The Annals of Mathematical Statistics
1969, Vol. 40, No. 2, 480–491

ON SOME RESULTS OF N. V. SMIRNOV CONCERNING LIMIT DISTRIBUTIONS FOR VARIATIONAL SERIES

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1. Introduction. Let $\xi_1, \cdots, \xi_n$ be n mutually independent and identically distributed random variables (rv). For every k $(1 \leqq k \leqq n)$ denote by ξ_{kn} the rv that assumes the kth value in descending order of size among the values assumed by $\xi_1, \cdots, \xi_n$. So, e.g., we have

$$\xi_{1n} = \max(\xi_1, \cdots, \xi_n).$$

Many authors (cf. [1], [2], [11]) have investigated the asymptotic behavior of the distribution function (df) of the maximal term ξ_{1n} as $n \to \infty$. The most complete results, which may be said to summarize in a sense this series of investigations, were obtained by B. V. Gnedenko [3]. In particular, he determined the class of all df's which can be a limit of the df of the normalized maximal term $(\xi_{1n} - b_n)/a_n$ as $n \to \infty$, where $a_n > 0$ and b_n are suitably chosen real numbers.

Gnedenko's results were generalized by N. V. Smirnov [12]. He showed that the class of all proper limit distribution laws for the normalized rv ξ_{kn} consists of the following:

$$\begin{aligned} \Phi_\alpha(x; k) &= 0 && \text{if } x < 0,\\ &= \exp(-x^{-\alpha}) \textstyle\sum_{s=0}^{k-1} x^{-s\alpha}/s! && \text{if } x \geqq 0;\\ \Psi_\alpha(x; k) &= \exp(-|x|^{\alpha}) \textstyle\sum_{s=0}^{k-1} |x|^{s\alpha}/s! && \text{if } x < 0,\\ &= 1 && \text{if } x \geqq 0; \end{aligned} \tag{1.1}$$

where $\alpha > 0$, and

$$\Lambda(x; k) = \exp(-e^{-x}) \textstyle\sum_{s=0}^{k-1} e^{-sx}/s!$$

The limit distributions for the maximal term are obtained by putting $k = 1$.

The variable ξ_{kn} is a well-defined function of the rv's $\xi_1, \cdots, \xi_n$ and the index k $(1 \leqq k \leqq n)$

$$\xi_{kn} = f(\xi_1, \cdots, \xi_n; k) \tag{1.2}$$

which satisfies the identity

$$f(\xi_1, \cdots, \xi_n; k) = -f(-\xi_1, \cdots, -\xi_n; n - k + 1).$$

This relation permits us to carry over results found for the df of ξ_{kn} to the df of $\xi_{n-k+1,n}$ and conversely.

Received 24 May 1968.

Gnedenko's problem was generalized in another direction in [4] and [7]–[10], where the limit distributions of the maximal term ξ_{1n} were considered in the case where the initial rv's $\xi_1, \cdots, \xi_n$ are not necessarily identically distributed. It is the purpose of the present paper to generalize Smirnov's result in this direction.

2. Statement of the problem. Let $\xi_1, \cdots, \xi_n$ be mutually independent rv's and let

$$F_i(x) = P(\xi_i \leqq x), \qquad i = 1, \cdots, n.$$

It is easy to see that for every k $(1 \leqq k \leqq n)$

$$(2.1) \quad \Phi_{kn}(x) = P(\xi_{kn} \leqq x) = \prod_{i=1}^{n} F_i(x) + \sum_{s=1}^{k-1} \sum_{(n*s)} [\prod_{j\varepsilon(n*s)^c} F_j(x) \prod_{i\varepsilon(n*s)} (1 - F_i(x))]$$

where $(n * s)$ is any subset of the set $(1, \cdots, n)$, consisting of s indices, $(n * s)^c$—its complement and the inner summation is over all such subsets. Since

$$(2.2) \qquad \Phi_{1n}(x) = \prod_{i=1}^{n} F_i(x)$$

then, whenever x is such that $F_i(x) > 0$ $(i = 1, \cdots, n)$, then the df $\Phi_{kn}(x)$ may be written also in the form

$$(2.3) \qquad \Phi_{kn}(x) = \Phi_{1n}(x) \sum_{s=0}^{k-1} \sum \prod_{(n*s)} (1/F_i(x) - 1),$$

where for the sake of brevity we write $\sum \prod_{(n*s)}$ instead of $\sum_{(n*s)} \prod_{i\varepsilon(n*s)}$ and put $\sum \prod_{(n*0)} \equiv 1$.

THEOREM 2.1. *For every* df $\Phi(x)$ *and every* k *there exists a sequence of* df's $F_i(x)$ *such that*

$$(2.4) \qquad \lim_{n\to\infty} \Phi_{kn}(x) = \Phi(x).$$

PROOF. Consider an auxiliary sequence of mutually independent rv's η_i with df's

$$(2.5) \qquad G_i(x) = P(\eta_i \leqq x) = 0, \qquad \text{if } x < 0,$$
$$= 1 - 2/(1 + e^{ix}), \qquad \text{if } x \geqq 0.$$

For arbitrary k and n $(1 \leqq k \leqq n)$ let us denote

$$\eta_{kn} = f(\eta_1, \cdots, \eta_n; k),$$

where the function f is the same as in (1.2), and let

$$(2.6) \qquad \Gamma_{kn}(x) = P(\eta_{kn} \leqq x).$$

It is clear that $\Gamma_{kn}(x) = 0$ if $x \leqq 0$, and for $x > 0$, in view of (2.3), the df $\Gamma_{kn}(x)$ may be represented in the form

$$(2.7) \qquad \Gamma_{kn}(x) = \sum_{s=0}^{k-1} T_{sn}(x),$$

where

$$T_{sn}(x) = 2^s \prod_{i=1}^{n} G_i(x) \sum \prod_{(n*s)} (e^{jx} - 1)^{-1}. \tag{2.8}$$

For fixed s and $a > 0$ consider the expression

$$\Delta_{mn}(x) = |T_{s,n+m}(x) - T_{sn}(x)|,$$

where $x \geqq a$. We have

$$\Delta_{mn}(x) < 2^s[\prod_{i=n+1}^{n+m} (1 + 2/(e^{ix} - 1)) - 1] \sum \prod_{(n*s)} (e^{jx} - 1)^{-1}$$
$$+ \sum \prod_{((n+m)*s)} (e^{ix} - 1)^{-1} - \sum \prod_{(n*s)} (e^{ix} - 1)^{-1}.$$

It is easy to see that

$$\prod_{i=n+1}^{n+m} (1 + 2/(e^{ix} - 1))$$
$$< \prod_{i=n+1}^{\infty} (1 + 2/(e^{ia} - 1)) < \exp\ (2 \sum_{i=n+1}^{\infty} (e^{ia} - 1)^{-1})$$
$$< \exp\ (4a^{-2} \sum_{i=n+1}^{\infty} i^{-2}),$$

$$\sum \prod_{(n*s)} (e^{ix} - 1)^{-1} \leqq (\sum_{i=1}^{\infty} (e^{ia} - 1)^{-1})^s \leqq (2a^{-2} \sum_{i=1}^{\infty} i^{-2})^s,$$

and for $s \geqq 1$ we have

$$\sum \prod_{((n+m)*s)} (e^{ix} - 1)^{-1} - \sum \prod_{(n*s)} (e^{ix} - 1)^{-1}$$
$$< \sum_{t=1}^{s} [(\sum_{i=1}^{n} (e^{ix} - 1)^{-1})^{s-t} (\sum_{j=n+1}^{n+m} (e^{jx} - 1)^{-1})^t$$
$$< s(2/a^2)^s (\sum_{i=1}^{\infty} i^{-2})^s (\sum_{j=n+1}^{\infty} j^{-2}).$$

Thus we obtain

$$\Delta_{mn}(x) < C(a, s; n) = o(1), \qquad (n \to \infty)$$

where $x \geqq a$ and m are arbitrary. This estimate proves the existence of the limit

$$\lim_{n\to\infty} \Gamma_{kn}(x) = \Gamma(x) \tag{2.9}$$

and the uniform convergence of the sequence $\Gamma_{kn}(x)$ in any interval of the form $[a, \infty)$, where $a > 0$. It follows in particular that the limit function $\Gamma(x)$ is a distribution, i.e. $\Gamma(+\infty) = 1$.

We will now show that

$$\Gamma(0+) = 0. \tag{2.10}$$

Because of (2.5) and (2.8), we have for any positive x

$$T_{sn}(x) < (2^s \sum \prod_{(n*s)} e^{-ix}) / \prod_{j=1}^{n} (1 + e^{-jx}),$$

and since

$$\sum \prod_{(n*s)} e^{ix} < (\sum_{i=1}^{n} e^{-ix})^s < 1/(e^x - 1)^s,$$

we also get

$$T_{sn}(x) < 2^s(e^x - 1)^{-s} / \prod_{i=1}^{n} (1 + e^{-ix}).$$

Therefore, if $0 < x < \ln 2$, then by (2.7) we get

$$\Gamma(x) < k2^k(e^x - 1)^{-k}/\prod_{i=1}^{\infty}(1 + e^{-ix}).$$

It is easy to verify that $1 + \alpha > e^{\alpha/2}$ if $0 < \alpha < 1$. Hence if $x > 0$, we have

$$\prod_{i=1}^{\infty}(1 + e^{-ix}) > \exp(2^{-1}\sum_{i=1}^{\infty} e^{-ix}) = \exp(2(e^x - 1))^{-1}.$$

Thus, for positive values of x close enough to 0 the inequality

$$\Gamma(x) < k2^k(e^x - 1)^{-k}/\exp(2(e^x - 1))^{-1}$$

holds—which proves (2.10).

The df's $G_i(x)$ defined by (2.5), as well as the $\Gamma_{kn}(x)$, are everywhere continuous. Therefore, by virtue of (2.10), and the uniform convergence in $[a, \infty)$, the limit $\Gamma(x)$ is also continuous (which shows, incidentally, that the convergence of $\Gamma_{kn}(x)$ is uniform in $-\infty < x < \infty$).

The function $\Gamma(x)$ is strictly increasing on $[0, \infty)$ so that the inverse function

$$g(x) = \Gamma^{-1}(x) \tag{2.11}$$

exists and is also increasing and continuous on $[0, 1)$ and satisfies the conditions

$$g(0) = 0, \qquad g(1) = +\infty. \tag{2.12}$$

Now let $\Phi(x)$ be an arbitrary df and consider a sequence of functions $F_i(x)$ defined as follows:

$$F_i(x) = G_i(g(\Phi(x))) \quad \text{if} \quad \Phi(x) < 1, \qquad = 1 \quad \text{if} \quad \Phi(x) = 1; \tag{2.13}$$

where the $G_i(x)$ are as in (2.5). Due to (2.12), the $F_i(x)$ are df's.

Let ξ_i be a sequence of mutually independent rv's such that $P(\xi_i \leqq x) = F_i(x)$. It follows from (2.5) and (2.13) that $P(\xi_i \leqq x) = P(\eta_i \leqq g(\Phi(x)))$, therefore we get for any k and n $(1 \leqq k \leqq n)$ also $P(\xi_{kn} \leqq x) = P(\eta_{kn} \leqq g(\Phi(x)))$ which by (2.6) can be written in the form

$$\Phi_{kn}(x) = \Gamma_{kn}(g(\Phi(x))).$$

Hence by (2.9) and (2.11) we conclude that (2.4) holds. This proves our theorem

Let us remark that the theorem becomes trivial in the case $k = 1$: for a given df $\Phi(x)$ we can take $F_i(x) = \Phi^{2^{-i}}(x)$. However, in the general case we have not been able to invent such a simple construction for the $F_i(x)$.

The theorem just proved shows that, if apart from the mutual independence no other restriction is imposed on the rv's ξ_i, then any df $\Phi(x)$ may be considered as a limit law for the kth term of some variational series. However, it is natural to require that the initial suitably normalized rv's ξ_i should—in some sense—be individually negligible in the limit, so that the role of a single component participating in the formation of the variable ξ_{kn} becomes vanishingly small as $n \to \infty$. Keeping the above notations, let us introduce the following definition:

We will say that the df $\Phi(x)$ belongs to the class G_k if there exist a sequence of df's $F_i(x)$ and real constants $a_n > 0$ and b_n such that

$$\lim_{n\to\infty} \Phi_{kn}(a_n x + b_n) = \Phi(x) \tag{2.14}$$

at each point of continuity of the function $\Phi(x)$, and such that for every x, for which $\Phi(x) > 0$,

$$\lim_{n\to\infty} F_i(a_n x + b_n) = 1 \tag{2.15}$$

uniformly in i $(1 \leqq i \leqq n)$.

(Let us remark that in the case when the rv's ξ_i are identically distributed then (2.15) is contained in (2.14).)

Our aim is to give an exact description of the class G_k.

For sake of brevity we will use the following notations

$$_*\Phi = \inf \{x:\Phi(x) > 0\}, \qquad \Phi_* = \sup \{x:\Phi(x) < 1\}.$$

If $\Phi(x)$ is a df, then $_*\Phi(\Phi_*)$ will be called its *left* (*right*) *end.*

Since each improper df trivially belongs to G_k, the limit distributions $\Phi(x)$ are assumed to be proper, i.e. $_*\Phi < \Phi_*$.

Finally, let us note that as a consequence of the weak convergence required in (2.14), every non-decreasing function $\Phi(x)$ which satisfies the conditions $\Phi(-\infty) = 0$, $\Phi(+\infty) = 1$ is a df, and equality of two df's means equality at their points of continuity.

3. The class G_1. Let P be the class of all df's $\Phi(x)$ that have the following property: for every $\beta > 0$ there exists a non-decreasing function $\phi_\beta(x)$, such that for all x

$$\Phi(x) = \Phi(x + \beta)\phi_\beta(x). \tag{3.1}$$

Let Q be the class of all df's $\Phi(x)$ that have the following properties:

$$\Phi_* < \infty \tag{3.2}$$

and for every α $(0 < \alpha < 1)$ there exists a non-decreasing function $\phi_\alpha(x)$, such that for all x

$$\Phi(x + \Phi_*) = \Phi(\alpha x + \Phi_*)\phi_\alpha(x) \tag{3.3}$$

(observe that $P \cap Q \neq \varnothing$).

Let R be the set of all $\Phi(x) \in Q$, which are continuous at the point $x = \Phi_*$.

The case $k = 1$ was studied in [7]–[10] under somewhat more restrictive requirements: it was assumed that a df is—by definition—continuous from the left and that the convergence in (2.4) holds at every point. With these assumptions, the class of the limit distributions was called *class G* and it was proved [10] that $G = P \cup R$. In the present—more general—situation, we get the following.

THEOREM 3.1.

$$G_1 = P \cup Q.$$

PROOF. By [7] it is sufficient to prove that $Q \subset G_1$. So let us suppose that the df $\Phi(x)$ belongs to Q. Without loss of generality we can assume, because of (3.2), that $\Phi_* = 0$. Let us first assume that ${}_*\Phi > -\infty$. Then, by (3.3) the function

$$H(x; \alpha) = 0, \quad \text{if } x < {}_*\Phi, \qquad = \Phi(x)/\Phi(\alpha x), \quad \text{if } x > {}_*\Phi, \tag{3.4}$$

is a df for every fixed α $(0 < \alpha < 1)$. Define

$$\begin{aligned} D(x) &= 0 && \text{if} \quad x < -1/e, \\ &= 1 + 1/\ln |x| && \text{if} \quad -1/e < x < 0, \\ &= 1 && \text{if} \quad x > 0, \end{aligned}$$

and

$$\Phi(0-) = a \qquad (0 < a \leqq 1).$$

We define the desired sequence $F_i(x)$ by

$$F_i(x) = H((i+1)x; \alpha_i)D^{t_i}(x), \qquad i = 1, 2, \cdots, \tag{3.5}$$

where $\alpha_i = i/(i+1)$, $t_i = (\ln \alpha_i) \ln a$. Taking $a_n = 1/(n+1)$, $b_n = 0$, we verify that for $x > {}_*\Phi$

$$\textstyle\prod_{i=1}^n D^{t_i}(a_n x) \to a \qquad (n \to \infty)$$

and

$$\textstyle\prod_{i=1}^n H((i+1)a_n x; \alpha_i) \to \Phi(x)/a \qquad (n \to \infty),$$

since

$$\textstyle\prod_{i=1}^n H((i+1)a_n x; \alpha_i) = \Phi(x)/\Phi((n+1)x).$$

Hence, it follows in virtue of (2.2) and (3.5) that (2.14) holds for all $x > {}_*\Phi$. On the other hand, by (3.4) and (3.5) it is clear that (2.14) holds also for $x < {}_*\Phi$.

It is easy to check that if $x > {}_*\Phi$, then (2.15) is fulfilled by any fixed i and by $i = n$. Hence, since the sequence a_n is monotone, we conclude that (2.15) is fulfilled uniformly in i $(1 \leqq i \leqq n)$. Thus the df $\Phi(x)$ belongs to G_1.

The case ${}_*\Phi = -\infty$ can be treated in the same way by defining the function $H(x; \alpha)$ by

$$\begin{aligned} H(x; \alpha) &= 0 && \text{if} \quad x < \alpha/(\alpha - 1), \\ &= \Phi(x)/\Phi(\alpha x) && \text{if} \quad x > \alpha/(\alpha - 1). \end{aligned}$$

An example of a df which belongs to G_1 but is discontinuous at its left end (and, therefore, does not belong to G) is given by

$$\begin{aligned} \Phi(x) &= \exp(x - 1) && \text{if} \quad x < 0, \\ &= 1 && \text{if} \quad x > 0. \end{aligned}$$

A more transparent characterization of the class G_1 is contained in

THEOREM 3.2. [9] *The* df $\Phi(x)$ *belongs to* G_1 *if and only if it is logarithmically convex* $(\Phi(x) \varepsilon P)$ *or the function* $\Phi({}_*\Phi - e^{-x})$ *is logarithmically convex* $(\Phi(x) \varepsilon Q)$.

It follows from this that the only possible points of discontinuity of a df belonging to G_1 are its ends. Moreover, the right end can be a point of discontinuity only if the df belongs to Q.

In the sequel we shall need the following;

LEMMA 3.1. *Let* $\Phi(x) \varepsilon G_1$ *and assume* ${}_*\Phi > -\infty$. *Then the sequences* $F_i(x)$, a_n *and* b_n *that appear in* (2.14) *and* (2.15), *can be chosen so, that for every* $x < {}_*\Phi$ *and* k *we will have*

$$F_{n-s}(a_n x + b_n) = 0, \qquad s = 0, 1, \cdots, k, \tag{3.6}$$

for all sufficiently large n.

PROOF. Let $\Phi(x) \varepsilon P$. Then by (3.1) the function

$$\begin{aligned} H(x;\beta) &= 0 && \text{if} \quad x < {}_*\Phi, \\ &= \Phi(x)/\Phi(x+\beta) && \text{if} \quad x > {}_*\Phi, \end{aligned}$$

is a df for every $\beta > 0$. Taking

$$F_i(x) = H(x - \textstyle\sum_{j=1}^{i} (1/j); 1/i), \qquad i = 1, 2, \cdots,$$

$$a_n = 1, \qquad b_n = \textstyle\sum_{j=1}^{n} (1/j),$$

we verify the validity of (2.14) and (2.15). On the other hand, for every s $(0 \leqq s \leqq n)$ we have

$$F_{n-s}(a_n x + b_n) = 0 \qquad \text{if} \quad x < {}_*\Phi - \textstyle\sum_{j=n-s+1}^{n} (1/j).$$

Thus (3.6) holds for $x < {}_*\Phi$ and $n > k/({}_*\Phi - x) + k - 1$. The case $\Phi(x) \varepsilon Q$ may be handled in the same way, by using the sequences that were constructed in course of the proof of Theorem 3.1.

4. The class $G_k (k \geqq 1)$**.** The characterization of the class G_k is given by

THEOREM 4.1. *The* df $\Phi(x)$ *belongs to* G_k *if and only if it can be represented in the form*

$$\Phi(x) = \phi(x) \textstyle\sum_{s=0}^{k-1} [(-\ln \phi(x))^s/s!], \tag{4.1}$$

where $\phi(x)$ *is a df of* G_1 *and* ${}_*\Phi = {}_*\phi$.

Let δ_{in} and λ_{in} $(1 \leqq i \leqq n; n = 1, 2, \cdots)$ be numerical sequences. We shall need the following two lemmas.

LEMMA 4.1. *Let*

$$\delta_{in} \geqq 0 \quad \textit{and} \quad \max_{1 \leqq i \leqq n} \delta_{in} \to 0 \qquad (n \to \infty). \tag{4.2}$$

If $\sum_{i=1}^{n} \delta_{in} \to \delta$ $(n \to \infty)$, *then, keeping the notation of* Section 2, *we have for every* s

$$\textstyle\sum \prod_{(n*s)} \delta_{in} \to \delta^s/s! \qquad (n \to \infty).$$

For a proof see [5], Section 15.

LEMMA 4.2. *Let*

$$0 < \lambda_{in} \leqq 1 \quad \textit{and} \quad \min_{1 \leqq i \leqq n} \lambda_{in} \to 1 \qquad (n \to \infty). \tag{4.3}$$

Denote

$$\pi_n = \prod_{i=1}^n \lambda_{in}, \qquad \sigma_n = \sum_{i=1}^n (1 - \lambda_{in}), \qquad \bar{\sigma}_n = \sum_{i=1}^n (1/\lambda_{in} - 1).$$

(a) *If one of the sequences* π_n, σ_n *or* $\bar{\sigma}_n$ *converges* (*to a finite or infinite limit*) then *the other two also converge and*

$$\lim_{n\to\infty} \pi_n = \lim_{n\to\infty} \exp(-\sigma_n) = \lim_{n\to\infty} (-\bar{\sigma}_n). \tag{4.4}$$

(b) *Let s be an arbitrary non-negative integer. If*

$$\pi_n \to \pi > 0 \qquad (n \to \infty), \tag{4.5}$$

then

$$\sum \prod_{(n*s)} (1/\lambda_{in} - 1) \to (-\ln \pi)^s/s! \qquad (n \to \infty), \tag{4.6}$$

while if

$$\pi_n \to 0 \qquad (n \to \infty), \tag{4.7}$$

then also

$$\pi_n \sum \prod_{(n*s)} (1/\lambda_{in} - 1) \to 0 \qquad (n \to \infty). \tag{4.8}$$

PROOF. (a) It is well known that for any $0 < \alpha \leqq 1$

$$\exp(1 - 1/\alpha) \leqq \alpha \leqq \exp(\alpha - 1).$$

Therefore, for every n

$$\exp(-\bar{\sigma}_n) \leqq \pi_n \leqq \exp(-\sigma_n). \tag{4.9}$$

On the other hand, for arbitrary $\epsilon (0 < \epsilon < 1)$ we will have, by (4.3), for sufficiently large n

$$\bar{\sigma}_n \leqq \sigma_n/(1 - \epsilon), \tag{4.10}$$

which, together with (4.9), proves this part of the lemma.

(b) Let us put $\delta_{in} = 1/\lambda_{in} - 1$, then by (4.3) the conditions (4.2) hold and $\sum_{i=1}^n \delta_{in} = \bar{\sigma}_n$. Since by (4.4) and (4.5) $\lim_{n\to\infty} \bar{\sigma}_n = -\ln \pi$, then according to Lemma 4.1 we get (4.6). Obviously

$$\sum \prod_{(n*s)} (1/\lambda_{in} - 1) \leqq \bar{\sigma}_n^s.$$

Hence, putting $\epsilon = \frac{1}{2}$ in (4.10) we have for sufficiently large n

$$\sum \prod_{(n*s)} (1/\lambda_{in} - 1) \leqq (2\sigma_n)^s.$$

Finally, using (4.9), we see that for large n

$$\pi_n \sum \prod_{(n*s)} (1/\lambda_{in} - 1) \leqq (2\sigma_n)^s \exp(-\sigma_n). \tag{4.11}$$

Suppose now that (4.7) holds. Then in virtue of (4.4) $\sigma_n \to \infty$ $(n \to \infty)$. Since for any positive s we have

$$x^s \exp(-x) \to 0 \qquad (x \to \infty),$$

then (4.8) follows immediately from (4.11).

Notice that both lemmas obviously remain valid if the sequence of natural indices n is replaced by any subsequence n'.

PROOF OF THEOREM 4.1. *Necessity*. Let $\Phi(x) \varepsilon G_k$. Consequently there exist $F_i(x)$, a_n and b_n such that (2.14) and (2.15) hold. Consider the sequence of the df's of the corresponding maximal terms and let $\phi(x)$ be any of its partial limits, i.e. $\Phi_{1n'}(a_{n'}x + b_{n'}) \to \phi(x) \quad (n' \to \infty)$.

We first prove

$$(4.12) \qquad {}_*\phi = {}_*\Phi.$$

Since for every $k \geqq 1$ we have $\Phi_{1n}(x) \leqq \Phi_{kn}(x)$, then clearly

$$(4.13) \qquad {}_*\phi \geqq {}_*\Phi.$$

Let $x > {}_*\Phi$ be an arbitrary fixed number. For given i and n $(1 \leqq i \leqq n)$ denote

$$\lambda_{in} = F_i(a_n x + b_n),$$

then by (2.15) our sequence λ_{in} satisfies all of the hypotheses of Lemma 4.2 for sufficiently large n. On the other hand, from some n on we can use the expression (2.3). Therefore, introducing the notations of Lemma 4.2, we have by (2.2)

$$(4.14) \qquad \Phi_{kn}(a_n x + b_n) = \pi_n \textstyle\sum_{s=0}^{k-1} \sum \prod_{(n*s)} (1/\lambda_{in} - 1).$$

Hence, should we assume

$$\pi_{n'} \to \phi(x) = 0 \qquad (n' \to \infty)$$

we would get according to Lemma 4.2 that $\Phi(x) = 0$ too, which is impossible, since $x > {}_*\Phi$. Thus ${}_*\phi \leqq {}_*\Phi$ and by (4.13) equality (4.12) is proved.

Now let x $(x > {}_*\Phi)$ be a point of continuity of the function $\Phi(x)$. By (4.12)

$$\phi(x) = \lim_{n' \to \infty} \pi_{n'} > 0.$$

Therefore, according to Lemma 4.2, and (4.14), the df $\Phi(x)$ has the form (4.1).

It is easy to verify that for any $k \geqq 1$ the function

$$(4.15) \qquad \psi(x) = x \textstyle\sum_{s=0}^{k-1} [(-\ln x)^s/s!], \qquad \psi(0) = 0,$$

is strictly increasing in [0, 1]. Hence the representation of a df $\Phi(x)$ by means of a non-decreasing function $\phi(x)$ in the form (4.1) is unique. Thus we conclude that $\phi(x)$ is a df and

$$(4.16) \qquad \phi(x) = \lim_{n \to \infty} \Phi_{1n}(a_n x + b_n).$$

Since (2.15) holds for all $x > {}_*\phi$, then $\phi(x) \varepsilon G_1$.

Sufficiency. Let the df $\Phi(x)$ have the form (4.1). Then there exist $F_i(x)$, a_n

and b_n such that (4.16) holds and so does (2.15) for $x > {}_*\phi$. Consider the sequence $\Phi_{kn}(a_n x + b_n)$. It clearly follows from the arguments used in the first part of the proof that this sequence converges to $\Phi(x)$ if $x > {}_*\phi$. Therefore, the proof will be complete if we show that

$$(4.17) \qquad \Phi_{kn}(a_n x + b_n) \to 0 \qquad (n \to \infty)$$

for $x < {}_*\phi$. For these values of x we shall make use of the expression (2.1).

It is clear that if $1 \leqq s \leqq k - 1$, then in each collection of indices of the form $(n * s)^c$ which consists of $n - s$ different indices, there exists at least one index i' such that $n - k \leqq i' \leqq n$. On the other hand, according to Lemma 3.1, there is no loss of generality if we assume that for $x < {}_*\phi$ and sufficiently large n equalities (3.6) hold. Therefore, if $x < {}_*\phi$ then from some n on all the terms of the sum in (2.1) vanish, i.e. we get (4.17) and the theorem was proved.

In the course of the proof we saw that if for some $k > 1$ and some $F_i(x)$, a_n and b_n (2.14) and (2.15) hold, then—for the same $F_i(x)$, a_n and b_n—the sequence $\Phi_{1n}(a_n x + b_n)$ converges too, i.e. (4.16), the left ends of both limits $\Phi(x)$ and $\phi(x)$ coincide and the equality (4.1) holds. However, if (4.16) holds and (2.15) is fulfilled for all $x > {}_*\phi$, then for $k > 1$ the convergence of the sequence $\Phi_{kn}(a_n x + b_n)$ and the equality (4.1) are guaranteed only in the interval $({}_*\phi, \infty)$, and we assert nothing concerning the behavior of the sequence in $(-\infty, {}_*\phi)$. (It is for this reason that we needed Lemma 3.1.) However, the following proposition can be easily established.

THEOREM 4.2. *If* (2.14) *and* (2.15) *are satisfied for some k and some $F_i(x)$, a_n and b_n and the left end of the limiting function is a point of continuity* (*or it is* $-\infty$), *then* (2.14) *and* (2.15) *are satisfied for each k, the left end of the limiting law is independent of k and* (4.1) *holds.*

REMARK 4.1. Now let the mutually independent rv's ξ_i have the same df $F(x)$, *then*

$$(4.18) \qquad \Phi_{kn}(x) = \sum_{s=0}^{k-1} \binom{n}{s} F^s(x)(1 - F(x))^{n-s}.$$

From Gnedenko's result concerning the class of the limit distributions for the maximal term ξ_{1n} we obtain immediately Smirnov's laws (1.1), by using Theorem 4.2, since $\Phi_\alpha(x; 1)$, $\Psi_\alpha(x; 1)$ and $\Lambda(x; 1)$ are everywhere continuous.

The class of df's $F(x)$ for which constants a_n and b_n may be found, such that (2.14) holds and $\Phi_{kn}(x)$ is given by (4.18), is called the domain of attraction of the law $\Phi(x)$.

The domains of attraction of the df's $\Phi_\alpha(x; 1)$, $\Psi_\alpha(x; 1)$ and $\Lambda(x; 1)$ were first studied by R. de Mises [11]. A complete solution of this problem was given by Gnedenko [3]. Another characterization of the domain of attraction of the law $\Lambda(x; 1)$ was given in [6].

Smirnov showed [12] that the domain of attraction of any df from (1.1) does not depend on k, i.e. it coincides with the domain of attraction of the corresponding df that is obtained by putting $k = 1$. It is easy to see that also this result of Smirnov is an immediate corollary of our Theorem 4.2.

REMARK 4.2. The expression (4.1) admits also of another interpretation. Consider normalized rv's

$$\xi_{in}' = (\xi_i - b_n)/a_n, \qquad i = 1, \cdots, n,$$

where ξ_i are mutually independent and $a_n > 0$, b_n are real numbers. For given x and n let us denote by $\eta_n(x)$ the number of ξ_{in}' whose values occur in the interval (x, ∞), and let

$$P_{kn}(x) = P(\eta_n(x) = k).$$

Then

$$\begin{aligned} P_{kn}(x) &= \Phi_{1n}(x) && \text{if } k = 0, \\ &= \Phi_{k+1,n}(x) - \Phi_{kn}(x) && \text{if } 1 \leqq k \leqq n-1, \\ &= 1 - \Phi_{nn}(x) && \text{if } k = n. \end{aligned} \tag{4.19}$$

Now let ξ_i, a_n and b_n be such that the limit (4.16) exists and (2.15) hold for $x > {}_*\phi$. Then it follows from (4.1) and (4.19) that for constant $k \geqq 0$ and $x > {}_*\phi$ we have

$$P_{kn}(x) \to \phi(x)(-\ln \phi(x))^k/k! \qquad (n \to \infty).$$

Thus the distribution of the rv $\eta_n(x)$ converges to the Poisson distribution whose parameter is equal to $-\ln \phi(x)$. In particular, if $\phi(x) = e^x$ $(x < 0)$ and $x_1 < x_2 < 0$, then the expected number of ξ_{in}' which occur in the interval (x_1, x_2) asymptotically equals the length of the interval.

We conclude with

THEOREM 4.3. *For every k we have $G_k \subset G_1$. In particular, if $\phi(x) \,\varepsilon\, P(Q)$ then also $\Phi(x) \,\varepsilon\, P(Q)$, where $\Phi(x)$ is defined by* (4.1).

PROOF. Let $\psi(x)$ be a non-decreasing function in [0, 1], $0 \leqq \psi(x) \leqq 1$. If $\psi(e^x)$ and $\phi(x)$ $(0 \leqq \phi(x) \leqq 1)$ are logarithmically convex, then so is the function $\psi(\phi(x))$.

Indeed, for any non-positive x and y we have

$$\psi(e^x)\psi(e^y) \leqq \psi^2(\exp((x + y)/2)),$$

hence

$$\psi(\phi(x))\psi(\phi(y)) \leqq \psi^2((\phi(x)\phi(y))^{\frac{1}{2}}).$$

But since

$$\phi(x)\phi(y) \leqq \phi^2((x + y)/2)$$

and $\psi(x)$ is non-decreasing then

$$\psi(\phi(x))\psi(\phi(y)) \leqq \psi^2(\phi((x + y)/2)).$$

By straightforward differentiation we verify that the function $\psi(x)$ given by (4.15) possesses all the properties formulated above. Thus, by what has just been proved, the theorem follows from Theorems 3.2 and 4.1.

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