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NOTE ON WEYL'S CRITERION AND THE UNIFORM DISTRIBUTION OF INDEPENDENT RANDOM VARIABLES

BY P. J. HOLEWIJN

University of Technology, Delft

1. Introduction. Let $x_1, x_2, \cdots$ be a sequence of real numbers and let $\{x_k\}$ be the value of x_k modulo 1, $k = 1, 2, \cdots$. Let χ_x stand for the indicator function of the interval $0 \leqq y < x$ for $x \varepsilon [0, 1)$. Then the sequence $x_1, x_2, \cdots$ is termed uniformly distributed modulo 1 iff $\lim_{N\to\infty} N^{-1}\sum_{k\leqq N} \chi_x(\{x_k\}) = x$ for each $x \varepsilon [0, 1)$. The well-known Weyl criterion states that the sequence $x_1, x_2, \cdots$ is uniformly distributed (mod 1) iff $\lim_{N\to\infty} N^{-1}\sum_{k\leqq N} \exp(2\pi i h x_k) = 0$, $h = 1, 2, \cdots$.

Now, a sequence of real numbers can be considered as a sequence of degenerate random variables. The object of this note is to generalize Weyl's criterion to the case of a sequence $X_1, X_2, \cdots$ of independent random variables.

To this end let $F_1, F_2, \cdots$ be the corresponding distribution functions and consider the probability space $(\Omega = \prod_{i\geqq 1} R_i, \mathcal{B} = \prod_{i\geqq 1} \mathcal{B}_i, P = \prod_{i\geqq 1} P_i)$, where R_i denotes the real line with borel subsets $\mathcal{B}_i$ and with probability measure P_i induced by F_i, $i = 1, 2, \cdots$. Consider the subset A of Ω consisting of all sequences of realizations of $X_1, X_2, \cdots$ that are uniformly distributed (mod 1). If $P(A) = 1$ we will say that the sequence $X_1, X_2, \cdots$ is uniformly distributed (mod 1) a.s.

2. Theorem. Our theorem turns out to be an immediate consequence of the strong law of large numbers that we state here in the following form.

THEOREM 1. *Let $Y_1, Y_2, \cdots$ be a sequence of independent random variables with finite variances $\sigma_1^2, \sigma_2^2, \cdots$ and let $\lim_{N\to\infty} N^{-1}\sum_{n\leqq N} \mathcal{E}(Y_n) = 0$, where "$\mathcal{E}$" stands for expectation, and let*

$$\sum_{n\geqq 1} \sigma_n^2 n^{-2} < \infty.$$

Then

$$\lim_{N\to\infty} N^{-1}\sum_{n\leqq N} Y_n = 0 \qquad \text{a.s.}$$

PROOF. [1], A, p. 238.

We now have

THEOREM 2. *Let $X_1, X_2, \cdots$ be a sequence of independent random variables with characteristic functions $\varphi_1, \varphi_2, \cdots$. Then the sequence is uniformly distributed* (mod 1) a.s., *if and only if*

$$\lim_{N\to\infty} N^{-1}\sum_{n\leqq N} \varphi_n(2\pi h) = 0, \qquad h = 1, 2, \cdots. \tag{1}$$

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PROOF. If the sequence $X_1, X_2, \cdots$ is uniformly distributed (mod 1) a.s. we have with probability one

$$\lim_{N\to\infty} N^{-1} \sum_{n \leqq N} \exp(2\pi i h X_n) = 0, \qquad h = 1, 2, \cdots, \tag{2}$$

and (1) follows by integration in (2) with respect to P under the limit sign which is justified by the bounded convergence theorem. On the other hand let (1) be satisfied and apply Theorem 1 to the real and imaginary parts of

$$\exp(2\pi i h X_n)$$

respectively. The result is then immediate.

REFERENCES

[1] LOÈVE, M. (1963). *Probability Theory.* Van Nostrand, New York.