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ON THE EXPECTED VALUE OF A STOPPED STOCHASTIC SEQUENCE[1]

BY WILLIAM F. STOUT AND Y. S. CHOW[2]

University of Illinois and Purdue University

1. Introduction. Let $(\Omega, \mathfrak{F}, P)$ be a probability space with an integrable stochastic sequence $(X_n, \mathfrak{F}_n, n \geqq 1)$ defined on it. By a stochastic sequence is meant that the $\mathfrak{F}_n$'s form an increasing sequence of σ-fields in $\mathfrak{F}$ and that each random variable X_n is $\mathfrak{F}_n$ measurable. A random variable t is called a stopping time if it is positive integer (possibly $+\infty$) valued and if the event $[t = n] \varepsilon \mathfrak{F}_n$ for each $n \geqq 1$. If $P[t < \infty] = 1$, then t is called a stopping rule. For any sequence of random variables $(Z_n, n \geqq 1)$ and a stopping time t, we define the expected value of the stopped sequence by $EZ_t = \int_{t<\infty} Z_t$ provided the integral exists (we permit $EZ_t = \infty$ or $EZ_t = -\infty$). We let Z^+ and Z^- denote respectively the positive and negative parts of a random variable Z, and $\mathfrak{B}(Z)$ denote the σ-field generated by a random variable Z (possibly vector valued). Given a collection of sets $\mathcal{G}$, a set A in $\mathcal{G}$ is said to be an atom of $\mathcal{G}$ if $B \varepsilon \mathcal{G}$ and $B \subset A$ implies that $P[B] = 0$ or $P[B] = P[A]$. $\mathcal{G}$ is said to be non-atomic if it contains no atoms.

Recently, Dubins and Freedman [4] established that

$$(1) \quad (X_n, \mathfrak{F}_n, n \geqq 1) \text{ a martingale with } \sup E|X_n| = \infty \text{ implies that there exists a stopping time } t \text{ such that } E|X_t| = \infty.$$

In [2], this result is extended to the submartingale case. One might suspect that (1) would hold for some stopping rule or that the hypotheses of (1) would imply the existence of a stopping time t such that $EX_t^+ = \infty$. However simple examples exist in both cases ([4], p. 1505 and [1], p. 270 respectively) showing that such is not the case. Here we show that results in both of these directions are possible by certain modifications of the hypotheses of (1). The techniques developed in [2] and [4] were found to be useful here also. The natural setting for the results stated below is that of the general stochastic sequence as opposed to martingales in [4] and submartingales in [2]. As a corollary to the stated results for general stochastic sequences, it is shown in Corollary 2 that (1) and the corresponding result in [2] can be improved in the case where the X_n's are partial sums of independent random variables.

2. Results. Obviously, a necessary condition for the existence of a stopping rule t such that $E|X_t| = \infty$ is the existence of an unbounded stopping rule. The following lemma gives sufficient conditions for the existence of an unbounded stopping rule.

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[2] Now at Columbia University.

LEMMA 1. *Assume either*

(2) $$\bigcup_{k=1}^{\infty}\mathfrak{F}_k \text{ non-atomic},$$

(3) *for each* $A \varepsilon \bigcup_{k=1}^{\infty}\mathfrak{F}_k$ *such that* $P[A] > 0$ *there exists infinitely many* $n \geqq 1$ *such that both* $\int_A X_n^+ > 0$ *and* $\int_A X_n^- > 0$,

or

(4) $P\{[E[X_n^+ \mid \mathfrak{F}_{m_k}] > 0] \cap [E[X_n^- \mid \mathfrak{F}_{m_k}] > 0]$ *infinitely often* $(n \geqq 1)\}$ $= 1$ *for some sequence* $(m_k, k \geqq 1)$ *of distinct positive integers.*

Then there exists an unbounded stopping rule s.

PROOF. We show (4) implies (3) implies (2) implies the existence of s.

(4) implies (3): Fix $A \varepsilon \bigcup_{k=1}^{\infty}\mathfrak{F}_k$ such that $P[A] > 0$. (4) implies by the Borel Cantelli lemma that

$$\sum_{n=1}^{\infty} P\{A \cap [E[X_n^+ \mid \mathfrak{F}_{m_k}] > 0] \cap [E[X_n^- \mid \mathfrak{F}_{m_k}] > 0]\} = \infty$$

implying that

$$P\{A \cap [E[X_n^+ \mid \mathfrak{F}_{m_k}] > 0] \cap [E[X_n^- \mid \mathfrak{F}_{m_k}] > 0]\} > 0 \text{ infinitely often } (n \geqq 1).$$

But for m_k sufficiently large $\int_A X_n^+ = \int_A E[X_n^+ \mid \mathfrak{F}_{m_k}]$ and

$$\int_A X_n^- = \int_A E[X_n^- \mid \mathfrak{F}_{m_k}]$$

for all $n \geqq m_k$ implying (3).

(3) implies (2): Assume $\bigcup_{k=1}^{\infty}\mathfrak{F}_k$ not non-atomic. Then there exists an atom B of $\bigcup_{k=1}^{\infty}\mathfrak{F}_k$ with $P[B] > 0$. But there exists an n such that $\int_B X_n^+ > 0$ and $\int_B X_n^- > 0$ by (3). Hence $0 < P[B \cap (X_n > 0)] < P[B]$ contradicting the assumption that B is an atom of $\bigcup_{k=1}^{\infty}\mathfrak{F}_k$.

(2) implies the existence of s: We proceed by induction. Let $B_0 = \Omega$. By hypothesis, there exists disjoint sets $A_{1,1}$ and $A_{1,2}$ such that $A_{1,1} \cup A_{1,2} = \Omega$, $0 < P[A_{1,1}] \leqq P[A_{1,2}] < 1$, and $A_{1,1}$ and $A_{1,2} \varepsilon \mathfrak{F}_{n_1}$ for some $n_1 \geqq 1$. Let $C_1 = A_{1,2}$. Assume that disjoint sets $C_1, C_2, \cdots, C_k$ have been chosen such that $C_i \varepsilon \mathfrak{F}_{n_i}$, $n_i \uparrow$ as $i \uparrow k$, and $0 < P[B_k] \leqq \frac{1}{2}^k$ where $B_k = (\bigcup_{i=1}^{k} C_i)^c$. By hypothesis, there exists disjoint sets $A_{k+1,1}$ and $A_{k+1,2}$ such that $A_{k+1,1} \cup A_{k+1,2} = B_k$, $0 < P[A_{k+1,1}] \leqq P[A_{k+1,2}] < P[B_k]$, $A_{k+1,1}$ and $A_{k+1,2} \varepsilon \mathfrak{F}_{n_{k+1}}$ for some $n_{k+1} > n_k$. Let $C_{k+1} = A_{k+1,2}$. Then $0 < P[B_{k+1}] = P[B_k] - P[C_{k+1}] \leqq P[B_k] - P[B_k]/2 \leqq \frac{1}{2}^{k+1}$.

Clearly C_{k+1} is disjoint from $C_1, C_2, \cdots$, and C_k. Thus by induction $(C_k, k \geqq 1)$ is a class of disjoint sets and $(n_k, k \geqq 1)$ is a strictly increasing sequence of positive integers such that $C_k \varepsilon \mathfrak{F}_{n_k}$ and $P[C_k] > 0$ for each $k \geqq 1$, and $P[\bigcup_{k=1}^{\infty} C_k] = 1$. Setting $[s = n_k] = C_k$ for all $k \geqq 1$ defines an unbounded stopping rule.

REMARK. Note as the proof shows, that (4), (3), and (2) are progressively weaker conditions.

THEOREM 1. *If there exists an unbounded stopping rule s and if either*

(5) $\int_A X_n^+ \geqq 1$ *infinitely often* $(n \geqq 1)$ *for each* $A \in \bigcup_{k=1}^{\infty} \mathfrak{F}_k$ *such that* $P[A] > 0$

or

(6) $\sup_{n \geqq 1} E[X_n^+ \mid \mathfrak{F}_{m_k}] = \infty$ a.s. *for some strictly increasing sequence of positive integers* $(m_k, k \geqq 1)$,

then there exists a stopping rule t such that $EX_t^+ = \infty$.

PROOF. Under (5), for each j such that $P[s = j] > 0$, there exists an integer $m_j \geqq j$ such that $\int_{[s=j]} X_{m_j}^+ \geqq 1$. We define $t = m_j$ on $[s = j]$ for each $j \geqq 1$. Then $P[t < \infty] = 1$, $[t = n] = \bigcup_{k=1}^{n}[m_k = n, \ s = k] \in \mathfrak{F}_n$ and $EX_t^+ = \sum_{j=1}^{\infty} \int_{[s=j]} X_{m_j}^+ \geqq \sum_{j=1}^{\infty} 1 = \infty$. Thus the result holds under (5).

Under (6), for each j such that $P[s = j] > 0$, on $[s = j]$ we define

$$t = \inf\{n \geqq m_j \mid E[X_n^+ \mid \mathfrak{F}_{m_j}] \geqq 1/P[s = j]\}$$

where $m_j \in (m_k, k \geqq 1)$ and $m_j \geqq j$. $[t = n] = \bigcup_{j=1}^{n}[t = n, s = j]$ and $[t = n, s = j] \in \mathfrak{F}_{m_j}$ for all $j \leqq n$ together imply that $[t = n] \in \mathfrak{F}_n$ for all $n \geqq 1$. Since $P[t < \infty] = 1$ by (6), t is thus a stopping rule.

$$\begin{aligned} EX_t^+ &= \sum_{j=1}^{\infty} \int_{[s=j]} X_t^+ = \sum_{j=1}^{\infty} \sum_{n=j}^{\infty} \int_{[s=j,\ t=n]} X_n^+ \\ &= \sum_{j=1}^{\infty} \sum_{n=j}^{\infty} \int_{[s=j,\ t=n]} E[X_n^+ \mid \mathfrak{F}_{m_j}] \geqq \sum_{j=1}^{\infty} 1 = \infty. \end{aligned}$$

Thus the result holds under (6), establishing the theorem.

COROLLARY 1. *If either* (2), (3), *or* (4) *and either* (5) *or* (6) *hold then there exists a stopping rule t such that* $EX_t^+ = \infty$.

PROOF. The result is immediate from Lemma 1 and Theorem 1.

COROLLARY 2. *Let* $(Y_k, k \geqq 1)$ *be an integrable sequence of independent random variables with a subsequence* $(Y_{n_k}, k \geqq 1)$ *of non-degenerate random variables where* $n_k \uparrow \infty$ *as* $k \to \infty$. *Let* $\mathfrak{F}_n = \mathcal{B}(Y_1, Y_2, \cdots Y_n)$ *for all* $n \geqq 1$.

(i) *Let* $X_n = \sum_{k=1}^{n} Y_k$ *and* $\sup EX_n^+ = \infty$. *Then there exists a stopping rule t such that* $EX_t^+ = \infty$.

(ii) *Let* $X_n = \sum_{k=1}^{n} Y_k$ *be divergent with* $EY_k = 0$ *for all* $k \geqq 1$. *Then there exists a stopping rule t such that* $EX_t^+ = \infty$.

(iii) *Let* $X_n = \sum_{k=1}^{n} Y_k/n$ *and* $\sup EX_n^+ = \infty$. *Then there exists a stopping rule t such that* $EX_t^+ = \infty$.

(iv) *Let* $X_n = \max(Y_1, Y_2, \cdots, Y_n) - n$ *and* $\sup EX_n^+ = \infty$. *Then there exists a stopping rule t such that* $EX_t^+ = \infty$.

PROOF. In each case ((i)–(iv)), $(X_n, \mathfrak{F}_n, n \geqq 1)$ is an integrable stochastic sequence. The existence of the non-degenerate sequence $(Y_{n_k}, k \geqq 1)$ implies $\bigcup_{k=1}^{\infty} \mathfrak{F}_k$ is non-atomic. For, assume not. Then, there exists a set A atomic in $\bigcup_{k=1}^{\infty} \mathfrak{F}_k$ such that $P[A] > 0$, $A \in \mathfrak{F}_n$ say. Choose $n_k > n$ such that Y_{n_k} is non-degenerate. Thus there exists an event $A_1 \in \mathcal{B}(Y_{n_k})$ such that $0 < P[A_1] < 1$.

But A and A_1 are independent, implying that $0 < P[A \cap A_1] < P[A]$ and thus contradicting the assumption that A is atomic in $\bigcup_{k=1}^{\infty} \mathfrak{F}_k$.

In (i), for $n \geqq k$, $E[X_n^+ \mid \mathfrak{F}_k] = E[(X_n - X_k + X_k)^+ \mid \mathfrak{F}_k] \geqq E[(X_n - X_k)^+ \mid \mathfrak{F}_k] - X_k^- = E(X_n - X_k)^+ - X_k^- \geqq EX_n^+ - EX_k^+ - X_k^-$ a.s. Hence $\sup_{n \geqq 1} E[X_n^+ \mid \mathfrak{F}_k] = \infty$ a.s. for every $k \geqq 1$ since $\sup_{n \geqq 1} EX_n^+ = \infty$ by hypothesis. Thus (2) and (6) hold and (i) is established by Corollary 1.

In (ii), $(X_n, \mathfrak{F}_n, n \geqq 1)$ is a martingale. Thus X_n diverging implies that $\sup EX_n^+ = \infty$ by the Doob martingale convergence theorem ([3], p. 319). Thus (ii) follows from (i).

In (iii), for $n \geqq k$, $E[X_n^+ \mid \mathfrak{F}_k] \geqq EX_n^+ - kn^{-1}(EX_k^+ + X_k^-)$ a.s. Hence $\sup_{n \geqq 1} E[X_n^+ \mid \mathfrak{F}_k] = \infty$ a.s. for every $k \geqq 1$ since $\sup EX_n^+ = \infty$ by hypothesis. Thus (2) and (6) hold and (iii) is established by Corollary 1.

In (iv), for $n > k$, $E[X_n^+ \mid \mathfrak{F}_k] = E[(\max(Y_1, Y_2, \cdots, Y_n) - n)^+ \mid \mathfrak{F}_k] \geqq E[(\max(Y_{k+1}, Y_{k+2}, \cdots, Y_n) - n)^+ \mid \mathfrak{F}_k] = E(\max(Y_{k+1}, Y_{k+2}, \cdots, Y_n) - n)^+$. Hence $\sup_{n \geqq 1} E[X_n^+ \mid \mathfrak{F}_k] = \infty$ a.s. for every $k \geqq 1$ since $\sup EX_n^+ = \infty$ by hypothesis thereby implying that $\sup_{n \geqq 1} E(\max(Y_{k+1}, Y_{k+2}, \cdots, Y_n) - n)^+ = \infty$ for every $k \geqq 1$. Thus (2) and (6) hold and (iv) is established by Corollary 1. The proof of Corollary 2 is complete.

In the following theorem we shall consider two new conditions which we state now for easy reference:

$$(7) \quad \int_A X_n^- \geqq 1 \text{ infinitely often } (n \geqq 1) \text{ for each } A \in \bigcup_{k=1}^{\infty} \mathfrak{F}_k \text{ such that } P[A] > 0$$

and

$$(8) \quad \sup_{n \geqq 1} E[X_n^- \mid \mathfrak{F}_{m_k}] = \infty \text{ a.s. for some strictly increasing sequence of positive integers } (m_k, k > 1).$$

THEOREM 2. *If there exists an unbounded stopping rule s, if either* (5) *or* (6) *holds, and if either* (7) *or* (8) *holds, then there exists a stopping rule t such that* $EX_t^+ = EX_t^- = \infty$.

PROOF. The details are omitted. Let $(m_k, k \geqq 1)$ be the essential range of s with $\Omega^1 = \bigcup_{k=1}^{\infty} (s = m_{2k-1})$ and $\Omega^2 = \bigcup_{k=1}^{\infty} (s = m_{2k})$. Using the idea of the proof of Theorem 1 we then define t on Ω^1 such that $EX_t^+ = \infty$ and t on Ω^2 such that $EX_t^- = \infty$.

LEMMA 2. *Let* $\mathfrak{F}_n$ *be non-atomic for some* $n \geqq 1$ *and let* A *be a subset of* $\mathfrak{F}_n$ *with* $P(A) > 0$, *and* $\sup_{n \geqq 1} \int_A X_n^+ = \infty$. *Then there exists a set* $F \subset A$ *such that* $P[A] > P[F] \geqq P[A]/2$, $F \in \mathfrak{F}_m$ *for some* $m \geqq n$, $\int_F X_m^+ \geqq 1$ *and* $\sup_{n \geqq 1} \int_{A \setminus F} X_n^+ = \infty$ *where* $A \setminus F = A \cap F^c$.

PROOF. Choose $m \geqq n$ such that $\int_A X_m^+ \geqq 2$. Since $\mathfrak{F}_m$ is non-atomic by hypothesis there exists disjoint sets B and $C \in \mathfrak{F}_m$ such that $B \cup C = A$, $\int_B X_m^+ \geqq 1$, and $\int_C X_m^+ \geqq 1$. Either $\sup \int_B X_n^+ = \infty$ or $\sup \int_C X_n^+ = \infty$. Without loss of generality we assume $\sup \int_B X_n^+ = \infty$. Possibly $P[B] > P[A]/2$. However, there exists disjoint sets D and $E \in \mathfrak{F}_m$ such that $D \cup E = B$, $0 < P[D] \leqq P[A]/2$

and $0 < P[E] \leqq P[A]/2$ by the non-atomicity of $\mathfrak{F}_m$. Either $\sup \int_D X_n^+ = \infty$ or $\sup \int_E X_n^+ = \infty$. Without loss of generality, we assume $\sup \int_D X_n^+ = \infty$. Letting $F = A \backslash D$ it follows that $\int_F X_m^+ \geqq 1$, $\sup \int_{A \backslash F} X_n^+ = \infty$ with $P[A] > P[F] \geqq P[A]/2$, establishing the lemma.

In Corollary 1, we needed $\bigcup_{k=1}^\infty \mathfrak{F}_k$ non-atomic (2) and roughly speaking a local moment condition ((5) or (6)) in order to conclude that $EX_t^+ = \infty$ for some stopping rule t. The question arises naturally whether the global condition $\sup EX_n^+ = \infty$ is sufficient if we strengthen (1). This yields:

THEOREM 3. *If* $\sup EX_n^+ = \infty$ *and* $\mathfrak{F}_k$ *is non-atomic for some* $k \geqq 1$, *then there exists a stopping rule t such that* $EX_t^+ = \infty$.

PROOF. We proceed by induction. By Lemma 2, there exists a subset F_1 of Ω such that $1 > P(F_1) \geqq \frac{1}{2}$, $F_1 \varepsilon \mathfrak{F}_{n_1}$, for some $n_1 \geqq k$, $\int_{F_1} X_{n_1}^+ \geqq 1$ and $\sup \int_{F_1^c} X_n^+ = \infty$. Now assume disjoint sets $F_1, F_2, \cdots, F_m$ and integers $k \leqq n_1 \leqq n_2 \cdots \leqq n_m$ such that $F_i \varepsilon \mathfrak{F}_{n_i}$ and $\int_{F_i} X_{n_i}^+ \geqq 1$, for all $1 \leqq i \leqq m$, $1 > P[A_m^c] \geqq 1 - \frac{1}{2}^m$ where $A_m = (\bigcup_{i=1}^m F_i)^c$, and $\sup_{n \geqq 1} \int_{A_m} X_n^+ = \infty$ have been chosen. Note $A_m \varepsilon \mathfrak{F}_{n_m}$ and $P(A_m) > 0$. By Lemma 2, there exists a subset F_{m+1} of A_m such that $P[A_m] > P[F_{m+1}] \geqq P[A_m]/2$, $F_{m+1} \varepsilon \mathfrak{F}_{n_{m+1}}$ for some $n_{m+1} \geqq n_m$, $\int_{F_{m+1}} X_{n_{m+1}}^+ \geqq 1$ and $\sup_{n \geqq 1} \int_{A_{m+1}} X_n^+ = \infty$. Then $1 > P[A_{m+1}^c] = P[A_m^c] + P[F_{m+1}] \geqq P[A_m^c] + P[A_m]/2 \geqq 1 - \frac{1}{2}^{m+1}$. Thus by induction for all $m \geqq 1$, F_m and n_m are well defined with $F_m \varepsilon \mathfrak{F}_{n_m}$, $\int_{F_m} X_{n_m}^+ \geqq 1$, $F_i \cap F_j = \emptyset$ for $i \neq j$ and $P[\bigcup_{m=1}^\infty F_m] = 1$. We define a stopping rule t by $[t = n_m] = F_m$ for all $m \geqq 1$. $EX_t^+ = \sum_{m=1}^\infty \int_{F_m} X_{n_m}^+ \geqq \sum_{m=1}^\infty 1 = \infty$, concluding the proof.

REMARK. Theorem 3 can be proved using other methods, for instance using the techniques of [4].

Arguing much as in the proof of Theorem 3, we can establish:

THEOREM 4. *If* $\sup EX_n^+ = \sup EX_n^- = \infty$ *and* $\mathfrak{F}_k$ *is non-atomic for some* $k \geqq 1$, *then there exists a stopping rule t such that* $EX_t^+ = EX_t^- = \infty$.

PROOF. Omitted.

It seemed plausible that we could deduce the existence of a stopping rule t such that $EX_t = \infty$ under reasonable hypotheses. However, even if $(Y_n, n \geqq 1)$ is an independent sequence of random variables with $EY_n = 0$ for all $n \geqq 1$, $X_n = \sum_{k=1}^n Y_k$, $\lim_{n \to \infty} EX_n^+ = \infty$, the following example shows that there need not exist a stopping rule such that $EX_t = \infty$.

EXAMPLE 1. We choose two valued independent random variables $(Y_n, n \geqq 1)$ such that $EY_n = 0$ for all $n \geqq 1$. Let $P[Y_n > 0] = \frac{1}{2}^n$ and $P[Y_n < 0] = 1 - \frac{1}{2}^n$, $X_n = \sum_{k=1}^n Y_n$, $\mathfrak{F}_n = \mathcal{B}(Y_1, Y_2, \cdots, Y_n)$, and $y_n^- \uparrow \infty$ for all $n \geqq 1$, where (y_n^+, y_n^-) is the range of Y_n. Note that

$$(9) \quad P[X_n \downarrow -\infty \text{ for } n \geqq N \mid X_N = c] \geqq \prod_{k=1}^\infty (1 - \tfrac{1}{2}^k) > 0$$

for every choice of N and c such that $P[X_N = c] > 0$.

Let $y_1^- = 2$ and $y_n^- = \prod_{k=1}^n 2^k + 2\sum_{k=1}^{n-1} y_k^+$ for $n \geqq 2$. $y_n^+ \geqq y_n^-$ for all $n \geqq 1$. Let $C_i = \{c_{ij}, j \geqq 1\}$ be the range of X_i for each $i \geqq 1$. Note that the c_{ij}'s are all distinct and that

$$(10) \qquad |c_{ij}| P[X_i = c_{ij}] \geqq 1 \quad \text{for all} \quad c_{ij} \varepsilon \bigcup_{i=1}^\infty C_i.$$

Further $EX_n^+ \to \infty$ a.s. as $n \to \infty$. Let t be a stopping rule such that $EX_t^+ = \infty$. There exists infinitely many $c_{ij} > 0$ such that $P[X_t = c_{ij}] > 0$. It follows that there exists infinitely many $c_{ij} < 0$ such that $P[X_t = c_{ij}] > 0$ by (9) and $P[t < \infty] = 1$. $EX_t^- = - \sum_{(i,j):c_{ij}<0, P[X_t=c_{ij}]>0} c_{ij} P[X_i = c_{ij}] = \infty$ by (10) and the atomicity of the $\mathfrak{F}_n$'s. Hence $EX_t^+ = \infty$ implies that $EX_t^- = \infty$ thereby implying the impossibility of $EX_t = \infty$ for any stopping rule t.

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