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A COMPARISON TEST FOR MARTINGALE INEQUALITIES

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1. Introduction. If $f = (f_1, f_2, \cdots)$ is a sequence of real valued functions on a probability space we define its difference sequence $d = (d_1, d_2, \cdots)$ by $d_1 = f_1$, $d_i = f_i - f_{i-1}$, $i > 1$, and use the following notation: $f_n^* = \max(|f_1|, \cdots, |f_n|)$, $f^* = \sup_n f_n^*$, $S_n(f) = [\sum_{i=1}^n d_i^2]^{\frac{1}{2}}$, $S(f) = S_\infty(f) = \sup_n S_n(f)$, and $\|f\|_p = \sup_n \|f_n\|_p$, where $\|f_k\|_p$ is the L^p norm of f_k. $\|f\|_p$ will be called the L^p norm of the sequence f and f will be said to be L^p bounded if it has finite L^p norm.

In [2] Burkholder derives a number of martingale inequalities from Theorem 6 of that paper, which states: There is a real number M such that if f and g are martingales relative to the same sequence of σ-fields and $S_n(g) \leqq S_n(f)$, $n \geqq 1$, then $\lambda P(g^* > \lambda) \leqq M \|f\|_1$, $\lambda > 0$.

The proof of this result is based on a widely applicable method, which yields, however, no information about the size of M. In [6] Gundy gives proofs capable of providing numerical bounds for M for several of the inequalities established in [2]. However, only a special case of Theorem 6 is obtained, that in which g is a transform of f under a uniformly bounded multiplier sequence. Here a proof providing numerical bounds for M is given for a strengthened version of Theorem 6 and an additional inequality is obtained for g^* if the f of Theorem 6 is uniformly integrable.

In the final section several existing results about the convergence of L^1 bounded martingales f are shown to follow easily from information concerning $S(f)$.

2. A comparison test for martingale inequalities. If $f = (f_1, f_2, \cdots)$ is a martingale with difference sequence d then

$$E(f_n^2) = E((\textstyle\sum_{i=1}^n d_i)^2) = E(\sum_{i=1}^n d_i^2) = E((S_n(f))^2).$$

Since $E(f_n^{*2}) \leqq 4E(f_n^2)$ by an inequality due to Doob ([4], page 317) we have upon taking limits the basic relation:

$$E(S(f)^2) \leqq E(f^{*2}) \leqq 4E(S(f)^2). \tag{1}$$

We will make use of the result proved in [2] and [4], if f is a martingale

$$\lambda P(S(f) > \lambda) \leqq 22 \|f\|_1, \qquad \lambda > 0, \tag{2}$$

(Burkholder proves there is a real number M for which $\lambda P(S(f) > \lambda) \leqq M \|f\|_1$, and Gundy's method gives numerical bounds for M). In particular (1) will be used to translate the information (2) gives us about $S(f)$ into information about f^*. No effort is made to minimize the constants involved.

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THEOREM 1. *If f and g are martingales relative to the same sequence of σ-fields and $S(g) \leqq S(f)$, then $\lambda P(g^* > \lambda) < 5000\,\|f\|_1$, $\lambda > 0$.*

PROOF. Assume without loss of generality that f is L^1 bounded. Then corresponding to any $\lambda > 0$ the martingale g can be decomposed into three martingales, $g = a + b + h$, satisfying

(i) $P(a^* > 0) \leqq 136\,\|f\|_1/\lambda$,

(ii) $E(\sum_{i=1}^{\infty} |\beta_i|) \leqq 180\,\|f\|_1$, where $\beta = (\beta_1, \beta_2, \cdots)$ is the difference sequence of b.

(iii) $\|h\|_2^2 \leqq 544\,\|f\|_1\,\lambda$.

The hypotheses of Theorem 1 imply $S_n(g) \leqq 44f_n' + 44f_n'' + S_n(f') + S_n(f'')$, with f' and f'' the martingales of the Krickeberg decomposition of f, proved with the aid of (2) just preceding the statement of Theorem 1 in [3]. Let t be the first time one of the terms $f_n', f_n'', S_n(f'), S_n(f'')$ exceeds λ and let $e = (e_1, e_2, \cdots)$ be the difference sequence of g. Define $p_n = I_{\{t=n\}}[-(e_n - 4\lambda)^+ + (e_n + 4\lambda)^-]$. Let τ be the first time n that $\sum_{i=1}^{n+1} E(|p_i|\,|\mathfrak{F}_{i-1})$ exceeds λ.

The martingales a, b, and h are defined as follows:

$$a_n = \sum_{i=1}^{n} \alpha_i, \quad \text{where} \quad \alpha_k = e_k I_{\{t<k \text{ or } \tau<k\}},$$

$$b_n = \sum_{i=1}^{n} \beta_i, \quad \text{where} \quad \beta_k = (E(p_k \mid \mathfrak{F}_{k-1}) - p_k) I_{\{\tau \geqq k\}},$$

$$h_n = \sum_{i=1}^{n} \theta_i, \quad \text{where} \quad \theta_k = (e_k I_{\{t \geqq k\}} + p_k - E(p_k \mid \mathfrak{F}_{k-1})) I_{\{\tau \geqq k\}}.$$

On $\{t = n, e_n > 4\lambda\}$ we have $(e_n - 4\lambda)^+ + 4\lambda = e_n \leqq S_n(g) \leqq 44f_n' + 44f_n'' + S_n(f') + S_n(f'') \leqq 44f_n' + 44f_n'' + (\lambda + |d_n'|) + (\lambda + |d_n''|) \leqq 45f_n' + 45f_n'' + 4\lambda$. Thus $(e_n - 4\lambda)^+ \leqq 45f_n' + 45f_n''$ on $\{t = n\}$. Similarly $(e_n + 4\lambda)^- \leqq 45f_n' + 45f_n''$ on $\{t = n\}$. Thus $E(\sum_{i=1}^{\infty} |p_i|) \leqq E((45f_t' + 45f_t'')I_{\{t<\infty\}}) \leqq 90\,\|f\|_1$.

Now $P(a^* > 0) \leqq P(t < \infty) + P(\tau < \infty) \leqq P(f'^* > \lambda) + P(f''^* > \lambda) + P(S(f') > \lambda) + P(S(f'') > \lambda) + P(\tau < \infty) \leqq \|f\|_1/\lambda + \|f\|_1/\lambda + 22\,\|f\|_1/\lambda + 22\,\|f\|_1/\lambda + E(\sum_{i=1}^{\infty} E(|p_i| \mid \mathfrak{F}_{i-1}))/\lambda \leqq 46\,\|f\|_1/\lambda + E(\sum_{i=1}^{\infty} |p_i|)/\lambda \leqq 46\,\|f\|_1/\lambda + 90\,\|f\|_1/\lambda = 136\,\|f\|_1/\lambda$, verifying (i).

And $E(\sum_{i=1}^{\infty} |\beta_i|) \leqq E(\sum_{i=1}^{\infty} |E(p_i|\,\mathfrak{F}_{i-1}) - p_i|) \leqq 2E(\sum_{i=1}^{\infty} |p_i|) \leqq 180\,\|f\|_1$, verifying (ii).

Finally

$$\begin{aligned}
\|h\|_2^2 &= \sum_{i=1}^{\infty} E(\theta_i^2) \\
&= \sum_{i=1}^{\infty} E(((e_i I_{\{i \leqq t\}} + p_i - E(p_i \mid \mathfrak{F}_{i-1}))I_{\{\tau \geqq i\}})^2) \\
&\leqq \sum_{i=1}^{\infty} 2[E((e_i I_{\{i \leqq t\}} + p_i)^2) + E((E(p_i \mid \mathfrak{F}_{i-1})I_{\{\tau \geqq i\}})^2)] \\
&= 2E(\sum_{i=1}^{\infty} (e_i I_{\{i \leqq t\}} + p_i)^2) + 2E(\sum_{i=1}^{\tau} E(p_i \mid \mathfrak{F}_{i-1})^2).
\end{aligned}$$

Now $E(\sum_{i=1}^{\tau} E(p_i|\mathfrak{F}_{i-1})^2) \leqq E((\sum_{i=1}^{\tau} |E(p_i|\mathfrak{F}_{i-1})|)^2) \leqq \lambda E(\sum_{i=1}^{\tau} E(|p_i||\mathfrak{F}_{i-1})) \leqq 90\,\|f\|_1\,\lambda$. If $x > 0$, $P(\sum_{i=1}^{\infty} (e_i I_{\{i \leqq t\}} + p_i)^2 > x) \leqq P(S(g)^2 > x) \leqq P(S(f)^2 > x) \leqq 22\,\|f\|_1/x^{\frac{1}{2}}$ by (2).

Also $\sum_{i=1}^{\infty} (e_i I_{\{i \leqq t\}} + p_i)^2 = \sum_{i=1}^{\infty} e_i^2 I_{\{i<t\}} + \sum_{i=1}^{\infty} (e_i + p_i)^2 I_{\{i=t\}} \leqq \lambda^2 + (4\lambda)^2$

$= 17\lambda^2$, so $E(\sum_{i=1}^{\infty} (e_i I_{\{i \leqq t\}} + p_i)^2) = \int_0^{17\lambda^2} P(\sum_{i=1}^{\infty} (e_i I_{\{i \leqq t\}} + p_i)^2 > x)\, dx \leqq \int_0^{17\lambda^2} 22 \, \|f\|_1 / x^{\frac{1}{2}} \, dx \leqq 182 \, \|f\|_1 \lambda$. Thus $\|h\|_2^2 \leqq 2[90 \, \|f\|_1 \lambda + 182 \, \|f\|_1 \lambda] = 544 \, \|f\|_1 \lambda$.

Since $g^* \leqq a^* + b^* + h^*$, $P(g^* > 2\lambda) \leqq P(h^* > \lambda) + P(b^* > \lambda) + P(a^* > 0) \leqq E(h^{*2})/\lambda^2 + E(\sum_{i=1}^{\infty} |\beta_i|)/\lambda + 136 \, \|f\|_1/\lambda \leqq 4 \, \|h\|_2^2/\lambda^2 + 180 \, \|f\|_1/\lambda + 136 \, \|f\|_1/\lambda \leqq 2176 \, \|f\|_1 \lambda/\lambda^2 + 180 \, \|f\|_1/\lambda + 136 \, \|f\|_1/\lambda = 2492 \, \|f\|_1/\lambda \leqq 5000 \, \|f\|_1/2\lambda$, completing the proof of Theorem 1.

Theorem 1 is of interest only when the martingale f is L^1 bounded. If f is also uniformly integrable we can get an additional inequality concerning g.

LEMMA 1. *Let* $f = (f_1, f_2, \cdots)$ *be a uniformly integrable martingale. Let* t *be the first time* f_n *exceeds* λ. Then $\lim_{\lambda \to \infty} E f_t I_{\{t > \infty\}} = 0$.

PROOF. Let $f_\infty = \lim_{n \to \infty} f_n$. Then $E(|f_t I_{\{t < \infty\}}|) = \sum_{k=1}^{\infty} E(|f_k I_{\{t=k\}}|) = \sum_{k=1}^{\infty} E(|E(f_\infty \mid \mathfrak{F}_k) I_{\{t=k\}}|) \leqq \sum_{k=1}^{\infty} E(E(|f_\infty \mid I_{\{t=k\}}| \, \mathfrak{F}_k)) = E(|f_\infty I_{\{t<\infty\}}|)$ which, since f_∞ is integrable and $P(t < \infty)$ approaches 0 as $\lambda \to \infty$, goes to 0 as $\lambda \to \infty$.

In particular this shows $\lim_{\lambda \to \infty} \lambda P(f^* > \lambda) = 0$ if f is a uniformly integrable martingale.

LEMMA 2. *If* f *is a uniformly integrable martingale* $\lim_{\lambda \to \infty} \lambda P(S(f) > \lambda) = 0$.

PROOF. Let $\epsilon > 0$. Pick n so large that $E(|f_n - \lim_k f_k|) \leqq \epsilon/22$. Then $(f_n - f_n, f_{n+1} - f_n, f_{n+2} - f_n, \cdots)$ is a martingale of L^1 norm not exceeding $\epsilon/22$. Thus $\lambda P((\sum_{i=n}^{\infty} (f_{i+1} - f_i)^2)^{\frac{1}{2}} > \lambda) \leqq \epsilon$ by (2). Since $\lim_{\lambda \to \infty} \lambda P(S_n(f) > \lambda) = 0$ and $S(f) \leqq S_n(f) + (\sum_{i=n}^{\infty} (f_{i+1} - f_i)^2)^{\frac{1}{2}}$, we have $\limsup_{\lambda \to \infty} \lambda P(S(f) > \lambda) \leqq \epsilon$ completing the proof.

REMARK. Theorem 8 of [2] states in part: There is a constant M such that if f is a martingale then $\lambda P(f^* > \lambda) \leqq M E(S(f))$. Using this, a proof very similar to the above can be given to show $E(S(f)) < \infty$ implies $\lim_{\lambda \to \infty} \lambda P(f^* > \lambda) = 0$.

THEOREM 2. *If* f *and* g *are martingales relative to the same sequence of* σ*-fields,* f *uniformly integrable, and* $S(f) \leqq S(g)$, *then* $\lim_{\lambda \to \infty} \lambda P(g^* > \lambda) = 0$.

PROOF. Again let $f = f' + f''$ be the Krickeberg decomposition of f, and let t, a, b and every other symbol similarly stand for the same things they did in the proof of Theorem 1. f' and f'' are both uniformly integrable since f is, for if $f_\infty = \lim_n f_n$ then $f_n' = E(f_\infty^+ \mid \mathfrak{F}_n)$ and $f_n'' = E(f_\infty^- \mid \mathfrak{F}_n)$.

$$E(\sum_{i=1}^{\infty} |p_i|) \leqq E((45 f_t' + 45 f_t'') I_{\{t<\infty\}}) = 45 E(|f_\infty| \, I_{\{t<\infty\}}),$$

the inequality holding in the same way it did in the proof of Theorem 1 and the equality by an argument similar to the proof of Lemma 1. Thus

$$E(\sum_{i=1}^{\infty} |p_i|) \to 0 \quad \text{as} \quad \lambda \to \infty. \tag{3}$$

The proof of Theorem 2 can be completed with the substitution of (3) and Lemmas 1 and 2 in the appropriate places in the proof of Theorem 1. Using (3) we get $E(\sum_{i=1}^{\infty} |\beta_i|) \to 0$ as $\lambda \to \infty$ and $P(\tau < \infty) = o(1/\lambda)$ as $\lambda \to \infty$. Lemmas 1 and 2 give $P(t < \infty) = o(1/\lambda)$ as $\lambda \to \infty$, so $P(a^* > 0) = o(1/\lambda)$ as $\lambda \to \infty$. And (3), together with Lemma 2 and the fact that $\int_0^{17\lambda^2} o(1/x^{\frac{1}{2}})\, dx = o(\lambda)$ as $\lambda \to \infty$ give $\|h\|_2^2 = o(\lambda)$ as $\lambda \to \infty$. Theorem 2 follows from the decomposition $g = a + b + h$.

3. A remark on martingale convergence. The finiteness of $S(f)$ if f is an L^1 bounded martingale, due to Austin ([1]) and a consequence of (2) implies the convergence of L^1 bounded martingales. For suppose f is an L^1 bounded martingale and $\epsilon > 0$. Let $\tau_0 \equiv 1$, and if $i > 1$ let τ_i be the first time after τ_{i-1} for which $|f_n - f_{i-1}| \geqq \epsilon$. Suppose $P(\tau_i < \infty \text{ for all } i) > 0$. Then there exists integers $n_0 < n_i < \cdots$ such that $P(\tau_i < \infty, \tau_i > n_i) \leqq P(\tau_i < \infty \text{ for all } i)/2^{i+2}$. Define $v_i = \min(\tau_i, n_i)$. Then $(f_{v_i}, f_{v_2}, \cdots)$ is an L^1 bounded martingale (Doob [4], page 302) and $\sum_{i=1}^{k} (f_{v_i} - f_{v_{i-1}})^2 \geqq k\epsilon^2$ on $\{\tau_0 \leqq n_0, \cdots, \tau_k \leqq n_k\} \supset \{\text{all } \tau_i \leqq n_i\}$, a set with probability at least $\frac{1}{2}P(\tau_i < \infty \text{ for all } i)$. Thus $\sum_{i=1}^{\infty} (f_{v_i} - f_{v_{i-1}})^2 = \infty$ on this set, contradicting its assumed positive probability. Thus $P(\tau_i < \infty \text{ for all } i) = 0$. Since ϵ was arbitrary the convergence of f is verified.

The inequality (1) sharpens this by giving in a sense bounds for the rate of convergence of L^1 bounded martingales. If $\epsilon > 0$ we will say the sequence $(a_1, a_2, \cdots)$ makes n ϵ-excursions if we can choose integers $i_1 < i_2 < \cdots < i_{n+1}$ such that $|a_{i_k} - a_{i_{k-1}}| > \epsilon$, $k > 1$. If a sequence has a limit the number of ϵ-excursions it makes is finite, and this number provides a measure of how much the sequence jumps around before converging. In particular if there no ϵ-excursions $|a_i - \lim_{n\to\infty} a_n| \leqq \epsilon$ for all i.

THEOREM 3. *If f is a martingale and ϵ, r are positive numbers the probability that f makes at least r ϵ-excursions is no larger than* $44\, \|f\|_1/\epsilon r^{\frac{1}{2}}$.

PROOF. Let $\tau_0 = 1$, and let τ_i be the first time after τ_{i-1} that $|f_n - f_{i-1}| \geqq \epsilon/2$. Then $P(\tau_r < \infty)$ is at least as large as the probability that f makes r ϵ-excursions. Define $v_i = \min(\tau_i, n^i)$. Then $\sum_{i=1}^{r} (f_{v_i} - f_{v_{i-1}})^2 \geqq r\epsilon^2/4$ on $\{\tau_0 \leqq n^0, \cdots, \tau_r \leqq n^r\} = A_n$. Thus $P(A_n) \leqq 22 \sup_n E|f_{v_n}|/(r\epsilon^2/4)^{\frac{1}{2}} \leqq 22\, \|f\|_1/(r\epsilon^2/4)^{\frac{1}{2}}$. Since $\lim_{n\to\infty} P(A_n) = P(\tau_r < \infty)$, Theorem 3 is proved.

Different but essentially sharper bounds for the rate of convergence of L^1 bounded martingales are given in [5].

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