of the sample averages, as his λ_i's are nothing but our $V_{N,i0}/V_{N,00}$, $i = 1, \cdots, p$. In the above discussion, $\mathbf{V}_N$ is assumed to be positive definite. Later on, it will be observed that $\mathbf{V}_N$ is positive definite, in probability (as $N \to \infty$) under mild restrictions on $G_1, \cdots, G_c$.

From the remark made just after (3.1), it follows that the permutation distribution of $\mathfrak{L}_N$ will not depend on the unknown G (when (1.1) and (1.3) hold), and hence, an exact (conditional) test of size α $(0 < \alpha < 1)$, can be based on $\mathfrak{L}_N$. The test will thus be a similar size α test for (1.3). To apply the above test in practice, we really require to study all possible $(N!/\prod_{k=1}^{c} n_k!)$ partitionings of $\mathfrak{C}_N$ into c subsets of strength $n_1, \cdots, n_c$ respectively, and for each such partitioning to compute the corresponding value of $\mathfrak{L}_N$. (Note that $\mathbf{V}_N$ is invariant under such partitionings; so we require only to compute the values of $T^{(k)}_{N,i}$'s). The labor involved in this scheme increases prohibitively with the increase in N. Hence, in large samples, we approximate the exact permutation distribution of $\mathfrak{L}_N$ by a chi-square distribution with $(c-1)$ degrees of freedom. This is the subject-matter of the next section.

4. Asymptotic permutation distribution of $\mathfrak{L}_N$. Let us denote by $G_{k[i]}(x)$ and $G_{k[i,j]}(x, y)$ the marginal cdf's of $X^{(k)}_{i\alpha}$ and $(X^{(k)}_{i\alpha}, X^{(k)}_{j\alpha})$, for $i \neq j = 0, 1, \cdots, p$; $k = 1, \cdots, c$. Let then

$$(4.1) \qquad H_{[i]}(x) = \textstyle\sum_{k=1}^{c} (n_k/N) G_{k[i]}(x),$$

$$H_{[i,j]}(x, y) = \textstyle\sum_{k=1}^{c} (n_k/N) G_{k[i,j]}(x, y),$$

for $i \neq j = 0, \cdots, p$. We denote by $H(\mathbf{z}) = \sum_{k=1}^{c} (n_k/N) G_k(\mathbf{z})$, $\mathbf{z} \varepsilon R^{p+1}$. Now, concerning $n_1, \cdots, n_c$, we assume that for all N,

$$(4.2) \quad 0 < \lambda_0 \leqq \lambda_N^{(k)} = n_k/N \leqq 1 - \lambda_0 < 1; \quad (0 < \lambda_0 \leqq 1/c), \quad k = 1, \cdots, c,$$

hold. Furthermore, we let

$$(4.3) \qquad E^{(i)}_{N,\alpha} = J_{N(i)}(\alpha/(N+1)), \qquad 1 \leqq \alpha \leqq N, \quad i = 0, \cdots, p,$$

where $J_{N(i)}$'s satisfy the extended Chernoff-Savage (1958) conditions, studied for the multivariate case by Puri and Sen (1966), in detail. For the sake of completeness, we state these briefly as follows:

(I) $\lim_{N\to\infty} J_{N(i)}(u) = J_{(i)}(u): 0 < u < 1$ exists and is not a constant, for all $i = 0, 1, \cdots, p$;

(II) $T_{N,i}^{(k)} - (1/n_k)\sum_{\alpha=1}^{n_k} J_{(i)}(R_{i\alpha}^{(k)}/(N+1)) = o_p(N^{-\frac{1}{2}})$, $k = 1, \cdots, c$; $i = 0, \cdots, p$;

(III) $J_{(i)}(u)$ is absolutely continuous in $u: 0 < u < 1$, and

$$(4.4) \qquad |(\partial^r/\partial u^r) J_{(i)}(u)| \leqq K\{u(1-u)\}^{-r-\frac{1}{2}+\delta} \qquad \text{for } r = 0, 1, 2,$$

where $\delta > 0$ and $K(0 < K < \infty)$ are constants independent of $i = 0, 1, \cdots, p$.

(IV) $(1/n_k)\sum_{\alpha=1}^{n_k} E_{N,R_{i\alpha}^{(k)}}^{(i)} E_{N,R_{j\alpha}^{(k)}}^{(j)} - (1/n_k)\sum_{\alpha=1}^{n_k} J_{(i)}(R_{i\alpha}^{(k)}/(N+1)) J_{(j)} \cdot (R_{j\alpha}^{(k)}/(N+1)) = o_p(1)$ for all $i, j = 0, \cdots, p$; $k = 1, \cdots, c$.

(V) The matrix $\boldsymbol{\nu}(H)$, defined below, is positive definite. Let

$$(4.5) \qquad \nu_{ii}(H) = \nu_{ii} = \int_0^1 J_{(i)}^2(u)\,du - \mu_i^2; \qquad \mu_i = \int_0^1 J_{(i)}(u)\,du;$$

$$(4.6) \qquad \nu_{ij}(H) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} J_{(i)}(H_{[i]}(x)) J_{(j)}(H_{[j]}(y))\,dH_{[i,j]}(x, y) - \mu_i\mu_j,$$

for $i \neq j = 0, \cdots, p$, and let

$$(4.7) \qquad \boldsymbol{\nu}(H) = ((\nu_{ij}(H)))_{i,j=0,1,\cdots,p}.$$

Then, we have the following theorem whose proof is an immediate consequence of Theorems 4.1 and 4.2 of Puri and Sen (1966), and is therefore omitted.

THEOREM 4.1. *Under the assumptions* (I) – (V), (i) $\mathbf{V}_N$ *is stochastically equivalent to* $\boldsymbol{\nu}(H)$, *defined by* (4.5)–(4.7), *and hence,* $\mathbf{V}_N$ *is positive definite, in probability* (*as* $N \to \infty$), *and* (ii) *the joint distribution* (*under* $\mathcal{P}_N$) *of the* $(p+1)\cdot(c-1)$ *random variables* $\{T_{N,i}^{(k)} - \bar{E}_N^{(i)}, i = 0, 1, \cdots, p; k = 1, \cdots, c\}$ *is asymptotically, in probability, normal with a null mean vector and a covariance matrix with elements defined by* (3.4). *Hence, the permutation distribution of the statistic* $\mathcal{L}_N$, *defined by* (3.6), *converges asymptotically, in probability,* (*as* $N \to \infty$) *to the chi-square distribution with* $(c-1)$ *degrees of freedom.*

By virtue of the above theorem, the permutation (conditional) test procedure based on the statistic $\mathcal{L}_N$ asymptotically reduces to the following:

(4.8) reject or accept H_0 in (1.3) according as $\mathcal{L}_N$ is $\geqq$ or $< \chi_{\alpha,c-1}^2$, where $\chi_{\alpha,r}^2$ is the upper $100\,\alpha\,\%$ point of the chi-square distribution with r degrees of freedom.

We shall see in the next section that this conditional test is also asymptotically equivalent to an asymptotically distribution-free unconditional test based on the same statistic $\mathcal{L}_N$.

5. Asymptotic distribution of $\mathcal{L}_N$ for shift alternatives. It follows from (3.5) that the study of the unconditional distribution of $\mathcal{L}_N$ requires the same for the statistics $T_{N,i}^{(k)}$, $i = 0, 1, \cdots, p$; $k = 1, \cdots, c$, and this has essentially been studied in detail by Puri and Sen (1966) for arbitrary $G_1, \cdots, G_c$). In the present situation, we have a relatively simpler model, where the marginals $F_1^{(1)}, \cdots, F_c^{(1)}$ are all identical. Thus, for any fixed $G_1, \cdots, G_c$, the joint asymptotic normality of $T_{N,i}^{(k)}$'s follow readily from Theorem 5.1 of Puri and Sen (1966). Since, the consistency of the test based on $\mathcal{L}_N$ can be easily established for any fixed (shift) alternative (i.e., referred to the model (1.1) and (1.2) with $\boldsymbol{\tau} \neq \mathbf{0}$), we shall consider here the following sequence of alternatives $\{K_N\}$, specified by

$$K_N: \boldsymbol{\tau} = \boldsymbol{\tau}_N = N^{-\frac{1}{2}}\boldsymbol{\theta}, \qquad [cf. (1.2)]. \tag{5.1}$$

where $\boldsymbol{\theta}$ is a p-vector with real and finite elements. Let us assume that

$$\lim_{N\to\infty} \lambda_N^{(k)} = \lambda^{(k)}: 0 < \lambda^{(k)} < 1 \qquad \text{for} \quad k = 1, \cdots, c. \tag{5.2}$$

Also, we assume the existence of the following integral

$$B(G_{[0]}) = \int_{-\infty}^{\infty} (d/dx) J_{(0)}(G_{[0]}(x))\, dG_{[0]}(x), \tag{5.3}$$

where $G_{[0]}$ refers to the cdf $G_{k[0]}$ when $G_{1[0]} = \cdots = G_{c[0]} = G_{[0]}$. Finally, let $\boldsymbol{\nu}(G) = \boldsymbol{\nu}(H)\,|_{H=G}$, and

$$\nu^{-1}(G) = ((\nu^{ij}(G))) = ((\nu_{ij}(G)))^{-1}_{i,j=0,1,\cdots,p}. \tag{5.4}$$

THEOREM 5.1. *Under the conditions* (I)–(V) *of Section* 4 *and* (5.1)–(5.3), *the statistic $\mathcal{L}_N$ has asymptotically a noncentral chi-square distribution with $c - 1$ degrees of freedom and the noncentrality parameter.*

$$\Delta_{\mathcal{L}} = \nu^{00}(G)B^2(G_{[0]}) \textstyle\sum_{k=1}^{c} \lambda^{(k)}(\theta_k - \bar{\theta})^2; \qquad \bar{\theta} = \sum_{k=1}^{c} \lambda^{(k)}\theta_k. \tag{5.5}$$

PROOF. From Theorem 5.1 of Puri and Sen (1966), it follows that under the conditions of the theorem, the random variables $N^{\frac{1}{2}}(T_{N,0}^{(k)} - \bar{E}_N^{(0)}, T_{N,i}^{(k)} - \bar{E}_N^{(i)}$, $i = 1, \cdots, p; k = 1, \cdots, c)$ have (jointly) a limiting multi-normal distribution with means $(\mu_0^{(k)}, \mu_i^{(k)}, i = 1, \cdots, p; k = 1, \cdots, c)$ and covariance matrix $\boldsymbol{\tau} = ((\tau_{ij}^{(k,q)}))$, where

$$\mu_0^{(k)} = (\theta_k - \bar{\theta})B(G_{[0]}), \qquad k = 1, \cdots, c, \tag{5.6}$$

$$\mu_i^{(k)} = 0, \qquad i = 1, \cdots, p; k = 1, \cdots, c; \tag{5.7}$$

and

$$\tau_{ij}^{(k,q)} = (\delta_{kq}/\lambda^{(k)} - 1)\nu_{ij}(G), \qquad i, j = 0, 1, \cdots, p, k, q = 1, \cdots, c, \tag{5.8}$$

δ_{kq} being the usual Kronecker delta. Since $T_{N,k}^*$ is a linear compound of $T_{N,i}^{(k)}$, $i = 0, 1, \cdots, p$, for $k = 1, \cdots, c$, [see (3.5)] and as by Theorem 4.1, under $\{K_N\}$ in (5.1), $V_{N,i0}/V_{N,00} \sim_p \nu^{i0}(G)/\nu^{00}(G)$ for all $i = 0, 1, \cdots, p$, we obtain from the above result on using Sverdrup's (1952) results (on the limiting distribution of a continuous function of random variables) that under $\{K_N\}$, $(N^{\frac{1}{2}}T_{N,k}^*, k = 1, \cdots, c)$ has asymptotically a (singular) multi-normal distribution with mean vector $((\theta_k - \bar{\theta})B(G_{[0]}), k = 1, \cdots, c)$ and a dispersion matrix with elements

$$(\delta_{kq}/\lambda^{(k)} - 1)/\nu^{00}(G), \qquad k, q = 1, \cdots, c. \tag{5.9}$$

Thus, under $\{K_N\}$ in (5.1), the statistic

$$\mathcal{L}_N^* = \nu^{00}(G) \textstyle\sum_{k=1}^{c} \lambda^{(k)}(T_{N,k}^*)^2 \tag{5.10}$$

has asymptotically a noncentral chi-square distribution with $c - 1$ degrees of freedom and the noncentrality parameter $\Delta_{\mathcal{L}}$, defined by (5.5). Again, by Theorem 4.1, we obtain from (3.6) and (5.10) that under (5.1) and (5.2),

$$\mathcal{L}_N \sim_p \mathcal{L}_N^*. \tag{5.11}$$

Hence the theorem.

In particular, under the null hypothesis $\tau_1 = \cdots = \tau_c = 0$, $\mathcal{L}_N$ has asymptotically a chi-square distribution with $c - 1$ degrees of freedom. Thus, from (4.9) and (5.11), we may conclude that the permutation test in Section 4 and the asymptotically distribution-free test based on the asymptotic (unconditional) chi-square distribution of $\mathcal{L}_N$ are asymptotically equivalent for the sequence of alternatives $\{K_N\}$ in (5.1). The next section is devoted to the study of the asymptotic power efficiency of either of these tests with respect to the standard parametric test.

6. Asymptotic efficiency of the proposed tests. Let us denote the covariance matrix of $\mathbf{Z}_\alpha^{(k)}$ by $\mathbf{\Sigma}_k$, $k = 1, \cdots, c$. Under (1.1) and (1.2), $\mathbf{\Sigma}_1 = \cdots = \mathbf{\Sigma}_c = \mathbf{\Sigma}$ (say). We assume that $\mathbf{\Sigma}$ is positive definite, and let $\mathbf{\Sigma}^{-1}(G) = ((\sigma^{ij}(G)))$ (where G denotes its dependence in the cdf G) denote the inverse of $\mathbf{\Sigma}$. The classical parametric test under the assumptions that G is normal, and the regression of $X_{0\alpha}^{(k)}$ on $\mathbf{X}_\alpha^{(k)}$ is linear, is based on the variance ratio criterion with proper adjustments for the concomitant variates [cf. Scheffé (1959), Chapter 6]. It can be shown that under (5.1) and (5.2), and the existence of moments of the order $2 + \delta$, $\delta > 0$ of G, $(c - 1)$ times the variance ratio criterion has asymptotically the noncentral chi-square distribution with $(c - 1)$ degrees of freedom, and the noncentrality parameter

$$\Delta_{\mathcal{F}} = \sigma^{00} \textstyle\sum_{k=1}^{c} \lambda^{(k)}(\theta_k - \bar{\theta})^2. \tag{6.1}$$

(The proof is a straight forward application of the multivariate central limit theorem, and the consistency of the least squares estimators of the regression parameters of X_0 on $\mathbf{X}$, and is therefore omitted.)

Thus, the asymptotic (Pitman) relative efficiency (ARE) of the test based on $\mathcal{L}_N$ with respect to the variance-ratio test is equal to

$$e_{\mathcal{L},\mathcal{F}} = \Delta_{\mathcal{L}}/\Delta_{\mathcal{F}} = (\nu^{00}/\sigma^{00})B^2(G_{[0]}). \tag{6.2}$$

It is clear that (6.2) dependes on $B(G_{[0]})$ as well as on $\nu(G)$ and $\mathbf{\Sigma}(G)$.

By virtue of the following two simple inequalities

$$\nu^{00}\nu_{00} \geqq 1 \qquad \text{and} \qquad \sigma^{00}\sigma_{00} \geqq 1, \tag{6.3}$$

we obtain

$$B^2(G_{[0]})/\nu_{00}\sigma^{00} \leqq e_{\mathcal{L},\mathcal{F}} \leqq \sigma_{00}\nu^{00}. \tag{6.4}$$

Now from the results of Puri (1964), it follows that the ARE of the $\mathcal{L}_N$ test with respect to the parametric ANOVA test (when there is no concomitant variable) is

$$e^0_{\mathfrak{L},\mathfrak{F}} = B^2(G_{[0]})\sigma_{00}/\nu_{00}. \tag{6.5}$$

From (6.2) and (6.5), we obtain

$$e_{\mathfrak{L},\mathfrak{F}} = e^0_{\mathfrak{L},\mathfrak{F}}(\nu^{00}\nu_{00})/(\sigma^{00}\sigma_{00}). \tag{6.6}$$

Using (6.3), we deduce

$$e^0_{\mathfrak{L},\mathfrak{F}}(\sigma_{00}\sigma^{00})^{-1} \leqq e_{\mathfrak{L},\mathfrak{F}} \leqq e^0_{\mathfrak{L},\mathfrak{F}}\nu_{00}\nu^{00}. \tag{6.7}$$

Now, since for some well-known statistics such as the rank-sum statistic or the normal scores statistic, the bounds for $e^0_{\mathfrak{L},\mathfrak{F}}$ are well-known (cf. Puri (1964)), (6.6) and (6.7) can be used to study the bounds for $e_{\mathfrak{L},\mathfrak{F}}$ for some specific cases.

(a) CASE 1. *Generalized rank sum test.* Let $E^{(i)}_{N,\alpha} = \alpha/(N+1)$, $\alpha = 1, \cdots, N$ and $i = 0, 1, \cdots, p$. In this case, it is well-known that

$$e^0_{\mathfrak{L},\mathfrak{F}} = 12\,\sigma_{00}\left(\int_{-\infty}^{\infty} g^2_{[0]}(x)\,dx\right)^2 \tag{6.8}$$

(where $g_{[0]}$ is the density of $G_{[0]}$) is bounded below by 0.864 for all $G_{[0]}$, is equal to $3/\pi$ when $G_{[0]}$ is standard normal cdf, and is greater than 1 for many non-normal $G_{[0]}$. Consequently, if $\mathbf{\Sigma}$ is nonsingular (so that $(\sigma_{00}\sigma^{00})^{-1} > 0$), (6.7) provides analogous lower bound for $e_{\mathfrak{L},\mathfrak{F}}$. However, it is not possible to find lower bound to $(\sigma_{00}\sigma^{00})^{-1}$ for all $G \in \mathcal{G} = \{G: \mathbf{\Sigma}(G) \text{ nonsingular}\}$, and hence to $e_{\mathfrak{L},\mathfrak{F}}$. However, some interesting results are given below for specific cases.

(i) Let X_0 be uncorrelated with the elements of $\mathbf{X}$. Then $\sigma_{0i} = 0$, $i = 1, \cdots, p$, and hence $\sigma_{00}\sigma^{00} = 1$. In such a case [cf. (6.7)],

$$e_{\mathfrak{L},\mathfrak{F}} \geqq 12\,\sigma_{00}\left(\int_{-\infty}^{\infty} g^2_{[0]}(x)\,dx\right)^2, \tag{6.9}$$

which implies that

$$e_{\mathfrak{L},\mathfrak{F}} \geqq 0.864 \qquad \text{for all} \quad G \in \mathcal{G}. \tag{6.10}$$

This case indicates that if the concomitant variates are (linearly) unrelated to X_0 (though there may or may not be a nonlinear defendence), the generalized rank sum test has a better behavior.

(ii) If G is a $(p+1)$-variate normal distribution, then (6.2) reduces to

$$e_{\mathfrak{L},\mathfrak{F}} = 3\pi^{-1}\cdot(|6\pi^{-1}\sin^{-1}(\rho_{ij}/2)|_{i,j=1,\cdots,p}/|6\pi^{-1}\sin^{-1}(\rho_{ij}/2)|_{i,j=0,\cdots,p}) \cdot(|\rho_{ij}|_{i,j=0,1,\cdots,p}/|\rho_{ij}|_{i,j=1,\cdots,p}) \tag{6.11}$$

where ρ_{ij} is the product moment correlation of X_i, X_j, $i, j = 0, 1, \cdots, p$.

For $p = 1$, Quade (1967) has shown that (6.11) has maximum value $3\pi^{-1}$ when $\rho_{01} = 0$ and decreases to 0.866 when ρ_{01} approaches to $+1$ or -1. It is easy to show that (6.11) is bounded above by $3\pi^{-1}$ for all $p \geqq 1$. To prove this it suffices to show that $((2\sin^{-1}\rho_{ij}/2 - \rho_{ij}))$ is positive semi definite for all $p \geqq 1$. Now writing

$$2\sin^{-1}\rho_{ij}/2 = \rho_{ij} + a_1\rho^3_{ij} + a_2\rho^5_{ij} + \cdots, \tag{6.12}$$

where $a_j > 0$ for all j, it follows that

$$(6.13) \quad ((2 \sin^{-1} \rho_{ij}/2 - \rho_{ij})) = \sum_{j=1}^{\infty} a_j((\rho_{ij}^{2j+1})), \qquad a_j > 0 \text{ for all } j.$$

Since $((\rho_{ij}))$ is positive (semi-) definite, so also are $((\rho_{ij}^{2j+1}))$, $j \geqq 1$, and hence the result. It may also be noted that (6.11) may be quite close to zero, as can always be shown by some pathological examples. We conclude that the asymptotic relative efficiency of the generalized rank sum test with respect to the classical parametric test can be very low when $p \geqq 1$ and the underlying cdf is normal.

(b) CASE 2. *Generalized normal scores test.* Let $E_{N,\alpha}^{(i)}$ denote the expected value of the αth order statistic in a sample of size N from the standard normal distribution, $\alpha = 1, \cdots, N$ and $i = 0, 1, \cdots, p$. In this case [cf, [3]],

$$(6.14) \qquad e_{\mathfrak{L},\mathfrak{F}}^{0} = \sigma_{00}(G_{[0]})(\textstyle\int_{-\infty}^{\infty} g_{[0]}^{2}(x) \; dx/\phi[\Phi^{-1}(G_{[0]}(x)])^2 \geqq 1$$

for all $G_{[0]}$, and is 1 if and only if $G_{[0]}$ is normal. Here $\phi(\cdot)$ is the density function of $\Phi(\cdot)$, the standard cumulative normal distribution function. Hence, if X_0 is uncorrelated with the elements of $\mathbf{X}$, (6.2) will be bounded below by 1. If G is a $(p + 1)$-variate normal cdf, then $\boldsymbol{\nu} = \boldsymbol{\Sigma}$ and hence (6.2) will be equal to 1 for all $p \geqq 1$. Thus the generalized normal scores test is asymptotically as efficient as the classical test based on the variance ratio criterion when the underlying distribution is normal. The ARE of the generalized normal scores test and the corresponding rank sum test may be studied in a similar manner.

Acknowledgment. The authors are grateful to the referee for his useful comments on the paper.

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