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QUASI-STATIONARY BEHAVIOUR OF A LEFT-CONTINUOUS RANDOM WALK

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1. Introduction. Consider a random walk $\{S_n\}$ $(n = 0, 1, \cdots)$ on the integers $\{\cdots, -1, 0, 1, \cdots\}$ for which $S_0 = 1$ and

$$\text{(1.1)} \qquad \operatorname{pr}\{S_{n+1} = S_n + k \mid S_n\} = p_k \qquad (\text{all } n, k)$$

such that

$$\text{(1.2)} \qquad p_{-1} > 0, \qquad p_k = 0 \ (k = -2, -3, \cdots), \qquad \textstyle\sum_{k=-1}^{\infty} p_k = 1.$$

The main object of this note is to study the limits as $n \to \infty$ of

$$\text{(1.3)} \qquad a_j^n = \operatorname{pr}\{S_n = j \mid \min(S_1, \cdots, S_n) > 0, S_0 = 1\}$$

when

$$\text{(1.4)} \qquad 0 < m = 1 + \textstyle\sum_{k=-1}^{\infty} kp_k < 1,$$

the limits being zero when $m \geqq 1$. In other words, if after a long time the process has not yet visited the set $\{\cdots, -1, 0\}$ what (if any) is its asymptotic behaviour? An extensive discussion of such questions in the context of Markov chains on a countable state space is given in papers by Seneta and others, the most refined results being given in Seneta and Vere-Jones (1966). This note may be regarded as an illustration of their work in the case of a moderately simple Markov chain, or as an addendum to what is already known on left-continuous simple random walks. To simplify our discussion, we assume that

$$\text{(1.5)} \qquad \{S_n\} \quad \text{is aperiodic, i.e.,} \quad \gcd\{j : p_{j-1} > 0\} = 1.$$

In the trivial case that $p_{-1} + p_0 = 1$ and $p_{-1} < 1$, $a_j^n = 1$ if $j = 1$ and $= 0$ otherwise, so to eliminate this exception we assume further that

$$\text{(1.6)} \qquad p_{-1} + p_0 < 1.$$

With this notation and

$$\text{(1.7)} \qquad f(s) = \textstyle\sum_{k=-1}^{\infty} p_k s^{k+1} \qquad (|s| \leqq 1),$$

we shall prove

Theorem 1. *For a left-continuous aperiodic random walk* $\{S_n\}$ *with mean step-length* $m - 1 < 0$,

$$\lim_{n\to\infty} a_j^n = \lim_{n\to\infty} \operatorname{pr}\{S_n = j \mid S_r > 0 \ (r = 1, \cdots, n), S_0 = 1\}$$

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exists and equals a_j. *Either*

(i) $f(z)$ *is not analytic at* $z = 1$ *and* $a_j = 0$ $(j = 1, 2, \cdots)$; *or*

(ii) $f(z)$ *is analytic at* $z = 1$, $a_j > 0$ $(j = 1, 2, \cdots)$, $\sum_{j=1}^{\infty} a_j = 1$, *and*

$$A(s) = \sum_{j=1}^{\infty} a_j s^j = (R - 1)s/[Rf(s) - s] \qquad (|s| \leqq 1), \tag{1.8}$$

where the positive constant R *satisfies* $1 < R < m^{-1}$ *and is defined by*

$$R = s_0/f(s_0) \leqq 1/f'(s_0) < 1/m, \tag{1.9}$$

s_0 *being the positive root* (*if any*) *of the equation* $sf'(s) = f(s)$ *and otherwise is the radius of convergence of the power series* $f(s)$.

2. Motivation and discussion. Defining sequences of random variables $\{N_n\}$ and $\{Z_n\}$ $(n = 0, 1, \cdots)$ by means of

$$N_0 = 0, \qquad Z_n = S_{N_n}, \qquad N_{n+1} = N_n + Z_n, \tag{2.1}$$

it is known (cf. Spitzer (1964) p. 234 and Harris (1963) Chapter I) that $\{Z_n\}$ is a Galton-Watson branching process with offspring distribution $\{f_j\}$ given by

$$f_j = p_{j-1} \qquad (j = 0, 1, \cdots). \tag{2.2}$$

The mean of the offspring distribution equals m as at (1.4), and when $m < 1$, Yaglom's theorem in its refined version (Heathcote, Seneta, and Vere-Jones (1967)) states that the limits g_j as $n \to \infty$ of

$$\begin{aligned} g_j^{\,n} &= \mathrm{pr}\,\{Z_n = j \mid Z_n > 0, Z_0 = 1\} \\ &= \mathrm{pr}\,\{S_{N_n} = j \mid \min(S_1, \cdots, S_{N_n}) > 0, S_0 = 1\} \end{aligned} \tag{2.3}$$

exist and form a probability distribution on $\{1, 2, \cdots\}$, the generating function $G(s) = \sum_{j=1}^{\infty} g_j s^j$ being the unique probability generating function solution with $G(0) = 0$ of the equation

$$1 - G(f(s)) = m(1 - G(s)). \tag{2.4}$$

There is a superficial resemblance between $g_j^{\,n}$ in the second form of (2.3) and $a_j^{\,n}$ in (1.3), and originally I had hoped that knowledge of the limits a_j would shed more light on the nature of the distribution $\{g_j\}$. Such was not to be the case, as is shown by comparison of Theorem 1 and Yaglom's theorem, and is further exemplified in the last section of this note.

The present work should be regarded then as a contribution to the study of the quasi-stationarity features of a left-continuous random walk, a special case of which (simple random walk) may be found in Seneta and Vere-Jones (1966). It is pertinent therefore to enquire further concerning the convergence properties of

$$p_j^{\,n} = \mathrm{pr}\,\{S_n = j, S_r > 0\ (r = 1, \cdots, n-1) \mid S_0 = 1\}, \tag{2.5}$$

and in particular, the radii of convergence of the power series

$$P_j(z) = \sum_{n=0}^{\infty} p_j^{\,n} z^n \qquad (j = 1, 2, \cdots) \tag{2.6}$$

and the related power series

$$P_{ij}(z) = \sum_{n=0}^{\infty} \operatorname{pr}\{S_n = j \mid S_0 = i\} z^n. \tag{2.7}$$

These questions are related to other work of Vere-Jones (1962), (1967) whose results apply immediately to the Markov chains $\{S_n'\}$ and $\{S_n\}$, where

(2.8) $\{S_n'\}$ denotes the process $\{S_n\}$ until its first exit from the set $\{1, 2, \cdots\}$.

Vere-Jones shows that the transition probability generating functions of an irreducible Markov chain have a common circle of convergence, on which circle either all or none of the generating functions converge everywhere. Using his terminology, we shall prove the following two theorems.

THEOREM 2. *The power series $P_j(z)$ $(j = 1, 2, \cdots)$ converge on their common circle of convergence $|z| = R$, where R is defined in Theorem 1 with $R = 1 = s_0$ if $f(z)$ is not analytic at $z = 1$. Thus, $\{S_n'\}$ is R-transient.*

THEOREM 3. *The power series $P_{ij}(z)$ $(i, j = 1, 2, \cdots)$ converge on their common circle of convergence $|z| = R = s_0/f(s_0)$ if and only if*

$$s_0 f'(s_0) < f(s_0). \tag{2.9}$$

If (2.9) holds, then $\{S_n\}$ is R-transient; otherwise, $s_0 f'(s_0) = f(s_0)$ and $\{S_n\}$ is R-null-recurrent.

Theorem 1 can be regarded as stating the conditions for the existence of a non-trivial non-negative left R-invariant vector (a_j) with $\sum a_j < \infty$ for the one-step sub-stochastic transition matrix

$$\mathbf{P} = (p_{ij}) = (p_{j-i}) \qquad (i, j = 1, 2, \cdots) \tag{2.10}$$

of the Markov chain $\{S_n'\}$. Our last theorem complements Theorem 1.

THEOREM 4. *With s_0 as in Theorem 1 and $R = s_0 = 1$ in case (i) of that theorem, $(b_j) = (j s_0^{j-1})$ is a non-trivial non-negative right R-subinvariant vector for the transition matrix $\mathbf{P}$, being right R-invariant if and only if $s_0 f'(s_0) = f(s_0)$. When $s_0 = 1$, the vector (a_j') with $a_j' = 1$ $(j = 1, 2, \cdots)$ is a left R-superinvariant vector for $\mathbf{P}$ with $\sum_{j=1}^{\infty} a_j' = \infty = \sum_{j=1}^{\infty} a_j' b_j$; when $s_0 > 1$, $\sum_{j=1}^{\infty} a_j b_j = \infty$.*

3. Proof of Theorem 1. The key steps in the proof rely on the left-continuity property of $\{S_n\}$ and on Kemeny's (1959) individual ratio limit theorem for random walks (cf. also Stone (1967)). Setting

$$\begin{aligned} q(z) &= \sum_{n=1}^{\infty} q_n z^n \\ &= \sum_{n=1}^{\infty} \operatorname{pr}\{S_n = 0, S_r > 0\ (r = 1, \cdots, n-1) \mid S_0 = 1\} z^n \qquad (|z| \leqq 1), \end{aligned} \tag{3.1}$$

it is known (e.g. Spitzer (1964), p. 234) that $w = q(z)$ is the unique root in $|w| < 1$ of

$$z = w/f(w) \qquad (|w| < 1). \tag{3.2}$$

Also, because the walk is left-continuous,

(3.3) $q_{n+1} = p_{-1} \operatorname{pr} \{S_n = 1, S_r > 0 \ (r = 1, \cdots, n-1) \mid S_0 = 1\} = p_{-1}p_1^n$

where p_1^n is defined at (2.1), so

(3.4) $$P_1(z) = q(z)/p_{-1}z.$$

Defining

(3.5) $$Q_n = q_{n+1} + q_{n+2} + \cdots, \qquad (n = 0, 1, \cdots),$$

the quantities a_j^n which we wish to study can be written in the form

(3.6) $$a_j^n = p_j^n Q_n^{-1} = p_j^n / p_{-1}(p_1^n + p_1^{n+1} + \cdots) \qquad (j = 1, 2, \cdots).$$

The quantities a_j^{n+1} and a_j^n are related by the forward Chapman-Kolmogorov equation, namely, from

$$p_j^{n+1} = \sum_{k=1}^{j+1} p_{j-k} p_k^n$$

there follows for $j = 1, 2, \cdots$

(3.7) $$p_{-1}a_{j+1}^n = p_{-1}p_{j+1}^n/Q_n = p_j^{n+1}/Q_{n+1})(Q_{n+1}/Q_n) - \sum_{k=1}^{j} p_{j-k}p_k^n/Q_n.$$

Thus to prove the existence of $\lim_{n\to\infty} a_j^n$, it suffices to prove the existence of

(3.8) $$a_1 = \lim_{n\to\infty} a_1^n \quad \text{and} \quad \rho = \lim_{n\to\infty} Q_{n+1}/Q_n.$$

From the assumption that $\{S_n\}$ is aperiodic, we have that $p_1^n > 0$ for all sufficiently large n, so for such n we can write from (3.6) with $j = 1$ that

(3.9) $$(p_{-1}a_1^n)^{-1} = 1 + r_n + r_n r_{n+1} + \cdots$$

where

(3.10) $$r_n = p_1^{n+1}/p_1^n.$$

Considering (3.9), we assert that

(3.11) $\lim_{n\to\infty} a_1^n$ *exists if and only if* $\lim_{n\to\infty} r_n = r$ *exists*, $\quad 0 < r \leqq 1$.

To justify the assertion, observe that $b_n = (p_{-1}a_1^n)^{-1}$ satisfies $b_n = 1 + r_n b_{n+1}$, and $\lim_{n\to\infty} b_n$ exists if and only if $\lim_{n\to\infty} a_1^n$ exists, interpreting $b_n \to \infty$ when $a_1^n \to 0$. Then $r_n = (b_n - 1)/b_{n+1}$ (which is defined for all sufficiently large n), and the convergence of a_1^n implies the convergence of r_n; we note that $a_1^n \to 0$ implies that $r_n \to 1$. Conversely, if $r_n \to r$, then for arbitrarily small $\epsilon > 0$ and for all sufficiently large n,

$$(p_{-1}a_1^n)^{-1} > 1 + (r - \epsilon) + (r - \epsilon)^2 + \cdots = 1/(1 - r + \epsilon),$$

so if $r_n \to r = 1$, $p_{-1}a_1^n \to 0$ $(n \to \infty)$. If $r < 1$, then when ϵ is such that $r < r + \epsilon < 1$, we have

$$(p_{-1}a_1^n)^{-1} < 1 + (r + \epsilon) + (r + \epsilon)^2 + \cdots = 1/(1 - r - \epsilon)$$

for all sufficiently large n. Hence, in both cases, if $r_n \to r$, we have proved (3.11), and indeed more, namely

$$(3.12) \qquad \lim_{n\to\infty} p_{-1}a_1^{\,n} = 1 - \lim_{n\to\infty} r_n$$

when either side exists.

Now for a left-continuous random walk it happens that

$$(3.13) \qquad \begin{aligned} q_n &= \mathrm{pr}\,\{S_n = 0, S_r > 0 \ (r = 1, \cdots, n-1) \mid S_0 = 1\} \\ &= \mathrm{pr}\,\{S_n = 0 \mid S_0 = 1\}/n \end{aligned}$$

(e.g. Spitzer (1964), p. 234), so (for all sufficiently large n)

$$(3.14) \qquad \begin{aligned} r_n &= p_{-1}p_1^{\,n+1}/p_{-1}p_1^{\,n} = q_{n+2}/q_{n+1} \\ &= (n+1)\,\mathrm{pr}\,\{S_{n+2} = 0 \mid S_0 = 1\}/(n+2)\,\mathrm{pr}\,\{S_{n+1} = 0 \mid S_0 = 1\}. \end{aligned}$$

We now appeal to Kemeny's (1959) generalization of the Chung and Erdös individual ratio limit theorem for aperiodic random walks; this shows that

$$(3.15) \qquad \lim_{n\to\infty} \mathrm{pr}\,\{S_{n+2} = 0 \mid S_0 = 1\}/\mathrm{pr}\,\{S_{n+1} = 0 \mid S_0 = 1\} = f(h(0))/h(0)$$

where $h(t)$ is the functional inverse of the strictly increasing function

$$(3.16) \qquad g(s) = (sf'(s) - f(s))/f(s) \qquad (0 \leqq s < s_1),$$

where s_1 is the radius of convergence of $f(s)$, and the range of definition of $h(\cdot)$ is made $(-\infty, \infty)$ by defining for $t < g(0) = -1$, $h(t) = h(g(0)) = 0$, and, if $\lim_{s\uparrow s_1} g(s) = g(s_1 - 0) < \infty$, for $t \geqq g(s_1 - 0)$, $h(t) = h(g(s_1 - 0)) = s_1 = \limsup\,\{s : f(s) < \infty, s > 0\}$. We find $h(0)$ as follows. If $f(s) \to \infty$ as $s \to s_1$, then there exists s_0 in $(0, s_1)$ such that

$$(3.17) \qquad s_0 f'(s_0) - f(s_0) = 0 = g(s_0),$$

in which case $h(0) = s_0$ and

$$(3.18) \qquad f(h(0))/h(0) = f(s_0)/s_0 = f'(s_0).$$

If $f(s) \to f(s_1 - 0) < \infty$ $(s \to s_1)$, and hence $f(s_1 - 0) = f(s_1)$ because $f(\cdot)$ has non-negative coefficients, there may still exist s_0 in $(0, s_1]$ such that (3.17) is satisfied, so (3.18) would again be true; if not, then $s_1 f'(s_1) < f(s_1)$, $h(0) = s_1$, and $f(h(0))/h(0) = f(s_1)/s_1$. Thus, with s_0 defined as in the theorem, and recalling (3.14), we have that

$$(3.19) \qquad r_n \to r = f(s_0)/s_0 \geqq f'(s_0) \geqq m,$$

the last inequality following from $g(1) = m - 1 < 0$ and hence $h(0) \geqq 1$. If $s_1 > 1$, since $g(s)$ is continuous and strictly increasing in $(0, s_1)$ with $g(1) = m - 1 < 0$, we have $s_0 > 1$ and so $f(s_0) < s_0$. If $s_1 = 1$ then $s_0 = s_1 = 1$ and $r = 1$, so we have that $r < 1$ if and only if $f(z)$ is analytic at $z = 1$, which by (3.12) proves that $a_1 > 0$ if and only if $f(z)$ is analytic at $z = 1$.

Next, by (3.3) and (3.6), $p_{-1}a_1^{\,n} = p_{-1}p_1^{\,n}/Q_n = q_{n+1}/Q_n = 1 - Q_{n+1}/Q_n$, so the convergence of $p_{-1}a_1^n$ to $1 - r$ implies that $Q_{n+1}/Q_n \to r(n \to \infty)$, and then by (3.8) we have proved the existence of the limits a_j for $j = 2, 3, \cdots$. Furthermore, these limits satisfy (from (3.7))

$$(3.20) \qquad ra_j = \sum_{k=1}^{j+1} p_{j-k}a_k ,$$

with $a_j = 0$ $(j = 1, 2, \cdots)$ when $r = 1$ and $a_j \geqq 0$, $a_1 > 0$ when $r < 1$. Since gcd $\{j : p_{j-1} > 0\} = 1$, the matrix $\mathbf{P} = (p_{jk}) = (p_{k-j})$ $(j, k = 1, 2, \cdots)$ is irreducible, and therefore for $r < 1$, the non-trivial non-negative left-eigenvector $(a_1, a_2, \cdots)$ has every element strictly positive (e.g. Seneta and Vere-Jones (1966), p. 408). Also by Fatou's lemma, $A = \sum_{j=1}^{\infty} a_j \leqq 1$, and to show that equality holds, we sum over $j = 1, 2, \cdots$ in (3.20), obtaining $rA = A - p_1 a_1 = A - 1 + r$, so when $r \neq 1$ we have $A = 1$. Forming the generating function $A(s) = \sum_{j=1}^{\infty} a_j s^j (|s| \leqq 1)$ from (3.20) yields (1.12) on identifying R in (1.12) with r^{-1}, and hence (1.13). Theorem 1 is proved.

Observe that the inequality $R < 1/m$ has a probabilistic interpretation, namely that if $R > 1$, then

$$1 < A'(1) = R(1 - m)/(R - 1)$$

and hence $Rm < 1$. Trivially, $R < 1/m$ when $R = 1$.

4. Proof of Theorems 2, 3 and 4. By Vere-Jones' (1962) work we know that none or all of $P_j(z)$ $(j = 1, 2, \cdots)$ remain finite as $z \uparrow R$, so it suffices to show that $\lim_{z \uparrow R} P_1(z) < \infty$, i.e., by (3.4) that

$$(4.1) \qquad \lim_{z \uparrow R} q(z) < \infty .$$

Let $\{s_\nu\}$ be a monotone increasing positive sequence converging to s_0. Then $z_\nu = s_\nu/f(s_\nu)$ is a monotone increasing sequence converging to $s_0/f(s_0) = R$, so $q(z_\nu) = s_\nu \to s_0 < \infty$ as $z_\nu \to R$, provided that $q(z)$ is in fact the inverse in $0 < w < s_0$ of $z = w/f(w)$. But this is readily seen by noting, first that $q(z)$, being a power series with non-negative coefficients, has its first singularity on the positive axis, and then that the range of definition of $q(z)$ can be extended by analytic continuation from $|z| < 1$ to a neighbourhood of that part of the positive axis corresponding to $1 \leqq w < w_0$ provided only that $f(w)$ is analytic and $(w/f(w))'$ does not vanish on $[1, w_0)$, i.e., provided $1 < w < s_0$. Hence (4.1), and Theorem 2 is proved.

In proving Theorem 3 we again use Vere-Jones' (1962) result in asserting that either none or all of the generating functions $P_{ij}(z)$ converge on their common circle of convergence $z = R$; this *is* the circle of convergence because by (3.13) it coincides with the circles of convergence of the generating functions $P_j(x)$. Consider next the left-continuous random walk $\{S_n^*\}$ with one-step transition probabilities

$$p_{ij}^* = p_{j-i}^* = p_{j-i}s_0^{j-i+1}/f(s_0) = s_0^{j-i}p_{ij}s_0/f(s_0).$$

Then the generating function $P_{ij}^*(z)$ of the transition probabilities pr $\{S_n^* =$

$j \mid S_0^* = i\}$ is given by

$$P_{ij}^*(z) = s_0^{j-i} P_{ij}(zs_0/f(s_0)).$$

Clearly $E(z^{S_1^*-S_0^*} \mid S_0^*) = f(s_0z)/zf(s_0)$, so

$$E(S_1^* - S_0^* \mid S_0^*) = -1 + s_0f'(s_0)/f(s_0).$$

Consequently $\lim_{z\uparrow 1} P_{ij}^*(z)$, which converges if and only if the one-dimensional random walk $\{S_n^*\}$ is transient, is finite only if $s_0f'(s_0) \neq f(s_0)$, and since $s_0f'(s_0) \leqq f(s_0)$, we obtain the assertion of the theorem. The walk $\{S_n^*\}$ is necessarily null, so when $s_0f'(s_0) = f(s_0)$, the walk $\{S_n\}$ is R-null-recurrent.

The proof of Theorem 4 consists of the statement,

$$\begin{aligned} R \textstyle\sum_{j=1}^{\infty} p_{ij}b_j &= [s_0/f(s_0)] \textstyle\sum_{j=1}^{\infty} p_{j-i}(j - i + 1 + i - 1)s_0^{j-1} \\ &= [s_0^{i-1}/f(s_0)][s_0f'(s_0) + (i-1)f(s_0)] \leqq b_i \end{aligned}$$

with equality holding if and only if $s_0f'(s_0) = f(s_0)$. The rest of the theorem is proved as easily, the only part needing explanation being the statement that $\sum_{j=1}^{\infty} a_jb_j = \infty$: this sum equals $\lim_{s\to s_0} A'(s)$, which equals infinity because $A(s) \to \infty$ for $s \to s_0$ (cf. (1.8) and (1.9)). Indeed, if we had $\sum a_jb_j < \infty$, we should have a contradiction of $\{S_n'\}$ being R-transient, for by Lemma 1 (iii) of Seneta and Vere-Jones (1966), if for the irreducible matrix (p_{ij}), (a_j) is a non-trivial non-negative left R-invariant vector and (b_j) is a non-trivial non-negative right R-subinvariant vector for which $\sum a_jb_j < \infty$, then the matrix is R-positive, and conversely.

5. Example. To illustrate case (ii) of Theorem 1, suppose that

$$(5.1) \qquad f(s) = a + bs/(1 - cs) \qquad (0 < a, b, c < 1; b = (1-a)(1-c)).$$

To ensure that $m < 1$ all that we require of a and c is that $1 > a > c > 0$. Noting that $f(s)$ is analytic in $|s| < s_1 = c^{-1}$ and that $f(s) \to \infty$ as $s \to s_1$, we have first to find s_0, the root in $(1, c^{-1})$ of

$$bs/(1 - cs)^2 = a + bs/(1 - cs).$$

We find that

$$\begin{aligned} s_0 &= \tfrac{1}{2}c^{-1} & (a + c = 1), \\ &= |\{ac - [ac(1-a)(1-c)]^{\frac{1}{2}}\}/c(a + c - 1)| & (a + c \neq 1). \end{aligned}$$

We do not give R in terms of a and c except when $a + c = 1$, in which case $\{p_j\}$ is a geometric distribution rather than a modified geometric distribution; when $a = 1 - c, R = 1/4c(1 - c)$, which is >1 because $a > c > 0$ and hence $2c < 1$. Not even with $a = 1 - c$ is $\{a_j\}$ a geometric or modified geometric distribution; rather it is related to the negative binomial in this special case. These results are more complicated algebraically than the corresponding results concerning $\{g_j\}$ for the embedded branching process, for which $G(s) = (a - c)s/(a - cs)$ and so $\{g_j\}$ is always a geometric distribution on $\{1, 2, \cdots\}$. It can be verified that both $G'(1) < A'(1)$ and $G'(1) > A'(1)$ are possible by suitable choice of a and c.

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