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VARIANCES OF VARIANCE-COMPONENT ESTIMATORS FOR THE UNBALANCED 2-WAY CROSS CLASSIFICATION WITH APPLICATION TO BALANCED INCOMPLETE BLOCK DESIGNS

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1. Introduction and summary. "Best" estimators of variance components for the unbalanced cases of random-effects models are not known. In fact, even for the very simplest of the unbalanced "designs", the balanced incomplete block designs, the question of the existence of minimum variance unbiased estimators remains open (Kapadia and Weeks [5]).

The traditional approach to the derivation of variance-component estimators for unbalanced cases has been to pick several quadratic functions of the data, set these functions equal to their expectations, and then solve the resulting system of equations for the variance components. Two of the estimators derived in this fashion for the variance components associated with the unbalanced two-way cross classification are those referred to as the Methods-1 and -3 estimators of Henderson [4]. Method-1 utilizes quadratics analogous to the sums of squares in a balanced analysis of variance. The quadratics employed in Method-3 represent differences between reductions in sums of squares due to fitting different models. Since in Method-3 more differences between reductions are available than one has variance components to estimate, the method is not uniquely defined. Here, the Method-3 estimators of the components associated with the two-way classification are taken to be those in Harville [3], which are the ones most commonly used.

Searle [9] obtained algebraic expressions for the sampling variances of the Method-1 estimators of the "two-way" components. Low [6] gave similar expressions for the Method-3 estimators for the zero-interaction case. Their results were obtained by applying well-known formulas for the variances and covariances of quadratic functions of multivariate-normal random variables. These formulas state that if $\mathbf{y}$ is a random vector having the multivariate normal distribution with mean $\boldsymbol{\mu}$ and variance-covariance matrix $\mathbf{V}$ and if $\mathbf{A}$ and $\mathbf{B}$ are square symmetric matrices of appropriate dimension having fixed elements, then

$$\text{var}\,[\mathbf{y}'\mathbf{A}\mathbf{y}] = 4\boldsymbol{\mu}'\mathbf{VA}\mathbf{V}\mathbf{A}\boldsymbol{\mu} + 2\,\text{tr}\,(\mathbf{VA})^2 \tag{1}$$

and

$$\text{cov}\,[\mathbf{y}'\mathbf{A}\mathbf{y}, \mathbf{y}'\mathbf{B}\mathbf{y}] = 4\boldsymbol{\mu}'\mathbf{AVB}\boldsymbol{\mu} + 2\,\text{tr}\,(\mathbf{VAVB}). \tag{2}$$

Searle [8], [10] and Mahamunulu [7] have also used these formulas to obtain algebraic expressions for the variances of commonly-used estimators of the components of variance associated with other unbalanced classifications.

In the present paper, results (supplementary to those of Searle) are given

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which lead to expressions for the sampling variances of Method-3 estimators of the variance components associated with the unbalanced two-way cross classification with interaction. By using these results in combination with those of Searle, the variances of Method-1 and Method-3 estimators can be directly compared for a given set of subclass numbers.

The results are shown to simplify when the "unbalancedness" is of the type associated with a balanced incomplete block design. Neither estimator of any component is uniformly better than the other for any such design. (Except for the estimators of the residual component which are identically equal.)

2. Preliminaries. n_{ij} will denote the number of observations in the ijth subclass. The observations y_{ijr} are taken as having the linear model

$$y_{ijr} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \epsilon_{ijr},$$

with $i = 1, \cdots, a; j = 1, \cdots, b$; and $r = 1, \cdots, n_{ij}$. μ is a general mean, the α_i and the β_j are main effects, the γ_{ij} are interaction effects, and the ϵ_{ijr} are residual effects. μ is regarded as fixed while the α_i, β_j, γ_{ij}, and ϵ_{ijr} are taken to be mutually-independent normal random variables with zero means and variances $\sigma_\alpha^2, \sigma_\beta^2, \sigma_\gamma^2$, and σ_ϵ^2.

Letting $n_{i\cdot} = \sum_j n_{ij}$, $n_{\cdot j} = \sum_i n_{ij}$, and $n_{\cdot\cdot} = \sum_i n_{i\cdot} = \sum_j n_{\cdot j}$ and using ordinary notation for means, take $R_0 = \sum_{ijr} y_{ijr}^2$, $R_\mu = n_{\cdot\cdot}\bar{y}_{\cdot\cdot\cdot}^2$, $R_\alpha = \sum_i n_{i\cdot}\bar{y}_{i\cdot\cdot}^2$, $R_\beta = \sum_j n_{\cdot j}\bar{y}_{\cdot j\cdot}^2$, and $R_\gamma = \sum_{ij} n_{ij}\bar{y}_{ij\cdot}^2$. **W** is taken to be a $b \times b$ matrix with elements

$$w_{jj} = n_{\cdot j} - \sum_i (n_{ij}^2/n_{i\cdot}), \qquad j = 1, \cdots, b,$$

and

$$w_{jr} = -\sum_i (n_{ij}n_{ir}/n_{i\cdot}), \qquad j \neq r = 1, \cdots, b.$$

m is defined to be the rank of **W**. (For most n_{ij}-patterns, $m = b - 1$; however, for certain designs that are not connected, $m < b - 1$.)

Take $\mathbf{W}_{11}$ to be the $m \times m$ matrix formed by deleting from **W** the last $b - m$ rows and columns. It can be assumed (without loss of generality) that this matrix is of full rank. Then, take

$$\hat{\boldsymbol{\beta}} = \mathbf{W}^* \mathbf{q};$$

where $\mathbf{q}$ is a $b \times 1$ vector with jth element

$$q_j = n_{\cdot j}\bar{y}_{\cdot j\cdot} - \sum_i n_{ij}\bar{y}_{i\cdot\cdot}$$

and, using **0** to represent any null matrix, the $b \times b$ matrix

$$\mathbf{W}^* = \begin{Vmatrix} \mathbf{W}_{11}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{Vmatrix}.$$

Now, take $R_{\alpha\beta} = R_\alpha + \hat{\boldsymbol{\beta}}' \mathbf{q}$.

The common value of the Method-1 and Method-3 estimators of σ_ϵ^2 is given by

$$\hat{\sigma}_\epsilon^2 = (R_0 - R_\gamma)/(n_{\cdot\cdot} - c)$$

where c denotes the total number of filled subclasses (subclasses such that $n_{ij} \geqq 1$). It is well known that

$$\text{var}[\hat{\sigma}_\epsilon^2] = 2\sigma_\epsilon^4/(n_{..} - c).$$

The Method-1 estimators of σ_α^2, σ_β^2, and σ_γ^2, which will be denoted by $\hat{\sigma}_\alpha^2$, $\hat{\sigma}_\beta^2$, and $\hat{\sigma}_\gamma^2$, respectively, are linear functions of $T_\alpha = R_\alpha - R_\mu$, $T_\beta = R_\beta - R_\mu$, $T_\gamma = R_\gamma - R_\alpha - R_\beta + R_\mu$, and $\hat{\sigma}_\epsilon^2$; and the Method-3 estimators, which will be denoted by $\tilde{\sigma}_\alpha^2, \tilde{\sigma}_\beta^2$, and $\tilde{\sigma}_\gamma^2$, are linear functions of $S_\alpha = R_{\alpha\beta} - R_\beta$, $S_\beta = R_{\alpha\beta} - R_\alpha$, $S_\gamma = R_\gamma - R_{\alpha\beta}$, and $\hat{\sigma}_\epsilon^2$, as in Harville [3]. Furthermore, by using the matrix notations introduced below and by applying Theorem 4.21 in Graybill [2], it can be readily shown that $\hat{\sigma}_\epsilon^2$ is distributed independently of T_α, T_β, T_γ, S_α, S_β, and S_γ. The variances and covariances of R_μ, R_α, R_β, and R_γ can be obtained from Searle's 1958 paper. [Actually, since Searle ignored the first term in the right hand sides of formulas (1) and (2), the expressions given by him represent the differences between these variances and covariances and the constant $\mu^2(\sigma_\alpha^2 \sum_i n_{i.}^2 + \sigma_\beta^2 \sum_j n_{.j}^2 + \sigma_\gamma^2 \sum_{ij} n_{ij}^2 + \sigma_\epsilon^2 n_{..})$. Nevertheless, the variances and covariances of T_α, T_β, and T_γ do not contain this term and consequently it can be disregarded in obtaining them from Searle's expressions.] Thus, to get expressions for the variances and covariances of the Method-3 estimators and for the covariances between Method-1 and -3 estimators, it suffices to derive the variance of S_β and its covariances with R_μ, R_α, R_β, and R_γ.

3. Variances and covariances. The procedure to be followed in obtaining expressions for the necessary variances and covariances will be to express R_μ, R_α, R_β, R_γ, and S_β as quadratic functions of the y_{ijr}'s using matrix notation; to then apply formulas (1) and (2); and finally to evaluate the right hand sides of these formulas.

Take $\mathbf{y}'$ to be the row vector of the $n_{..}$ y_{ijr}'s arrayed in r-order within j-classes within each i-class; i.e.,

$$\mathbf{y}' = (y_{111}, \cdots, y_{11n_{11}}, y_{121}, \cdots, y_{12n_{12}}, \cdots, y_{ab1}, \cdots, y_{abn_{ab}}).$$

Then,

$$(3) \qquad R_\mu = \mathbf{y}'\mathbf{Q}_\mu\mathbf{y}; \qquad R_\alpha = \mathbf{y}'\mathbf{Q}_\alpha\mathbf{y};$$
$$R_\beta = \mathbf{y}'\mathbf{Q}_\beta\mathbf{y}; \qquad R_\gamma = \mathbf{y}'\mathbf{Q}_\gamma\mathbf{y},$$

where $\mathbf{Q}_\mu$, $\mathbf{Q}_\alpha$, $\mathbf{Q}_\beta$, and $\mathbf{Q}_\gamma$ are $n_{..} \times n_{..}$ symmetric matrices which can be obtained from Searle's paper [9].

It is straightforward to show that

$$S_\beta = \sum_{ijv} \sum_{tsp} \phi_{it,js} y_{ijv} y_{tsp},$$

where, taking w_{js}^* to be the jsth element of the matrix $\mathbf{W}^*$ defined earlier,

$$\phi_{it,js} = \phi_{ti,sj} = w_{js}^* - \sum_r (n_{tr}/n_{t.}) w_{jr}^* - \sum_r (n_{ir}/n_{i.}) w_{sr}^*$$
$$+ \sum_{ru} (n_{ir}/n_{i.})(n_{tu}/n_{t.}) w_{ru}^*.$$

Thus,

$$S_\beta = \mathbf{y}'\mathbf{Q}_{\alpha\beta}\mathbf{y}, \tag{4}$$

where, taking $\mathbf{U}_{ij,ts}$ to be an $n_{ij} \times n_{ts}$ matrix with all elements equal to one, $\mathbf{Q}_{\alpha\beta}$ is an $n_{..} \times n_{..}$ symmetric matrix with $n_{ij} \times n_{ts}$ submatrices $\phi_{it,js}\mathbf{U}_{ij,ts}$.

The vector $\mathbf{y}$ has mean $\boldsymbol{\mu}$ and variance-covariance matrix $\mathbf{V}$, where $\boldsymbol{\mu}$ is an $n_{..} \times 1$ vector with all elements equal to μ and $\mathbf{V}$ is as given by Searle [9]. Thus, by using the matrix formulations (3) and (4) and applying formulas (1) and (2), matrix expressions for the variance of S_β and its covariances with R_μ, R_α, R_β, and R_γ can be readily obtained.

The necesary techniques for evaluating the resulting matrix expressions have been well illustrated by Searle. The first step is to carry out the matrix multiplications $\mathbf{VQ}_\mu$, $\mathbf{VQ}_\alpha$, $\mathbf{VQ}_\beta$, $\mathbf{VQ}_\gamma$, and $\mathbf{VQ}_{\alpha\beta}$. Searle has performed the multiplications for the first four of the products. Carrying out the multiplication for $\mathbf{VQ}_{\alpha\beta}$ gives an $n_{..} \times n_{..}$ matrix with $n_{ij} \times n_{ts}$ submatrices $\theta_{it,js}\mathbf{U}_{ij,ts}$, where

$$\theta_{it,js} = \sigma_\beta^2 \textstyle\sum_v n_{vj}\phi_{vt,js} + \sigma_\gamma^2 n_{ij}\phi_{it,js} + \sigma_\epsilon^2\phi_{it,js} .$$

It is straightforward to show that

$$\boldsymbol{\mu}'\mathbf{Q}_{\alpha\beta}\mathbf{V}\mathbf{Q}_{\alpha\beta}\boldsymbol{\mu} = \boldsymbol{\mu}'\mathbf{Q}_{\mu}\mathbf{V}\mathbf{Q}_{\alpha\beta}\boldsymbol{\mu} = \boldsymbol{\mu}'\mathbf{Q}_{\alpha}\mathbf{V}\mathbf{Q}_{\alpha\beta}\boldsymbol{\mu} = \boldsymbol{\mu}'\mathbf{Q}_{\beta}\mathbf{V}\mathbf{Q}_{\alpha\beta}\boldsymbol{\mu} = \boldsymbol{\mu}'\mathbf{Q}_{\gamma}\mathbf{V}\mathbf{Q}_{\alpha\beta}\boldsymbol{\mu} = 0.$$

Then,

$$\begin{aligned}
\operatorname{var}[S_\beta] &= 2\operatorname{tr}(\mathbf{VQ}_{\alpha\beta})^2 = 2\textstyle\sum_{it}\sum_{jp} n_{ij}n_{tp}\theta_{it,jp}\theta_{ti,pj} ;\\
\operatorname{cov}[R_\mu, S_\beta] &= 2\operatorname{tr}(\mathbf{VQ}_{\alpha\beta}\mathbf{VQ}_\mu)\\
&= (2/n_{..})\textstyle\sum_{it}\sum_{jp} n_{ij}n_{tp}\theta_{it,jp}(n_{t.}\sigma_\alpha^2 + n_{.p}\sigma_\beta^2 + n_{tp}\sigma_\gamma^2 + \sigma_\epsilon^2);\\
\operatorname{cov}[R_\alpha, S_\beta] &= 2\operatorname{tr}(\mathbf{VQ}_{\alpha\beta}\mathbf{VQ}_\alpha)\\
&= 2\{\textstyle\sum_i\sum_{jp} n_{ij}n_{ip}\theta_{ii,jp}[\sigma_\alpha^2 + (n_{ip}/n_{i.})\sigma_\gamma^2 + (1/n_{i.})\sigma_\epsilon^2]\\
&\quad + \sigma_\beta^2\textstyle\sum_{it}\sum_{jp} n_{ip}n_{tj}(n_{tp}/n_{t.})\theta_{ti,jp}\} ;
\end{aligned}$$

$$\begin{aligned}
\operatorname{cov}[R_\beta, S_\beta] &= 2\operatorname{tr}(\mathbf{VQ}_{\alpha\beta}\mathbf{VQ}_\beta)\\
&= 2\{[\textstyle\sum_{it}\sum_{j,p\neq j} n_{ij}n_{tp}(n_{tj}/n_{.j})\theta_{it,jp}\\
&\quad + \textstyle\sum_{it}\sum_j n_{ij}^2(n_{tj}/n_{.j})\theta_{it,jj}]\sigma_\alpha^2\\
&\quad + [\textstyle\sum_{it}\sum_j n_{ij}n_{tj}\theta_{it,jj}]\sigma_\beta^2 + [\sum_{it}\sum_j n_{ij}^2(n_{tj}/n_{.j})\theta_{it,jj}]\sigma_\gamma^2\\
&\quad + [\textstyle\sum_{it}\sum_j n_{ij}(n_{tj}/n_{.j})\theta_{it,jj}]\sigma_\epsilon^2\} ;
\end{aligned} \tag{5}$$

and

$$\begin{aligned}
\operatorname{cov}[R_\gamma, S_\beta] &= 2\operatorname{tr}(\mathbf{VQ}_{\alpha\beta}\mathbf{VQ}_\gamma)\\
&= 2\{[\textstyle\sum_i\sum_{jp} n_{ij}n_{ip}\theta_{ii,jp}]\sigma_\alpha^2 + [\sum_{it}\sum_j n_{ij}n_{tj}\theta_{it,jj}]\sigma_\beta^2\\
&\quad + \textstyle\sum_i\sum_j n_{ij}\theta_{ii,jj}(n_{ij}\sigma_\gamma^2 + \sigma_\epsilon^2)\}.
\end{aligned}$$

4. Balanced incomplete block designs. Now take the pattern of filled subclasses to be one associated with some balanced incomplete block design and take the number of observations per filled subclass to be a constant, say n.

Accordingly, $\sum_i (n_{ip}/n)$, $\sum_i (n_{ip}/n)(n_{ir}/n)$, and $\sum_j (n_{tj}/n)$ are constants when regarded as functions of p, r, and t, and their values will be denoted by s, λ, and k, respectively. Also, for $t \neq i$, set

$$\delta_{it} = \sum_j (n_{ij}/n)(n_{tj}/n).$$

The following, well-known properties of balanced incomplete block designs will be needed in the sequel: (i) $ak = bs$; (ii) $b > k \geqq 2$; (iii) $a > s \geqq 2$; (iv) $\lambda = s(k-1)/(b-1)$; (v) $s > \lambda$; (vi) $s \geqq k$ or, equivalently, $a \geqq b$; (vii) $\sum_{t \neq i} \delta_{it} = k(s-1)$, for all i; and (viii) $\sum_{t \neq i} \delta_{it}^2 = k[s - 1 + (k-1)(\lambda - 1)]$, for all i. Properties (vi), (vii), and (viii) were noted by Fisher [1].

Now, $m = b - 1$ and $\mathbf{W}_{11}^{-1}$ has diagonal elements $2k/(\lambda bn)$ and off-diagonal elements $k/(\lambda bn)$. Using this result, we obtain, for $t \neq i$ and $p \neq j$,

$$\begin{aligned}
\phi_{ii,jj} &= (k+1)/(\lambda bn), && n_{ij} = 0,\\
&= (k-1)/(\lambda bn), && n_{ij} = n;\\
\phi_{ii,jp} &= 1/(\lambda bn), && n_{ij} = n_{ip} = 0,\\
&= -1/(\lambda bn), && n_{ij} = n_{ip} = n,\\
&= 0, && \text{otherwise};\\
\phi_{it,jj} &= (k^2 + \delta_{it})/(\lambda bkn), && n_{ij} = n_{tj} = 0,\\
&= (k^2 + \delta_{it} - 2k)/(\lambda bkn), && n_{ij} = n_{tj} = n,\\
&= (k^2 + \delta_{it} - k)/(\lambda bkn), && \text{otherwise};\\
\phi_{it,jp} &= \delta_{it}/(\lambda bkn), && n_{ip} = n_{tj} = 0,\\
&= (\delta_{it} - 2k)/(\lambda bkn), && n_{ip} = n_{tj} = n,\\
&= (\delta_{it} - k)/(\lambda bkn), && \text{otherwise}.
\end{aligned}$$

Based on the above, we have (still taking $t \neq i$ and $p \neq j$)

$$\begin{aligned}
\theta_{ii,jj} &= \sigma_\beta^2 + [(k+1)/(\lambda bn)]\sigma_\epsilon^2, && n_{ij} = 0,\\
&= [(k-1)/k]\sigma_\beta^2 + [(k-1)/(\lambda bn)](n\sigma_\gamma^2 + \sigma_\epsilon^2), && n_{ij} = n;\\
\theta_{ii,jp} &= [1/(\lambda bn)]\sigma_\epsilon^2, && n_{ij} = n_{ip} = 0,\\
&= 0, && n_{ij} = 0, n_{ip} = n,\\
&= -(1/k)\sigma_\beta^2, && n_{ij} = n,\ \ n_{ip} = 0,\\
&= -(1/k)\sigma_\beta^2 - [1/(\lambda bn)](n\sigma_\gamma^2 + \sigma_\epsilon^2), && n_{ij} = n_{ip} = n;
\end{aligned}$$

$$
\begin{aligned}
\theta_{it,jj} &= \sigma_\beta^2 + [(k^2 + \delta_{it})/(\lambda bkn)]\sigma_\epsilon^2, & n_{ij} = n_{tj} = 0,\\
&= [(k-1)/k]\sigma_\beta^2 + [(k^2 + \delta_{it} - k)/(\lambda bkn)]\sigma_\epsilon^2, & n_{ij} = 0, \quad n_{tj} = n,\\
&= \sigma_\beta^2 + [(k^2 + \delta_{it} - k)/(\lambda bkn)](n\sigma_\gamma^2 + \sigma_\epsilon^2), & n_{ij} = n, \quad n_{tj} = 0,\\
&= [(k-1)/k]\sigma_\beta^2 + [(k^2 + \delta_{it} - 2k)/(\lambda bkn)](n\sigma_\gamma^2 + \sigma_\epsilon^2), & n_{ij} = n_{tj} = n;\\
\theta_{it,jp} &= [\delta_{it}/(\lambda bkn)]\sigma_\epsilon^2, & n_{ij} = n_{ip} = n_{tj} = 0,\\
&= -(1/k)\sigma_\beta^2 + [(\delta_{it} - k)/(\lambda bkn)]\sigma_\epsilon^2, & n_{ij} = n_{ip} = 0, \quad n_{tj} = n,\\
&= [(\delta_{it} - k)/(\lambda bkn)]\sigma_\epsilon^2, & n_{ij} = n_{tj} = 0, \quad n_{ip} = n,\\
&= -(1/k)\sigma_\beta^2 + [(\delta_{it} - 2k)/(\lambda bkn)]\sigma_\epsilon^2, & n_{ij} = 0, \quad n_{ip} = n_{tj} = n,\\
&= [\delta_{it}/(\lambda bkn)](n\sigma_\gamma^2 + \sigma_\epsilon^2), & n_{ij} = n, \quad n_{ip} = n_{tj} = 0,\\
&= -(1/k)\sigma_\beta^2 + [(\delta_{it} - k)/(\lambda bkn)](n\sigma_\gamma^2 + \sigma_\epsilon^2), & n_{ij} = n_{tj} = n, \quad n_{ip} = 0,\\
&= [(\delta_{it} - k)/(\lambda bkn)](n\sigma_\gamma^2 + \sigma_\epsilon^2), & n_{ij} = n_{ip} = n, \quad n_{tj} = 0,\\
&= -(1/k)\sigma_\beta^2 + [(\delta_{it} - 2k)/(\lambda bkn)](n\sigma_\gamma^2 + \sigma_\epsilon^2), & n_{ij} = n_{ip} = n_{tj} = n.
\end{aligned}
$$

Upon substituting these θ-values into the general expressions (5) and after some algebraic manipulation, we find

$$
\begin{aligned}
\operatorname{var}[S_\beta] &= 2n^2(b-1)[(a\lambda/s)\sigma_\beta^2 + (\sigma_\gamma^2 + \sigma_\epsilon^2/n)]^2,\\
\operatorname{cov}[R_\mu, S_\beta] &= 0,\\
\operatorname{cov}[R_\alpha, S_\beta] &= 2n^2a[(k-1)/k](s-\lambda)\sigma_\beta^4,\\
\operatorname{cov}[R_\beta, S_\beta] &= 2n^2(a/s)(k-1)[s\sigma_\beta^2 + (\sigma_\gamma^2 + \sigma_\epsilon^2/n)]^2,
\end{aligned}
$$

and

$$\operatorname{cov}[R_\gamma, S_\beta] = \operatorname{var}[S_\beta] + 2n^2(a/s)(a-s)\lambda\sigma_\beta^4.$$

The general expressions for the variances and covariances of R_μ, R_α, R_β, and R_γ, which were given by Searle [9], also simplify for designs of the type described above, but since the simplifications are very elementary and straightforward, they will not be given here.

Also, the equations for the Method-1 and -3 estimators of the components now have the simple forms

$$\hat{\sigma}_\epsilon^2 = [R_0 - R_\gamma]/[ak(n-1)],$$

$$
\left\|\begin{matrix}\hat{\sigma}_\alpha^2\\ \hat{\sigma}_\beta^2\\ \hat{\sigma}_\gamma^2\end{matrix}\right\| = \mathbf{C}\left\|\begin{matrix}T_\alpha - (a-1)\hat{\sigma}_\epsilon^2\\ T_\beta - (b-1)\hat{\sigma}_\epsilon^2\\ T_\gamma - (ak - a - b + 1)\hat{\sigma}_\epsilon^2\end{matrix}\right\|,
$$

and

$$\begin{Vmatrix} \tilde{\sigma}_\alpha^2 \\ \tilde{\sigma}_\beta^2 \\ \tilde{\sigma}_\gamma^2 \end{Vmatrix} = \mathbf{D} \begin{Vmatrix} S_\alpha - (a-1)\hat{\sigma}_\epsilon^2 \\ S_\beta - (b-1)\hat{\sigma}_\epsilon^2 \\ S_\gamma - (ak-a-b+1)\hat{\sigma}_\epsilon^2 \end{Vmatrix};$$

where

$$\mathbf{C} = (1/\rho_c) \begin{Vmatrix} a(b-1)(s-1)(k-1) & -b(a-s)(s-1) & -a(b-1)(s-1) \\ -a(b-k)(k-1) & b(a-1)(s-1)(k-1) & -b(a-1)(k-1) \\ a(b-k)(k-1) & b(a-s)(s-1) & [ks(a-1)(b-1)-(b-k)(a-s)] \end{Vmatrix}$$

and

$$\mathbf{D} = (1/\rho_d) \begin{Vmatrix} a(k-1)(ak-a-b+1) & 0 & -a(k-1)(a-1) \\ 0 & b(s-1)(ak-a-b+1) & -b(b-1)(s-1) \\ 0 & 0 & ab(s-1)(k-1) \end{Vmatrix},$$

with $\rho_c = abn(k-1)(s-1)(ak-k-s+1)$ and $\rho_d = abn(k-1)(s-1) \cdot (ak-a-b+1)$.

Set $\tau = ak - k - s + 1$ and $\chi = ak - a - b + 1$. Straightforward, though (in some cases) lengthy and tedious, algebraic manipulations now give

$$\begin{aligned}
\text{var}\,[\hat{\sigma}_\epsilon^2] &= 2\sigma_\epsilon^4/[ak(n-1)], \\
\text{var}\,[\tilde{\sigma}_\gamma^2] &= 2(\sigma_\gamma^2 + \sigma_\epsilon^2/n)^2/\chi + 2\sigma_\epsilon^4/[akn^2(n-1)], \\
\text{var}\,[\hat{\sigma}_\gamma^2] &= 2\sigma_\alpha^4 k^2(a-1)(a-b)(b-k)^2/[b^2\tau^2(b-1)(s-1)^2] \\
&\quad + 4\sigma_\alpha^2\sigma_\beta^2 k(a-s)/\tau^2 \\
&\quad + 2(\sigma_\gamma^2 + \sigma_\epsilon^2/n)^2(k\tau - b + k)/[b\tau(k-1)(s-1)] \\
&\quad + 4\sigma_\alpha^2(\sigma_\gamma^2 + \sigma_\epsilon^2/n)k(a-1)(a-s)/[a\tau^2(s-1)] \\
&\quad + 4\sigma_\beta^2(\sigma_\gamma^2 + \sigma_\epsilon^2/n)s(b-1)(b-k)/[b\tau^2(k-1)] \\
&\quad + 2\sigma_\epsilon^4/[akn^2(n-1)],
\end{aligned}$$

$$(6)\quad \begin{aligned} \operatorname{var}[\tilde{\sigma}_\beta^2] &= 2\sigma_\beta^4/[a(k-1)] \\ &\quad + 2\chi[\sigma_\beta^2 + (b-1)(\sigma_\gamma^2 + \sigma_\epsilon^2/n)/\chi]^2/[a(k-1)(b-1)], \\ \operatorname{var}[\hat{\sigma}_\beta^2] &= 2\sigma_\alpha^4 k^2(a-1)(a-b)(b-k)^2/[b^2\tau^2(b-1)(s-1)^2] \\ &\quad + 2\sigma_\beta^4/(b-1) \\ &\quad + 4\sigma_\alpha^2\sigma_\beta^2 k(a-s)/\tau^2 + 2(\sigma_\gamma^2 + \sigma_\epsilon^2/n)^2(a-1)/[a\tau(s-1)] \\ &\quad + 4\sigma_\alpha^2(\sigma_\gamma^2 + \sigma_\epsilon^2/n)k(a-1)(a-s)/[a\tau^2(s-1)] \\ &\quad + 4\sigma_\beta^2(\sigma_\gamma^2 + \sigma_\epsilon^2/n)[a\tau - (ak-a-k+1)]/(a\tau^2), \\ \operatorname{var}[\tilde{\sigma}_\alpha^2] &= \{2/[b(s-1)]\}\{\sigma_\alpha^4[k(s-1)(b-1) \\ &\quad - (k-1)(b-k)]/[(s-1)(b-1)] \\ &\quad + (\sigma_\gamma^2 + \sigma_\epsilon^2/n)^2(a-1)/\chi + 2\sigma_\alpha^2(\sigma_\gamma^2 + \sigma_\epsilon^2/n)\}, \end{aligned}$$

and

$$\begin{aligned} \operatorname{var}[\hat{\sigma}_\alpha^2] &= 2\sigma_\alpha^4[k\tau(b-1)^2 + k(b-k)(b-1) \\ &\quad - (b-k)(k-1)]/[b\tau^2(b-1)] \\ &\quad + 4\sigma_\alpha^2\sigma_\beta^2 k(a-s)/\tau^2 + 2(\sigma_\gamma^2 + \sigma_\epsilon^2/n)^2(b-1)/[b\tau(k-1)] \\ &\quad + 4\sigma_\alpha^2(\sigma_\gamma^2 + \sigma_\epsilon^2/n)[b\tau - (bs-b-s+1)]/(b\tau^2) \\ &\quad + 4\sigma_\beta^2(\sigma_\gamma^2 + \sigma_\epsilon^2/n)k(b-1)(a-s)/[b\tau^2(k-1)]. \end{aligned}$$

The differences between the variances of the Method-1 and -3 estimators are given by

$$(7)\quad \begin{aligned} \operatorname{var}[\hat{\sigma}_\gamma^2] - \operatorname{var}[\tilde{\sigma}_\gamma^2] &= 2\sigma_\alpha^4 k^2(a-1)(a-b)(b-k)^2/[b^2\tau^2(b-1)(s-1)^2] \\ &\quad + 4\sigma_\alpha^2\sigma_\beta^2 k(a-s)/\tau^2 \\ &\quad - 2(\sigma_\gamma^2 + \sigma_\epsilon^2/n)^2(a-s)[(b-1)(s-1) \\ &\quad + (a-1)(k-1)]/[a\tau\chi(k-1)(s-1)] \\ &\quad + 4\sigma_\alpha^2(\sigma_\gamma^2 + \sigma_\epsilon^2/n)k(a-1)(a-s)/[a\tau^2(s-1)] \\ &\quad + 4\sigma_\beta^2(\sigma_\gamma^2 + \sigma_\epsilon^2/n)s(b-1)(b-k)/[b\tau^2(k-1)], \\ \operatorname{var}[\hat{\sigma}_\beta^2] - \operatorname{var}[\tilde{\sigma}_\beta^2] &= \operatorname{var}[\hat{\sigma}_\gamma^2] - \operatorname{var}[\tilde{\sigma}_\gamma^2] \\ &\quad - 8\sigma_\beta^2(\sigma_\gamma^2 + \sigma_\epsilon^2/n)(b-k)/[b\tau(k-1)], \end{aligned}$$

and

$$\begin{aligned} \operatorname{var}[\hat{\sigma}_\alpha^2] - \operatorname{var}[\tilde{\sigma}_\alpha^2] &= \operatorname{var}[\hat{\sigma}_\gamma^2] - \operatorname{var}[\tilde{\sigma}_\gamma^2] \\ &\quad - \{4(b-k)/[b\tau(s-1)]\}\{2\sigma_\alpha^2(\sigma_\gamma^2 + \sigma_\epsilon^2/n) \\ &\quad + \sigma_\alpha^4 k(a-b)(b-k)/[b(b-1)(s-1)]\}. \end{aligned}$$

It is clear from the above that the differences between the variances of the estimators are quadratic functions of the variance components and that each coefficient of each function has the same sign for every balanced incomplete block design. It is then also clear that neither estimator of any of the three components σ_α^2, σ_β^2, and σ_γ^2 is uniformly better than the other for any such design. One interesting property of the first difference function given above is that it is an increasing function of both σ_α^2 and σ_β^2. The second difference function is an increasing function of σ_α^2.

The simplified expressions (6) for the variances of the estimators are also valid for balanced data (data such that $n_{ij} = n$ for all i and j) if we replace s and λ by a and replace k by b. For the balanced case, the Method-1 and -3 estimators are identically equal. Thus, one check on the corectness of the results of this paper is to verify that the expressions (7) are identically equal to zero when the above substitutions are made. It is easy to show that this condition is indeed satisfied.

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