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PHASE FREE ESTIMATION OF COHERENCE[1]

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0. Summary. A phase free estimate of the coherence of a bivariate Gaussian process is presented. The technique is based on the usual independent, complex normal approximation to the distribution of the finite Fourier transform of a multivariate stationary time series, and the complex Wishart approximation to the distribution of spectrum estimates. If the spectral densities and coherence can be assumed to be constant over a wider frequency band than the phase can be assumed to be constant, the concept of inner and outer spectral windows would seem appropriate. Maximum likelihood estimates of the coherence are obtained using phase free marginal distributions at the inner window level. The results of simulations are presented showing the likelihood for various inner windows.

1. Introduction. It has been noted in certain physical multivariate stationary processes that estimates of coherence are biased towards zero because of varying phase within the spectral window. While the spectral densities of the two component processes and the coherence may remain fairly constant over a frequency band, changing phase can cause the co- and quadrature spectrum to change sign within a spectral window so that the smoothed estimates are badly biased. This problem is discussed, with references, by Akaike and Yamanouchi (1962) and Nettheim (1966). One approach is to shift the time axis of one series in order to reduce rapid variation in phase. Tick (1967) suggests estimating the coherence using a narrower spectral window than would be used on the spectral densities and gives several methods of combining the high resolution estimates, introducing the concept of inner and outer window. This paper derives maximum likelihood estimates based on two levels of smoothing.

The derivation is based on the distribution of independent bivariate complex normal variables which, as is well known, approximates the distribution of the Fourier transform of the multivariate stationary process for large sample size (Goodman, 1963).

2. The estimate. Let $[x_1, y_1, x_2, y_2]$ be a four dimensional normal variable with mean zero and covariance matrix

$$\frac{1}{2}\begin{bmatrix} \sigma_1^2 & 0 & \alpha\sigma_1\sigma_2 & -\beta\sigma_1\sigma_2 \\ 0 & \sigma_1^2 & \beta\sigma_1\sigma_2 & \alpha\sigma_1\sigma_2 \\ \alpha\sigma_1\sigma_2 & \beta\sigma_1\sigma_2 & \sigma_2^2 & 0 \\ -\beta\sigma_1\sigma_2 & \alpha\sigma_1\sigma_2 & 0 & \sigma_2^2 \end{bmatrix} \tag{2.1}$$

$$\alpha^2 + \beta^2 < 1, \qquad \sigma_1 > 0, \qquad \sigma_2 > 0.$$

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Forming a complex random vector

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 + iy_1 \\ x_2 + iy_2 \end{bmatrix}, \tag{2.2}$$

the probability density function is

$$p(z) = (\pi^2|\Sigma|)^{-1}e^{-z^*\Sigma^{-1}z} \tag{2.3}$$

where * denotes complex conjugate transpose and the Hermitian covariance matrix is

$$Ez z^* = \Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_1\sigma_2(\alpha + i\beta) \\ \sigma_1\sigma_2(\alpha - i\beta) & \sigma_2^2 \end{bmatrix} \tag{2.4}$$

(see Goodman, 1963 and Khatri, 1965).

When referring to the background problem in stationary processes, σ_1^2 and σ_2^2 correspond to the spectral densities of the two component processes, the coherence corresponds to

$$R^2 = \alpha^2 + \beta^2 \tag{2.5}$$

and the phase to

$$\theta = \text{arc tan } \beta/\alpha. \tag{2.6}$$

Let $z_1, z_2, \cdots z_n$ be an independent sample from this distribution. Let this sample be divided into m groups of size $n_k \geqq 2$,

$$n = \textstyle\sum_{k=1}^{m} n_k . \tag{2.7}$$

From each group obtain the sum of product matrix,

$$A_k = \textstyle\sum_j z_j z_j^*, \tag{2.8}$$

the sum being over the corresponding subgroup. It is well known (Goodman, 1963) that the distinct elements of A have a complex Wishart distribution

$$P(A_k) = |A_k|^{n_k-2}[\pi\Gamma(n_k)\Gamma(n_k - 1)|\Sigma|^{n_k}]^{-1}e^{-\text{tr}\Sigma^{-1}A_k}. \tag{2.9}$$

Writing this in terms of the four distinct real elements, and dropping the subscript k temporarily,

$$\begin{aligned} p(a_{11}, a_{22}, a_{12R}, a_{12I}) &= (a_{11}a_{22} - a_{12R}^2 - a_{12I}^2)^{n-2} \\ &\cdot[\pi\Gamma(n)\Gamma(n-1)(1-R^2)^n\sigma_1^{2n}\sigma_2^{2n}]^{-1} \\ &\cdot\exp\left(-\frac{\sigma_2^2 a_{11} - 2\alpha\sigma_1\sigma_2 a_{12R} - 2\beta\sigma_1\sigma_2 a_{12I} + \sigma_1^2 a_{22}}{(1-R^2)\sigma_1^2\sigma_2^2}\right). \end{aligned} \tag{2.10}$$

Introducing a change of variable, let

$$\begin{aligned} a_{12R} &= |a_{12}| \cos \phi, \\ a_{12I} &= |a_{12}| \sin \phi. \end{aligned} \tag{2.11}$$

The density function in terms of the variables a_{11}, a_{22}, $|a_{12}|^2$ and ϕ is, (the Jacobian is $\frac{1}{2}$)

$$p(a_{11}, a_{22}, |a_{12}|^2, \phi)$$
$$= (a_{11}a_{22} - |a_{12}|^2)^{n-2}[2\pi\Gamma(n)\Gamma(n-1)(1-R^2)^n\sigma_1^{2n}\sigma_2^{2n}]^{-1} \cdot \exp\{-(1-R^2)^{-1}[a_{11}\sigma_1^{-2} - 2|a_{12}|(\sigma_1\sigma_2)^{-1} R\cos(\theta-\phi) + a_{22}\sigma_2^{-2}]\}. \tag{2.12}$$

Consider now the phase free marginal distribution,

$$p(a_{11}, a_{22}, |a_{12}|^2) = \int_0^{2\pi} p(a_{11}, a_{22}, |a_{12}|^2, \phi)\, d\phi. \tag{2.13}$$

This integral can be evaluated by noting that

$$(2\pi)^{-1}\int_0^{2\pi} e^{z\cos u}\, du = I_0(z), \tag{2.14}$$

a modified Bessel function of order zero.
Now

$$p(a_{11}, a_{22}, |a_{12}|^2)$$
$$= |A|^{n-2} I_0(2R|a_{12}|/(1-R^2)\sigma_1\sigma_2)[\Gamma(n)\Gamma(n-1)|\Sigma|^n]^{-1} \cdot \exp(-(1-R^2)^{-1}[a_{11}(\sigma_1^{-2}) + a_{22}(\sigma_2^{-2})]). \tag{2.15}$$

This distribution eliminates phase at the inner window level and a_{11}, a_{22}, and $|a_{12}|^2$ (which are sufficient for σ_1^2, σ_2^2 and R^2 at the inner window level) will be used as basic data when combining the inner window data to form the final estimate.

Reintroducing the subscript k to denote the group, the likelihood is

$$L = |\Sigma|^{-n}\left(\prod_{k=1}^m [|A_k|^{n_k-2}\Gamma^{-1}(n_k)\Gamma^{-1}(n_k-1)I_0(2R|a_{12}^{(k)}|/(1-R^2)\sigma_1\sigma_2)]\right) \cdot \exp\left(-(1-R^2)^{-1}\sum_{k=1}^m [a_{11}^{\kappa}/\sigma_1^2 + a_{22}^{(k)}/\sigma_2^2]\right). \tag{2.16}$$

The log likelihood is

$$\log L = K - n\log[\sigma_1^2\sigma_2^2(1-R^2)] + \sum_{k=1}^m \log I_0(2R|a_{12}^{(k)}|/(1-R^2)\sigma_1\sigma_2) - (1-R^2)^{-1}\sum_{k=1}^m [a_{11}^{(k)}\sigma_1^{-2} + a_{22}^{(k)}\sigma_2^{-2}], \tag{2.17}$$

where K is a constant which does not depend on the parameters σ_1^2, σ_2^2 and R^2.

Setting the derivatives with respect to σ_1^2, σ_2^2 and R^2, to zero, and using

$$(d/dz)I_0(z) = I_1(z), \tag{2.18}$$

gives

$$1 - R^2 + R(n\sigma_1\sigma_2)^{-1}\sum_{k=1}^m |a_{12}^{(k)}| I_1 I_0^{-1} = (n\sigma_1^2)^{-1}\sum_{k=1}^m a_{11}^{(k)},$$
$$1 - R^2 + R(n\sigma_1\sigma_2)^{-1}\sum_{k=1}^m |a_{12}^{(k)}| I_1 I_0^{-1} = (n\sigma_2^2)^{-1}\sum_{k=1}^m a_{22}^{(k)}, \tag{2.19}$$
$$1 - R^2 + (1+R^2)(nR\sigma_1\sigma_2)^{-1}\sum_{k=1}^m |a_{12}^{(k)}| I_1 I_0^{-1} = n^{-1}\sum_{k=1}^m [a_{11}^{(k)}\sigma_1^{-2} + a_{22}^{(k)}\sigma_2^{-2}],$$

where the arguments of $I_0(\ )$ and $I_1(\ )$ have been dropped for convenience and are the same as in (2.17). The maximum likelihood estimates of σ_1^2 and σ_2^2 from the entire sample (without grouping) also satisfy these equations.

$$(2.20) \qquad \hat{\sigma}_1^2 = n^{-1} \sum_{k=1}^m a_{11}^{(k)}, \qquad \hat{\sigma}_2^2 = n^{-1} \sum_{k=1}^m a_{22}^{(k)}$$

giving the following equation for R^2,

$$(2.21) \qquad (n\hat{\sigma}_1\hat{\sigma}_2)^{-1} \sum_{k=1}^m |a_{12}^{(k)}| I_1 I_0^{-1} = R.$$

The likelihood, evaluated at the maximum of σ_1^2 and σ_2^2, as a function of R^2 is

$$(2.22) \quad L \propto (1 - R^2)^{-n}[\prod_{k=1}^n I_0(2R|a_{12}^{(k)}|/(1 - R^2)\hat{\sigma}_1\hat{\sigma}_2)]e^{-2n/(1-R^2)}$$

where the argument of I_0 has been returned. Since programs are available for calculating modified Bessel functions, the maximum of (2.22) can be found by numerical methods. Before pursuing this aspect, it will be shown that the restriction that the group size n_K be two or more is not necessary.

3. Single observation for the inner window. It may seem surprising that an inner window consisting of a single observation can be used, since the estimate of coherence from one observation is always one. To show that an inner window at the periodogram level is possible, we return to the complex normal distribution of the observations. Equation (2.3) can be written

$$(3.1) \quad p(z) = (\pi^2|\Sigma|)^{-1} \cdot \exp(-|\Sigma|^{-1}(|z_1|^2\sigma_2^2 - 2\sigma_1\sigma_2\mathcal{R}(z_1^*(\alpha + i\beta)z_2) + |z_2|^2\sigma_1^2)),$$

where $\mathcal{R}(\ )$ denotes the real part. Transforming to polar coordinates

$$(3.2) \qquad z_j = \rho_j e^{i\theta_j}, \qquad j = 1, 2,$$

$$(3.3) \quad p(\rho_1, \rho_2, \theta_1, \theta_2) = \rho_1\rho_2\pi^{-2}|\Sigma|^{-1} \exp(-|\Sigma|^{-1}(\rho_1^2\phi_2^2 - 2\rho_1\rho_2\sigma_1\sigma_2 R \cos(\theta + \theta_2 - \theta_1) + \rho_2^2\phi_1^2).$$

The phase free marginal distribution is

$$(3.4) \quad \int_0^{2\pi}\int_0^{2\pi} p(\rho_1, \rho_2, \theta_1, \theta_2)\, d\theta_1\, d\theta_2 = 4\rho_1\rho_2|\Sigma|^{-1}I_0(2\rho_1\rho_2 R/\sigma_1\sigma_2(1 - R^2)) \exp(-(1 - R^2)^{-1}(\rho_1^2\sigma_1^{-2} + \rho_2^2\sigma_2^{-2})).$$

Again changing variables to be consistent with the last section, let

$$(3.5) \qquad a_{11} = \rho_1^2, \qquad a_{22} = \rho_2^2$$

$$(3.6) \quad p(a_{11}, a_{22}) = |\Sigma|^{-1}I_0(2R(a_{11}a_{22})^{\frac{1}{2}},/(1 - R^2)\sigma_1\sigma_2) \cdot \exp(-(1 - R^2)^{-1}(a_{11}\sigma_1^{-2} + a_{22}\sigma_2^{-2})).$$

This distribution is analogous to (2.15) with $|a_{12}|$ replaced by $(a_{11}a_{22})^{\frac{1}{2}}$. With $n = 1$, the complex Wishart is singular since

$$(3.7) \qquad |a_{12}| = (a_{11}a_{22})^{\frac{1}{2}}.$$

Proceeding as before the remaining equations of Section 2 hold. It is not necessary to remember (3.7) since if $n_k = 1$, $|a_{12}|$ calculated from the data satisfies this equation.

4. The numerical problem. The maximum likelihood estimate of R^2 is obtained by finding the maximum of (2.22) in the range $0 \leqq R \leqq 1$. Letting

$$u_k = |a_{12}^{(k)}|/\hat{\sigma}_1\hat{\sigma}_2, \qquad k = 1, m, \tag{4.1}$$

gives m sufficient statistics for R^2, and the log of the likelihood evaluated at $\sigma_1 = \hat{\sigma}_1$ and $\sigma_2 = \hat{\sigma}_2$ is

$$\log L = K - 2n(1 - R^2)^{-1} - n \log (1 - R^2) + \textstyle\sum_{k=1}^m \log I_0[2Ru_k/(1 - R^2)] \tag{4.2}$$

where K is a constant which does not depend on R^2. The slope of (4.2) is

$$[d/d(R^2)] \log L = [(1 + R^2)/(1 - R^2)^2][R^{-1} \textstyle\sum_{k=1}^m u_k I_1/I_0 - n], \tag{4.3}$$

where, again, the arguments of I_0 and I_1 have been deleted and are the same as in (4.2). The series expansions for I_0 and I_1 are,

$$I_0(z) = \textstyle\sum_{k=0}^\infty (z/2)^{2k}(k!)^{-2}, \qquad I_1(z) = \textstyle\sum_{k=0}^\infty (z/2)^{2k+1}/(k!)(k + 1)!. \tag{4.4}$$

The slope of (4.2) at $R^2 = 0$ is therefore

$$\textstyle\sum_{k=1}^m u_k^2 - n, \tag{4.5}$$

which can be positive or negative. It is possible, therefore, that the maximum likelihood estimate of R^2 is the endpoint $R^2 = 0$. The behavior of the log likelihood as $R^2 \to 1$ can be obtained using the asymptotic expression for the modified Bessel functions, as $z \to \infty$,

$$I_0(z) \sim (2\pi z)^{-\frac{1}{2}}e^z, \qquad I_1(z) \sim (2\pi z)^{-\frac{1}{2}}e^z. \tag{4.6}$$

Therefore,

$$R^{-1} \textstyle\sum_{k=1}^m u_k I_1/I_0 \to \textstyle\sum_{k=1}^m u_k \tag{4.7}$$

as $R^2 \to 1$. The behavior of the likelihood function as $R^2 \to 1$ is determined by the sign of

$$\textstyle\sum_{k=1}^m u_k - n. \tag{4.8}$$

If (4.8) is positive, the likelihood diverges to $+\infty$ as $R \to 1$. In this case the maximum likelihood will be $R = 1$. If (4.8) is negative, the likelihood approaches zero as $R \to 1$. (4.8) will take the value zero, even in the limiting situation when the true value of $R = 1$, with probability zero. The possibility of a maximum at both zero and one is ruled out by the following inequality

$$m \textstyle\sum_{k=1}^m u_k^2 \geqq (\textstyle\sum_{k=1}^m u_k)^2. \tag{4.9}$$

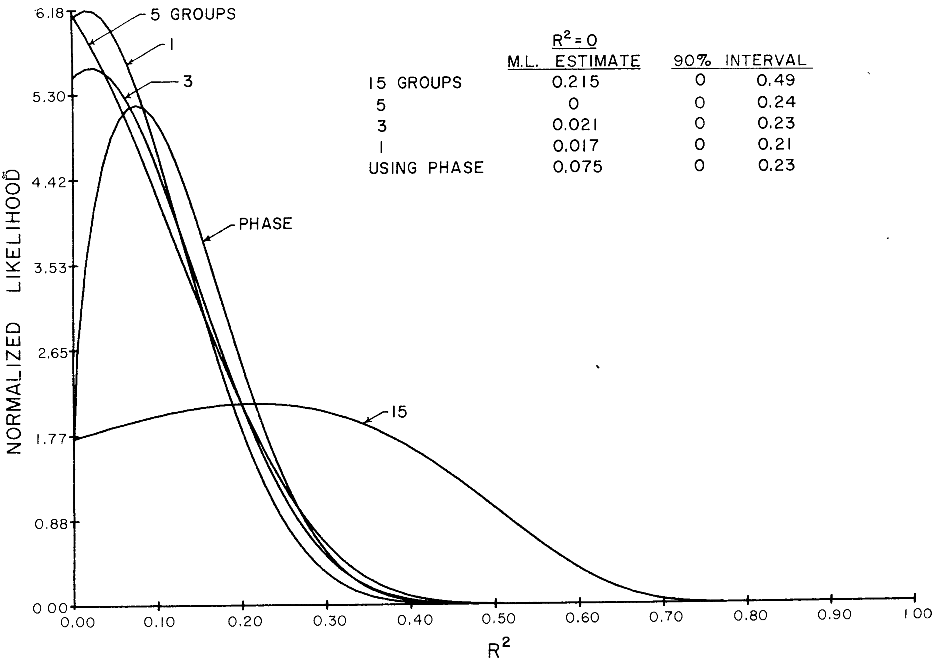

FIG. 1. Normalized likelihood as a function of R^2 evaluated at the maximum of the other parameters for various groupings and the usual estimate using phase. The maximum likelihood estimate is shown for each case together with the minimum length interval containing 90% of the area. The true R^2 is 0 and the sample size is 15.

5. Simulations. To obtain an idea of what is lost by grouping when it is not necessary, simulations were carried out. Random samples from a complex normal distribution with constant phase were generated and analyzed. The samples had zero means, unit variances, zero phase and specified coherence. The means were assumed known, corresponding to spectrum estimation, and the variances, phase and coherence were assumed unknown. Since the distributions of the estimated coherence do not depend on the actual variances and phase, the values used did not affect the simulations. Sample sizes of 15 were used corresponding to spectrum estimates with 30 degrees of freedom. Group sizes of 1, 3, 5 and 15 were used.

A group the size of the sample itself would not be used in practice, since in this case the usual estimate would be used, but is presented for comparison. Various values of coherence in the interval [0, 1] were used. In every case when the true coherence was one, (4.8) was positive for all groupings so the estimated coherence was one. Results are presented for values of R^2 equal to 0.0, 0.25, 0.5 and 0.75.

For each sample and grouping, the log likelihood was calculated from (4.2)

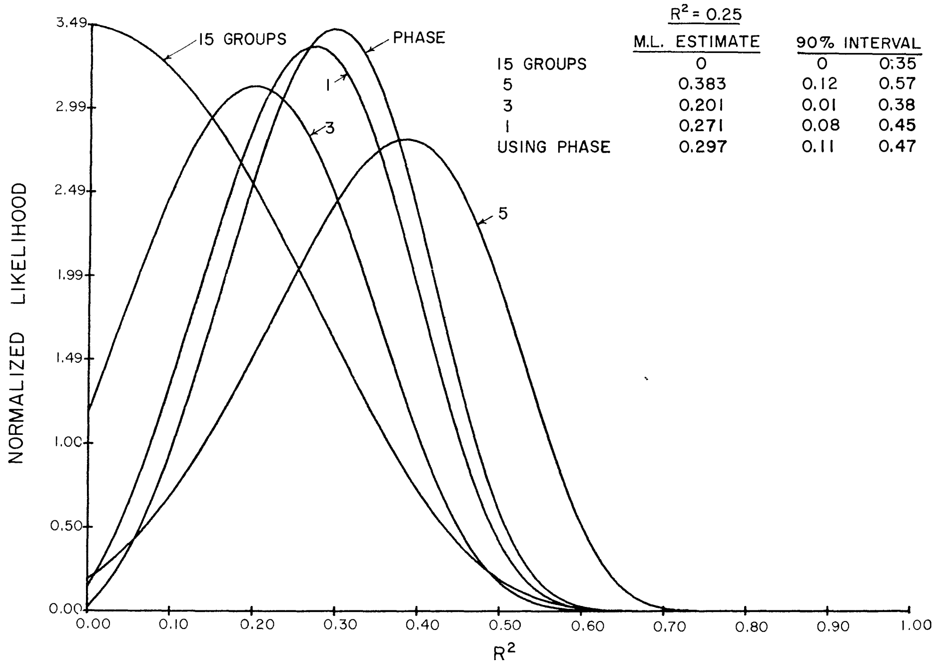

FIG. 2. Normalized likelihood as a function of R^2 evaluated at the maximum of the other parameters for various groupings and the usual estimate using phase. The maximum likelihood estimate is shown for each case together with the minimum length interval containing 90% of the area. The true R^2 is 0.25 and the sample size is 15.

for 100 equally spaced values of R^2, 1/200, 3/200, 5/200, $\cdots$, 199/200. Standard modified Bessel function routines are available for these calculations. The value of K was arbitrarily put equal to $2n$ making the log likelihood equal to zero at $R^2 = 0$. The likelihood was then calculated and normalized numerically to have unit area by dividing each value by the mean of the 100 values. Bayesian confidence limits, assuming a uniform [0, 1] prior on R^2 were calculated from the normalized likelihood.

For comparison purposes, the normalized likelihood for the usual estimate without grouping is presented. This is calculated from

$$(1 - R^2)^{-n} e^{-2n(1-R\hat{R})/(1-R^2)} \tag{5.1}$$

where

$$\hat{R} = |a_{12}|/(a_{11}a_{22})^{\frac{1}{2}}, \tag{5.2}$$

the usual maximum likelihood estimate of R. (5.1) is the likelihood function of the sample evaluated at the maximum of σ_1^2, σ_2^2 and θ. This is also presented normalized to unit area.

Maximum likelihood estimates were calculated by first checking the endpoints using (4.5) and (4.8). If (4.5) was positive and (4.8) negative, the slope at the

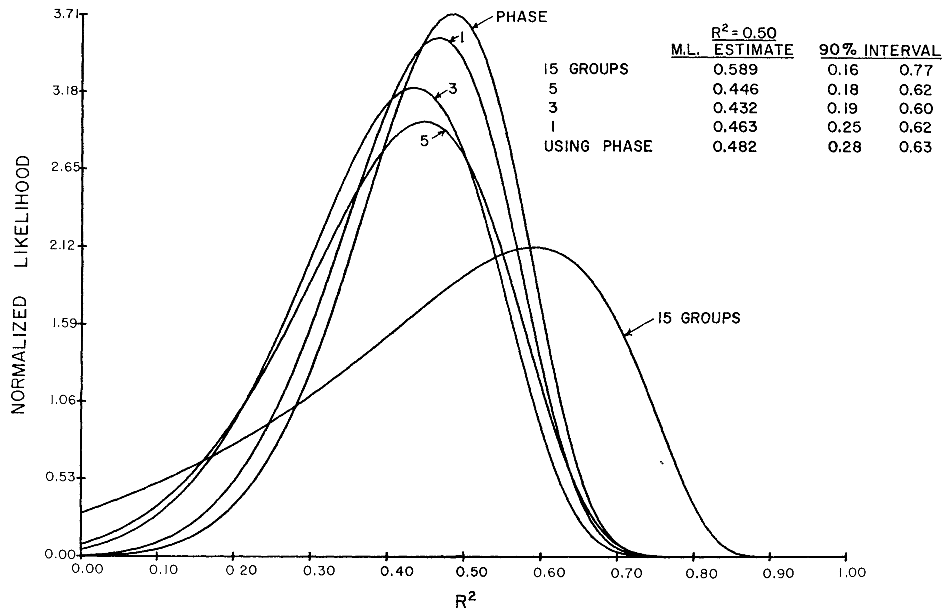

FIG. 3. Normalized likelihood as a function of R^2 evaluated at the maximum of the other parameters for various groupings and the usual estimate using phase. The maximum likelihood estimate is shown for each case together with the minimum length interval containing 90% of the area. The true R^2 is 0.50 and the sample size is 15.

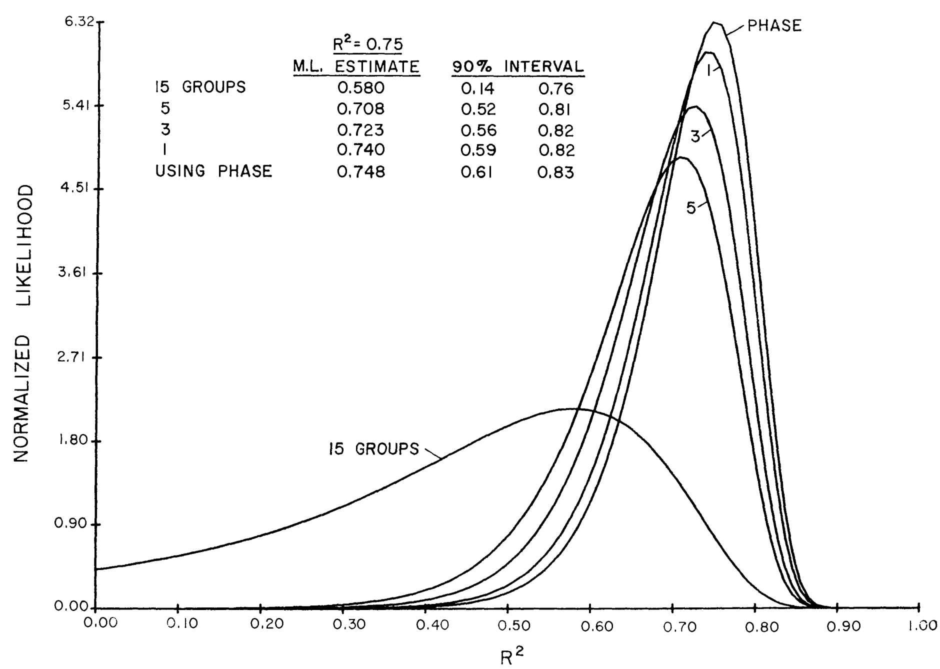

FIG. 4. Normalized likelihood as a function of R^2 evaluated at the maximum of the other parameters for various groupings and the usual estimate using phase. The maximum likelihood estimate is shown for each case together with the minimum length interval containing 90% of the area. The true R^2 is 0.75 and the sample size is 15.

midpoint, $R^2 = \frac{1}{2}$ was checked using (4.3). By repeating this procedure ten times, in order to locate the change in slope, the maximum likelihood estimate was taken to be the center of the final interval which had length 2^{-10}, giving three significant figures. The results of the simulations are shown in Figures 1 through 4.

6. Conclusion. The Figures show a clear loss of precision when group sizes of one are used. However, for an outer window of fifteen elementary frequency bands, the groupings of size three or five are acceptable. Therefore, it is necessary in a given situation, to decide whether the decrease in efficiency caused by grouping is worth the possible decrease in bias caused by varying phase.

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