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ON FINITE PRODUCTS OF POISSON-TYPE CHARACTERISTIC FUNCTIONS OF SEVERAL VARIABLES

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1. Introduction. A characteristic function f of the n variables $t = (t_1, \cdots, t_n)$ is a Poisson-type characteristic function if it is of the form

$$f(t) = \exp\{iP(t) + \textstyle\sum_\epsilon \lambda_{\epsilon_1,\cdots,\epsilon_n}(e^{i(\epsilon_1\alpha_1 t_1+\cdots+\epsilon_n\alpha_n t_n)} - 1)\},$$

where P is a polynomial of degree one without constant term and with real coefficients, the λ are non-negative constants, $\alpha = (\alpha_1, \cdots, \alpha_n)$ is a real vector, $\epsilon_j = 0$ or $1 (j = 1, \cdots, n)$ and $\sum_\epsilon$ indicates the summation on the $2^n - 1$ values of $\epsilon = (\epsilon_1, \cdots, \epsilon_n)$ different from $(0, \cdots, 0)$.

Therefore, the product f of two Poisson-type characteristic functions is of the form

$$(1.1)\quad f(t) = \exp\{iP(t) + \textstyle\sum_\epsilon [\lambda_{\epsilon_1,\cdots,\epsilon_n}(e^{i(\epsilon_1\alpha_1 t_1+\cdots+\epsilon_n\alpha_n t_n)} - 1) + \mu_{\epsilon_1,\cdots,\epsilon_n}(e^{i\,\epsilon_1\beta_1 t_1+\cdots+\epsilon_n\beta_n t_n)} - 1)]\}$$

with evident conditions on P, $\alpha = (\alpha_1, \cdots, \alpha_n)$, $\beta = (\beta_1, \cdots, \beta_n)$ and the constants λ and μ. In the case $n = 2$, we modify the notations and write (1.1) in the form

$$(1.2)\quad f(t) = \exp\{iP(t) + \lambda_1(e^{i\alpha_1 t_1} - 1) + \mu_1(e^{i\alpha_2 t_2} - 1) + \nu_1(e^{i(\alpha_1 t_1+\alpha_2 t_2)} - 1) + \lambda_2(e^{i\beta_1 t_1} - 1) + \mu_2(e^{i\beta_2 t_2} - 1) + \nu_2(e^{i(\beta_1 t_1+\beta_2 t_2)} - 1)\}.$$

In the case of one variable, it is known since P. Lévy [3] that the product of two Poisson-type characteristic functions has no indecomposable factor (in the sense of the decomposition of characteristic functions). But in the case of two variables, it is not the same: There are products of two Poisson-type characteristic functions which have indecomposable factors as it is shown in Section 2. Nevertheless, it is possible to find simple conditions assuring that the product of two Poisson-type characteristic functions has no indecomposable factor as it is shown in Sections 3 and 4. Finally, in Section 5, we give some results on the finite product of Poisson-type characteristic functions.

2. A counter-example. Let f be the product of two Poisson-type characteristic functions defined by

$$f(t_1, t_2) = \exp\{\lambda_1(e^{it_1} - 1) + \mu_1(e^{2it_2} - 1) + \nu_1(e^{i(t_1+2t_2)} - 1) + \lambda_2(e^{2it_1} - 1) + \mu_2(e^{it_2} - 1) + \nu_2(e^{i(2t_1+t_2)} - 1)\},$$

where λ_j, μ_j, ν_j $(j = 1, 2)$ are all positive. Then f has an indecomposable factor.

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The proof is almost identical to the one showing that the product of three Poisson-type characteristic functions of one variable may have an indecomposable factor (see, for instance, [4], pp. 178–179). Let P be the polynomial defined by

$$P(x, y) = 1 + \lambda_1 x + \mu_1 y^2 + \nu_1 xy^2 + \lambda_2 x^2 + \mu_2 y + \nu_2 x^2 y - kxy, \quad (k > 0).$$

If k is taken small enough, the expansion of

$$\exp [P(x, y)] = \sum_{j=0}^{\infty} [P(x, y)]^j/j!$$

in an entire series of the two variables x and y has only non-negative coefficients. Indeed we may choose k small enough so that the polynomials P^2 and P^3 have only non-negative coefficients. In this case, all the polynomials P^m $(m > 1)$ have only non-negative coefficients and only the coefficient of xy in $\exp [P(x, y)]$ can be negative. But this coefficient is

$$\lambda_1 \mu_2 - 2k + C,$$

C being non-negative, and therefore if

$$k \leqq \tfrac{1}{2}\lambda_1\mu_2 ,$$

the expansion of $\exp [P(x, y)]$ has only non-negative coefficients. The function defined by $\exp [P(x, y) - P(1, 1)]$ is then a generating function so that the function g defined by

$$\begin{aligned} g(t_1, t_2) &= \exp [P(e^{it_1}, e^{it_2}) - P(1, 1)] \\ &= \exp \{\lambda_1(e^{it_1} - 1) + \mu_1(e^{2it_2} - 1) + \nu_1(e^{i(t_1+2t_2)} - 1) + \lambda_2(e^{2it_1} - 1) \\ &\qquad + \mu_2(e^{it_2} - 1) + \nu_2(e^{i(2t_1+t_2)} - 1) - k(e^{i(t_1+t_2)} - 1)\} \end{aligned}$$

is a characteristic function which cannot be infinitely divisible from the Lévy's representation ([1], Chapter 1, Section 2). Therefore ([1], Theorem 1.6), g has an indecomposable factor. Since g divides f, f is a product of two Poisson-type characteristic functions which has an indecomposable factor.

3. A general theorem. Case $n = 2$. Recall (cf. [1], Chapter 4) that a function φ of the n complex variables $z = (z_1, \cdots, z_n)$ is said to be a ridge function if it is an entire function satisfying the condition

$$(3.1) \qquad |\varphi(z)| \leqq \varphi(\operatorname{Re} z), \qquad \operatorname{Re} z = (\operatorname{Re} z_1, \cdots, \operatorname{Re} z_n); \qquad z \in C^n.$$

THEOREM 1. *Let φ_1 and φ_2 be two ridge functions of the two variables $z = (z_1, z_2)$ such that*

$$(3.2) \quad \varphi_1(z)\varphi_2(z) = \exp \{\pi(z) + \lambda_1 e^{\alpha_1 z_1} + \mu_1 e^{\alpha_2 z_2} + \nu_1 e^{\alpha_1 z_1 + \alpha_2 z_2} + \lambda_2 e^{\beta_1 z_1} + \mu_2 e^{\beta_2 z_2} + \nu_2 e^{\beta_1 z_1 + \beta_2 z_2}\},$$

where π is a polynomial of degree one, λ_j, μ_j, ν_j $(j = 1, 2)$ are non-negative constants *and* $\alpha = (\alpha_1, \alpha_2)$, $\beta = (\beta_1, \beta_2)$ *are real vectors. If one of the following conditions*[2]

(a) $$\alpha_1\beta_1 \leqq 0;$$

(b) $$0 < \alpha_1 < \beta_1, \quad 0 \leqq \alpha_2 \leqq \beta_2;$$

(c) $$0 < \alpha_1 < \beta_1, \quad 0 < \beta_2 < \alpha_2, \quad \beta_1\beta_2 + \alpha_1\alpha_2 - \beta_1\alpha_2 > 0;$$

(d) $$0 < \alpha_1 < \beta_1, \quad \beta_1/\alpha_1 \text{ irrational}$$

is satisfied, then

$$\varphi_1(z) = \exp\{P(z) + l_1e^{\alpha_1z_1} + m_1e^{\alpha_2z_2} + n_1e^{\alpha_1z_1+\alpha_2z_2} + l_2e^{\beta_1z_1} + m_2e^{\beta_2z_2} + n_2e^{\beta_1z_2+\beta_2z_2}\},$$

where P, l_j, m_j, n_j *have respectively the same properties as* π, λ_j, μ_j, ν_j.

Proof. The idea is the same as the one used for the proof of the Theorem 5.1 of [1]. We use the following theorem which can be deduced from Theorem VII of Ostrovskiy [5] (see also [6], p. 122).

Let φ_1 and φ_2 be two ridge functions of the variable z such that

$$\varphi_1(z)\varphi_2(z) = \exp\{\pi(z) + \lambda e^{\alpha z} + \mu e^{\beta z}\},$$

where π is a polynomial of degree one, α and β are real constants and λ and μ are non-negative constants. Then

$$\varphi_1(z) = \exp\{P(z) + le^{\alpha z} + me^{\beta z}\},$$

where P, l, m have respectively the same properties as π, λ, μ.

We may suppose, without loss of generality, that $\alpha_1 < \beta_1$, $\beta_1 > 0$, $\beta_2 > 0$. We fix z_2 real. φ_1 and φ_2 are ridge functions of z_1 which satisfy the conditions of the above theorem. Therefore

$$\varphi_1(z) = \exp\{a + bz_1 + pe^{\alpha_1z_1} + qe^{\beta_1z_1}\}, \tag{3.3}$$

where a, b, p, q are functions of z_2, real for z_2 real. Then, fixing z_1 real, we obtain the representation

$$\varphi_1(z) = \exp\{a' + b'z_2 + p'e^{\alpha_2z_2} + q'e^{\beta_2z_2}\}, \tag{3.4}$$

where a', b', p', q' are functions of z_1, real for z_1 real. Therefore we obtain from (3.3) and (3.4) the equation for any z_1 and z_2 real

$$a + bz_1 + pe^{\alpha_1z_1} + qe^{\beta_1z_1} = a' + b'z_2 + p'e^{\alpha_2z_2} + q'e^{\beta_2z_2}.$$

Using the well-known properties of linear independence of polynomials and exponentials, we can solve this equation and obtain the representation for z_1, z_2 real

$$\begin{aligned}\varphi_1(z) = \exp\{&P(z) + cz_1z_2 + l_1e^{\alpha_1z_1} + m_1e^{\alpha_2z_2} + n_1e^{\alpha_1z_1+\alpha_2z_2} + l_2e^{\beta_1z_1} \\ &+ m_2e^{\beta_2z_2} + n_2e^{\beta_1z_1+\beta_2z_2} + r_1e^{\alpha_1z_1+\beta_2z_2} + r_2e^{\beta_1z_1+\alpha_2z_2} + s_1z_2e^{\alpha_1z_1} \\ &+ t_1z_2e^{\beta_1z_1} + s_2z_1e^{\alpha_2z_2} + t_2z_1e^{\beta_2z_2}\}.\end{aligned} \tag{3.5}$$

[2] We have not written the symmetrical conditions obtained by exchanging the roles of z_1 and z_2.

Since, in the two members of (3.5), we have two entire functions of z_1 and z_2, the representation (3.5) is also valid for z_1 and z_2 complex. Setting

$$u(x, y) = \operatorname{Re} \log [\varphi_1(x + iy)], \qquad x, y \in R^2,$$

we obtain, by a simple calculation,

$$\begin{aligned} u(x, 0) - u(x, y) & \\ = c y_1 y_2 &+ 2[l_1 e^{\alpha_1 x_1} \sin^2 \tfrac{1}{2}(\alpha_1 y_1) + m_1 e^{\alpha_2 x_2} \sin^2 \tfrac{1}{2}(\alpha_2 y_2) \\ &+ n_1 e^{\alpha_1 x_1 + \alpha_2 x_2} \sin^2 \tfrac{1}{2}(\alpha_1 y_1 + \alpha_2 y_2) + l_2 e^{\beta_1 x_1} \sin^2 \tfrac{1}{2}(\beta_1 y_1) \\ &+ m_2 e^{\beta_2 x_2} \sin^2 \tfrac{1}{2}(\beta_2 y_2) + n_2 e^{\beta_1 x_1 + \beta_2 x_2} \sin^2 \tfrac{1}{2}(\beta_1 y_1 + \beta_2 y_2) \\ &+ r_1 e^{\alpha_1 x_1 + \beta_2 x_2} \sin^2 \tfrac{1}{2}(\alpha_1 y_1 + \beta_2 y_2) + r_2 e^{\beta_1 x_1 + \alpha_2 x_2} \sin^2 \tfrac{1}{2}(\beta_1 y_1 + \alpha_2 y_2) \\ &+ s_1 x_2 e^{\alpha_1 x_1} \sin^2 \tfrac{1}{2}(\alpha_1 y_1) + t_1 x_2 e^{\beta_1 x_1} \sin^2 \tfrac{1}{2}(\beta_1 y_1) \\ &+ s_2 x_1 e^{\alpha_2 x_2} \sin^2 \tfrac{1}{2}(\alpha_2 y_2) + t_2 x_1 e^{\beta_2 x_2} \sin^2 \tfrac{1}{2}(\beta_2 y_2)] \\ &+ s_1 y_2 e^{\alpha_1 x_1} \sin \alpha_1 y_1 + t_1 y_2 e^{\beta_1 x_1} \sin \beta_1 y_1 + s_2 y_1 e^{\alpha_2 x_2} \sin \alpha_2 y_2 \\ &+ t_2 y_1 e^{\beta_2 x_2} \sin \beta_2 y_2 , \end{aligned} \tag{3.6}$$

and, from the definition (3.1) of a ridge function, we must have

$$u(x, 0) - u(x, y) \geqq 0. \tag{3.7}$$

Letting $y_1 = y_2 \to +\infty$ in (3.6) and using (3.7), we obtain $c \geqq 0$; similarly, letting $y_1 = -y_2 \to +\infty$, we obtain $c \leqq 0$. Hence

$$c = 0.$$

Let x_1 and y_1 be arbitrary, but fixed, and $|y_2| \to \infty$. Then we can conclude from (3.6) and (3.7) that

$$s_1 e^{\alpha_1 x_1} \sin \alpha_1 y_1 + t_1 e^{\beta_1 x_1} \sin \beta_1 y_1 = 0.$$

Hence

$$s_1 = t_1 = 0.$$

In the same way, letting $|y_1| \to \infty$, we find

$$s_2 = t_2 = 0.$$

We obtain now for $y_1 = 0$

$$\begin{aligned} u(x, 0) - u(x, y) = {} & 2 e^{\alpha_2 x_2} \sin^2 \tfrac{1}{2}(\alpha_2 y_2)[m_1 + n_1 e^{\alpha_1 x_1} + r_2 e^{\beta_1 x_1}] \\ &+ 2 e^{\beta_2 x_2} \sin^2 \tfrac{1}{2}(\beta_2 y_2)[m_2 + n_2 e^{\beta_1 x_1} + r_1 e^{\alpha_1 x_1}] \end{aligned} \tag{3.8}$$

and for $y_2 = 0$

$$\begin{aligned} u(x, 0) - u(x, y) = {} & 2 e^{\alpha_1 x_1} \sin^2 \tfrac{1}{2}(\alpha_1 y_1) [l_1 + n_1 e^{\alpha_2 x_2} + r_1 e^{\beta_2 x_2}] \\ &+ 2 e^{\beta_1 x_1} \sin^2 \tfrac{1}{2}(\beta_1 y_1)[l_2 + n_2 e^{\beta_2 x_2} + r_2 e^{\alpha_2 x_2}]. \end{aligned} \tag{3.9}$$

We distinguish now the different cases.

Case (a). In the cases $\alpha_2 = \beta_2$ and $\alpha_1 = 0$, there is nothing to prove. We suppose that $\alpha_2 < \beta_2$, $\alpha_1 < 0$ (the proof is the same if $\alpha_2 > \beta_2$, $\alpha_1 < 0$). Comparing (3.7) and (3.8), we obtain for $x_2 \to +\infty$

$$m_2 + n_2 e^{\beta_1 x_1} + r_1 e^{\alpha_1 x_1} \geqq 0 \tag{3.10}$$

and for $x_2 \to -\infty$

$$m_1 + n_1 e^{\alpha_1 x_1} + r_2 e^{\beta_1 x_1} \geqq 0. \tag{3.11}$$

From (3.10), we obtain for $x_1 \to -\infty$

$$r_1 \geqq 0,$$

and therefore $r_1 = 0$, since the corresponding term in φ_2 has the same sign and since their sum is zero from (3.2). We obtain then for $x_1 \to -\infty$,

$$m_2 \geqq 0$$

and for $x_1 \to +\infty$

$$n_2 \geqq 0.$$

From (3.11), we obtain in the same way

$$r_2 = 0, \qquad m_1 \geqq 0, \qquad n_1 \geqq 0.$$

Comparing (3.7) and (3.9), we obtain

$$l_1 \geqq 0, \qquad l_2 \geqq 0,$$

and the theorem is demonstrated in this case.

Case (b). We can suppose $0 < \alpha_2 < \beta_2$, the cases $\alpha_2 = 0$ and $\alpha_2 = \beta_2$ being trivial. Comparing (3.7) and (3.8), we obtain in the same way

$$r_2 = 0, \qquad m_1 \geqq 0, \qquad n_1 \geqq 0, \qquad m_2 \geqq 0, \qquad n_2 \geqq 0,$$

and comparing (3.7) and (3.9), we obtain

$$r_1 = 0, \qquad l_1 \geqq 0, \qquad l_2 \geqq 0,$$

and the theorem is demonstrated.

Case (c). Comparing (3.7) and (3.8), we obtain as in the case (b)

$$r_2 = 0, \qquad m_1 \geqq 0, \qquad n_1 \geqq 0, \qquad m_2 \geqq 0, \qquad n_2 \geqq 0$$

and comparing (3.7) and (3.9), we obtain

$$l_1 \geqq 0, \qquad l_2 \geqq 0.$$

It remains to demonstrate that $r_1 = 0$ and it is sufficient, from the remark made in the Case (a) to demonstrate that $r_1 \geqq 0$. Because of condition (c), we can choose y_1 and y_2 such that

$$\alpha_1 y_1 + \alpha_2 y_2 = 2\pi,$$

$$\beta_1 y_1 + \beta_2 y_2 = 2\pi.$$

We have then

$$u(x, 0) - u(x, y)$$
$$(3.12) \qquad = 2[l_1 e^{\alpha_1 x_1} \sin^2 \tfrac{1}{2}(\alpha_1 y_1) + m_1 e^{\alpha_2 x_2} \sin^2 \tfrac{1}{2}(\alpha_2 y_2) + l_2 e^{\beta_1 x_1} \sin^2 \tfrac{1}{2}(\beta_1 y_1) + m_2 e^{\beta_2 x_2} \sin^2 \tfrac{1}{2}(\beta_2 y_2) + r_1 e^{\alpha_1 x_1 + \beta_2 x_2} \sin^2 \tfrac{1}{2}(\alpha_1 y_1 + \beta_2 y_2)].$$

From the condition

$$\alpha_1 \alpha_2 + \beta_1 \beta_2 - \beta_1 \alpha_2 > 0$$

we may choose $x_2 = kx_1$ (k constant) so that

$$\alpha_1 x_1 + \beta_2 x_2 > \alpha_2 x_2,$$
$$\alpha_1 x_1 + \beta_2 x_2 > \beta_1 x_1.$$

Letting then $x_1 \to \infty$, we obtain from (3.7) and (3.12)

$$u(x, 0) - u(x, y) = e^{\alpha_1 x_1 + \beta_2 x_2}[r_1 \sin^2 \tfrac{1}{2}(\alpha_1 y_1 + \beta_2 y_2) + o(1)]$$

and since $\alpha_1 y_1 + \beta_2 y_2 \neq 2m\pi$ for any integer m,

$$r_1 \geqq 0$$

and the theorem is demonstrated in this case.

Case (d). We may suppose that $0 < \beta_2 < \alpha_2$ (the other cases are contained in the cases (a) and (b). As in the case (c), we obtain from (3.7) and (3.8)

$$r_2 = 0, \qquad m_1 \geqq 0, \qquad n_1 \geqq 0, \qquad m_2 \geqq 0, \qquad n_2 \geqq 0,$$

and from (3.7) and (3.9)

$$l_1 \geqq 0, \qquad l_2 \geqq 0,$$

and it remains to demonstrate that r_1 is zero and for that it is sufficient to show that $r_1 \geqq 0$. Setting $y_2 = 2\pi/\alpha_2$, since β_1/α_1 is irrational, from a theorem of Kronecker ([2], Theorem 444), it is possible to find $y_1 = y_1(x_2)$ such that

$$\sin \tfrac{1}{2}(\alpha_1 y_1) = o(e^{-\frac{1}{2}(\alpha_2 x_2)}), \qquad \sin \tfrac{1}{2}(\beta_1 y_1 + \beta_2 y_2) = o(e^{-\frac{1}{2}(\beta_2 x_2)})$$

when $x_2 \to +\infty$. We have then

$$u(x, 0) - u(x, y) = e^{\beta_2 x_2}[m_2 \sin^2 \tfrac{1}{2}(\beta_2 y_2) + r_1 e^{\alpha_1 x_1} \sin^2 \tfrac{1}{2}(\alpha_1 y_1 + \beta_2 y_2) + o(1)]$$

when $x_2 \to +\infty$, so that letting x_1 great enough, we obtain $r_1 \geqq 0$ and the theorem is demonstrated.

From this theorem and from the relations between entire characteristic functions and ridge functions (see [1], Chapter 4), we deduce almost immediately the

COROLLARY. *Let f be the product of two Poisson-type characteristic functions of two variables $t = (t_1, t_2)$ defined by (1.2). If the vectors $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$ satisfy one of the conditions of the Theorem 1, f has no indecomposable factor.*

REMARK. The example of the Section 2 satisfies the condition

$$0 < \alpha_1 < \beta_1, \qquad 0 < \beta_2 < \alpha_2, \qquad \beta_1\beta_2 + \alpha_1\alpha_2 - \beta_1\alpha_2 = 0.$$

4. The case n arbitrary.

THEOREM 2. *Let φ_1 and φ_2 be two ridge functions of the n variables $z = (z_1, \cdots, z_n)$ such that*

$$\varphi_1(z)\varphi_2(z) = \exp\{\pi(z) + \textstyle\sum_\epsilon [\lambda_{\epsilon_1,\cdots,\epsilon_n} e^{\epsilon_1\alpha_1 z_1+\cdots+\epsilon_n\alpha_n z_n} + \mu_{\epsilon_1,\cdots,\epsilon_n} e^{\epsilon_1\beta_1 z_1+\cdots+\epsilon_n\beta_n z_n}]\}$$

where π is a polynomial of degree one, $\alpha = (\alpha_1, \cdots, \alpha_n)$ and $\beta = (\beta_1, \cdots, \beta_n)$ are real vectors (we may suppose, without loss of generality, that β is positive), the λ and μ are non-negative constants, $\epsilon_j = 0$ or 1 ($j = 1, \cdots, n$) and $\sum_\epsilon$ indicates the summation on the $2^n - 1$ values of $\epsilon = (\epsilon_1, \cdots, \epsilon_n)$ different from $(0, \cdots, 0)$. If for all couples of indices (j, k), $\alpha_j, \beta_j, \alpha_k, \beta_k$ satisfy one the conditions of the Theorem 1, then

$$\varphi_1(z) = \exp\{P(z) + \textstyle\sum_\epsilon [l_{\epsilon_1,\cdots,\epsilon_n} e^{\epsilon_1\alpha_1 z_1+\cdots+\epsilon_n\alpha_n z_n} + m_{\epsilon_1,\cdots,\epsilon_n} e^{\epsilon_1\beta_1 z_1+\cdots+\epsilon_n\beta_n z_n}]\}$$

where P, the l and m have respectively the same properties as π, the λ and μ.

PROOF. We proceed by induction and suppose that this theorem is true for the ridge functions of $k(<n)$ variables. We suppose also that n is greater than 2.

If we fix z_1 real, we may apply the induction hypothesis and obtain the representation

$$\begin{aligned}(4.1)\quad \varphi_1(z) = \exp\{a &+ \textstyle\sum_{j=2}^n b_j z_j \\ &+ \textstyle\sum_{\epsilon'} [\xi_{\epsilon'} e^{\epsilon_2\alpha_2 z_2+\cdots+\epsilon_n\alpha_n z_n} + \eta_{\epsilon'} e^{\epsilon_2\beta_2 z_2+\cdots+\epsilon_n\beta_n z_n}]\}\end{aligned}$$

where a, b_j, $\xi_{\epsilon'}$, $\eta_{\epsilon'}$ are functions of z_1, real for z_1 real and where $\sum_{\epsilon'}$ indicates a summation on the $2^{n-1} - 1$ values of $\epsilon' = (\epsilon_2, \cdots, \epsilon_n)$ different from $(0, \cdots, 0)$.

If we fix $z_2, \cdots, z_n$ real, we may apply the Ostrovskiy's theorem cited above and obtain

$$(4.2)\qquad \varphi_1(z) = \exp\{\gamma + \delta z_1 + \rho e^{\alpha_1 z_1} + \sigma e^{\beta_1 z_1}\}$$

where γ, δ, ρ, σ are functions of the variables $z_2, \cdots, z_n$, real for $z_2, \cdots, z_n$ real.

If we compare (4.1) and (4.2), we obtain an equation which can be solved by a use of the linear independence of polynomials and exponentials. We obtain for $z_1, \cdots z_n$ real the representation

$$\begin{aligned}\varphi_1(z) = \exp\{&P(z) + \textstyle\sum_{k=2}^n C_k z_1 z_k \\ &+ \textstyle\sum_\epsilon [l_\epsilon e^{\epsilon_1\alpha_1 z_1+\cdots+\epsilon_n\alpha_n z_n} + m_\epsilon e^{\epsilon_1\beta_1 z_1+\cdots+\epsilon_n\beta_n z_n}] \\ (4.3)\qquad &+ \textstyle\sum_{\epsilon'} [p_{\epsilon'} e^{\alpha_1 z_1+\epsilon_2\beta_2 z_2+\cdots+\epsilon_n\beta_n z_n} + q_{\epsilon'} e^{\beta_1 z_1+\epsilon_2\alpha_2 z_2+\cdots+\epsilon_n\alpha_n z_n} \\ &+ r_{\epsilon'} z_1 e^{\epsilon_2\alpha_2 z_2+\cdots+\epsilon_n\alpha_n z_n} + s_{\epsilon'} z_1 e^{\epsilon_2\beta_2 z_2+\cdots+\epsilon_n\beta_n z_n}] \\ &+ \textstyle\sum_{k=2}^n [v_k z_k e^{\alpha_1 z_1} + w_k z_k e^{\beta_1 z_1}]\}\end{aligned}$$

where P is a polynomial of degree one and where all the constants C_k, l_ϵ, m_ϵ,

$p_{\epsilon'}$, $q_{\epsilon'}$, $r_{\epsilon'}$, $s_{\epsilon'}$, v_k, w_k are real. Since in the two members of (4.3) we have entire functions of $z_1, \cdots, z_n$, the representation (4.3) is also valid for $z_1, \cdots z_n$ complex.

If we take all the variables other than z_1 and z_j fixed and real, we may apply the Theorem 1 and obtain the representation

$$(4.4)\quad \varphi_1(z) = \exp\{\tau + \omega z_1 + \omega' z_j + f_1 e^{\alpha_1 z_1} + g_1 e^{\alpha_j z_j} + h_1 e^{\alpha_1 z_1 + \alpha_j z_j} + f_2 e^{\beta_1 z_1} + g_2 e^{\beta_j z_j} + h_2 e^{\beta_1 z_1} e^{\beta_1 z_1 + \beta_j z_j}\}.$$

Comparing (4.3) and (4.4), we obtain from the independence of polynomials and exponentials

$$C_j = 0, \qquad v_j = 0, \qquad w_j = 0$$

and

$$p_{\epsilon'} = q_{\epsilon'} = r_{\epsilon'} = s_{\epsilon'} = 0$$

for all ϵ' such that $\epsilon_j = 1$. Since j is arbitrary, we obtain finally the representation

$$(4.5)\quad \varphi_1(z) = \exp\{P(z) + \textstyle\sum_\epsilon [l_\epsilon e^{\epsilon_1 \alpha_1 z_1 + \cdots + \epsilon_n \alpha_n z_n} + m_\epsilon e^{\epsilon_1 \beta_1 z_1 + \cdots + \epsilon_n \beta_n z_n}]\}.$$

It remains to demonstrate that

$$(4.6)\qquad l_\epsilon \geqq 0, \qquad m_\epsilon \geqq 0.$$

Fixing z_j, we may apply again the induction hypothesis and obtain the representation

$$\begin{aligned}\varphi_1(z) = \exp\{&P_j(z) \\ &+ \textstyle\sum_{\epsilon''} [l_{\epsilon''} \exp(\epsilon_1\alpha_1 z_1 + \cdots + \epsilon_{j-1}\alpha_{j-1}z_{j-1} + \epsilon_{j+1}\alpha_{j+1}z_{j+1} + \cdots + \epsilon_n\alpha_n z_n) \\ &+ m_{\epsilon''} \exp[\epsilon_1\beta_1 z_1 + \cdots + \epsilon_{j-1}\beta_{j-1}z_{j-1} + \epsilon_{j+1}\beta_{j+1}z_{j+1} + \cdots + \epsilon_n\beta_n z_n)]\}\end{aligned}$$

with evident notations. Moreover

$$l_{\epsilon''} \geqq 0, \qquad m_{\epsilon''} \geqq 0.$$

Since, from (4.5)

$$l_{\epsilon''} = l_{\epsilon_1,\epsilon_2,\cdots,\epsilon_{j-1},0,\epsilon_{j+1},\cdots,\epsilon_n} + l_{\epsilon_1,\epsilon_2,\cdots,\epsilon_{j-1},1,\epsilon_{j+1},\cdots,\epsilon_n} e^{\alpha_j z_j}$$
$$m_{\epsilon''} = m_{\epsilon_1,\epsilon_2,\cdots,\epsilon_{j-1},0,\epsilon_{j+1},\cdots,\epsilon_n} + m_{\epsilon_1,\epsilon_2,\cdots,\epsilon_{j-1},1,\epsilon_{j+1},\cdots,\epsilon_n} e^{\beta_j z_j}$$

we obtain easily (4.6).

From this theorem, we obtain easily the

COROLLARY. *Let f be the product of two Poisson-type characteristic functions of the n variables $t = (t_1, \cdots t_n)$ defined by* (1.1). *If for all couples of indices (j, k), α_j, β_j, α_k, β_k satisfy one of the conditions of the Theorem* 1, *f has no indecomposable factor.*

5. Other results. With the method used here, we can also deduce from the corresponding results of Ostrovskiy in the case of one variable ([5]) the following results on the finite products of Poisson-type characteristic functions.

THEOREM 3. *Let f be the product of three Poisson-type characteristic functions defined by*

$$f(t) = f(t_1, \cdots, t_n) = \exp \{iP(t) + \textstyle\sum_{j=1}^{3} \sum_{\epsilon} \lambda_{j,\epsilon_1,\cdots,\epsilon_n}[e^{i(\alpha_{j,1}\epsilon_1 t_1+\cdots+\alpha_{j,n}\epsilon_n t_n)} - 1]\}.$$

If for $k = 1, 2, \cdots n$, the $\alpha_{j,k}$ satisfy one of the following conditions

(a) $\alpha_{1,k} < 0, \alpha_{3,k} > 0, 0 < \alpha_{2,k} < \min(\alpha_{3,k}, -\alpha_{1,k})$;

(b) $\alpha_{1,k} < 0, \alpha_{3,k} > 0, 0 > \alpha_{2,k} > \max(-\alpha_{3,k}, \alpha_{1,k})$;

(c) $0 < \alpha_{1,k} < \alpha_{2,k} < \min(2\alpha_{1,k}, \alpha_{3,k})$;

(d) $0 > \alpha_{1,k} > \alpha_{2,k} > \max(2\alpha_{1,k}, \alpha_{3,k})$

then f has no indecomposable factor.

THEOREM 4. *Let f be the product of four Poisson-type characteristic functions defined by*

$$f(t) = f(t_1, \cdots, t_n) = \exp \{iP(t) + \textstyle\sum_{j=1}^{4} \sum_{\epsilon} \lambda_{j,\epsilon_1,\cdots,\epsilon_n}[e^{i(\alpha_{j,1}\epsilon_1 t_1+\cdots+\alpha_{j,n}\epsilon_n t_n)} - 1)]\}.$$

If for $k = 1, 2, \cdots, n$, the $\alpha_{j,k}$ satisfy the following condition: There exist integer numbers m_k and n_k and incommensurable numbers $\rho_k > 0$, $\sigma_k > 0$ such that $\alpha_{1,k} = (n_k + 1)\sigma_k$, $\alpha_{2,k} = n_k\sigma_k$, $\alpha_{3,k} = m_k\rho_k$, $\alpha_{4,k} = (m_k + 1)\rho_k$ and $\max\{(m_k - 1)\rho_k, n_k\sigma_k\} < \min\{m_k\rho_k, (n_k - 1)\sigma_k\}$ *then f has no indecomposable factor.*

We prove also the

THEOREM 5. *Let f be the product of p Poisson-type characteristic functions of the n variables $t = (t_1, \cdots, t_n)$ defined by*

$$(5.1)\quad f(t) = \exp \{i\pi(t) + \textstyle\sum_{j=1}^{p} \sum_{\epsilon} [\lambda_{j,\epsilon_1,\cdots,\epsilon_n}(e^{i(\epsilon_1\alpha_{j,1}t_1+\cdots+\epsilon_n\alpha_{j,n}t_n)} - 1)]\}$$

where π is an homogeneous polynomial of degree one, $\lambda_{j,\epsilon_1,\cdots,\epsilon_n}$ are non-negative constants, the vectors $(\alpha_{j,1}, \cdots, \alpha_{j,n})$ are real ($j = 1, \cdots, p$), $\epsilon_j = 0$ or 1 and $\sum_{\epsilon}$ indicates the summation on the $2^n - 1$ values of $(\epsilon_1, \cdots, \epsilon_n)$ different from $(0, \cdots, 0)$. If for $k = 1, \cdots, n$, the components $\alpha_{1,k}, \cdots, \alpha_{p,k}$ are rationally independent then f has no indecomposable factor.

PROOF. The theorem in the case $n = 1$ has been obtained by P. Lévy ([3], p. 56). It is sufficient to demonstrate it in the case $n = 2$, the transition of these cases to the case n arbitrary following the same lines as the proof of Theorem 2. In the case $n = 2$, we change our notations and write (5.1) as

$$(5.2)\quad f(t) = f(t_1, t_2) = \exp \{i\pi(t) + \textstyle\sum_{j=1}^{p} [\lambda_j(e^{i\alpha_j t_1} - 1) + \mu_j(e^{i\beta_j t_2} - 1) + \nu_j(e^{i(\alpha_j t_1+\beta_j t_2)} - 1)]\}$$

($\lambda_j, \mu_j, \nu_j \geqq 0$). Let f_1 and f_2 be two characteristic functions such that for t_1 and t_2 real

$$(5.3)\quad f(t_1, t_2) = f_1(t_1, t_2)f_2(t_1, t_2).$$

We show that f_1 satisfies a relation of the kind (5.2). From the Theorem 2.3 of

[1], it follows that f_1 and f_2 are entire characteristic functions and that (5.3) is satisfied for any t_1 and t_2 complex. On the other hand, it follows from the Theorem 2.8 of [1] that f is defined by (5.2) for t_1 and t_2 complex. We use also the following lemma which is an evident consequence of the Theorem 2.2 of [1].

LEMMA. *If f is an entire characteristic function and t_2^0 a real constant, the function $f_{t_2^0}$ defined by*

$$f_{t_2^0}(t_1) = f(t_1, it_2^0)/f(0, it_2^0)$$

is an entire characteristic function.

For the following, it is simpler to introduce now the "moment generating functions" φ and φ_j $(j = 1, 2)$ defined by

$$f(-iz) = \varphi(z); \qquad f_j(-iz) = \varphi_j(z).$$

If we fix z_2 real, applying the lemma and the theorem of P. Lévy, we obtain

$$\varphi_1(z) = \exp \{a + bz_1 + \textstyle\sum_{j=1}^{p} q_j e^{\alpha_j z_1}\} \tag{5.4}$$

where a, b and q_j are functions of z_2, real for z_2 real. Then fixing z_1 real, we obtain

$$\varphi_1(z) = \exp \{a' + b' z_2 + \textstyle\sum_{k=1}^{p} q_k' e^{\beta_k z_2}\} \tag{5.5}$$

where a', b' and q_k' are functions of z_1, real for z_1 real. From (5.4) and (5.5), we deduce as in the proof of the Theorem 1 the relation

$$\begin{aligned}\varphi_1(z) = \exp \{h + P(z) + cz_1z_2 + \textstyle\sum_{j=0}^{p} \sum_{k=0}^{p} n_{j,k} e^{\alpha_j z_1 + \beta_k z_2} \\ + \textstyle\sum_{j=1}^{p} s_j z_1 e^{\beta_j z_2} + t_j z_2 e^{\alpha_j z_1}\}\end{aligned} \tag{5.6}$$

where all the constants and coefficients of P are real with the convention $\alpha_0 = \beta_0 = n_{0,0} = 0$. If we introduce for $x, y \varepsilon R^2$ the function

$$u(x, y) = \text{Re}\, \varphi_1(x + iy)$$

we have the ridge property

$$u(x, 0) - u(x, y) \geqq 0 \tag{5.7}$$

and from this property, it follows as in the proof of Theorem 1 that

$$c = s_j = t_j = 0.$$

We have now

$$u(x, 0) - u(x, y) = 2 \textstyle\sum_{j=0}^{p} \sum_{k=0}^{p} n_{j,k} e^{\alpha_j x_1 + \beta_k x_2} \sin^2 \tfrac{1}{2}(\alpha_j y_1 + \beta_k y_2).$$

If we show the relations

$$n_{j,k} \geqq 0 \tag{5.8}$$

it follows from the remark made during the proof of the Theorem 1 that

$$n_{j,k} = 0$$

for $j \neq 0$, $k \neq 0$, $j \neq k$, and the theorem will be demonstrated, the value of h being determined by the condition $\varphi_1(0) = 1$.

We can suppose without loss of generality that

$$\alpha_1 > \alpha_2 > \cdots > \alpha_p, \qquad \beta_k > 0.$$

We suppose that the relation (5.8) is satisfied for $j = 1, \cdots, m - 1$ and we show that

$$n_{m,k} \geqq 0.$$

From the theorem of Kronecker ([2], Theorem 444), we may choose $y_2 = y_2(x_2)$ such that

$$\sin \tfrac{1}{2}(\beta_q y_2) = o(e^{-\frac{1}{2}(\beta_q x_2)}), \qquad q \neq k, \quad x_2 \to \infty$$

and

$$\sin \tfrac{1}{2}(\beta_k y_2) \geqq 1 - \epsilon.$$

Then, we choose $y_1 = y_1(x_2)$ such that when $x_2 \to +\infty$

$$\sin \tfrac{1}{2}(\alpha_j y_1 + \beta_j y_2) = o(e^{-\frac{1}{2}(\beta_j x_2)}), \qquad j = 1, \cdots, k - 1$$

and

$$\sin \tfrac{1}{2}(\alpha_j y_1 + \beta_q y_2) = o(e^{-\frac{1}{2}(\beta_q x_2)}), \qquad j = k, \cdots, n;\ q \neq k.$$

We have then when $x_2 \to \infty$

$$u(x, 0) - u(x, y) = O(e^{\beta_k x_2}(\textstyle\sum_{j=m}^{n} n_{j,k} e^{\alpha_j x_1} \sin^2 \tfrac{1}{2}(\alpha_j y_1 + \beta_k y_2))). \tag{5.9}$$

Comparing (5.7) and (5.9), we obtain if x_1 is great enough

$$n_{m,k} \geqq 0$$

and the theorem is demonstrated.

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