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ON THE A PRIORI DISTRIBUTION OF THE COVARIANCE MATRIX[1]

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The purpose of this note is to give a fiducial argument in support of the well known non-informative *a priori* distribution of a covariance matrix. Although the fiducial argument was introduced and promoted energetically by R. A. Fisher as an inference method which presumably would lead to necessary (and therefore unique) *a posteriori* distributions [4], it was discovered soon that different fiducial arguments could lead to entirely different *a posteriori* distributions [2], and, in the particular case of fiducial inferences about a covariance matrix, interesting problems arise which require further research [3].

Let X be a random $p \times p$ matrix, which has the multivariate normal distribution with mean value equal to 0 (the null matrix) and covariance matrix Σ. That is, the density of X is proportional to

$$|\Sigma|^{-p/2} \exp -\tfrac{1}{2} \operatorname{tr} \Sigma^{-1}XX'. \tag{1}$$

Consider the change of variable

$$Y = \Sigma^{-\frac{1}{2}}X. \tag{2}$$

As is well known, the random matrix Y has the multivariate normal distribution with mean value equal to 0 and covariance matrix I (the identity matrix). Obviously, the transpose Y' has the same distribution, and

$$W = Y'Y = X'\Sigma^{-1}X \tag{3}$$

has the Wishart distribution with a mean value equal to pI, covariance matrix I, and p degrees of freedom, the density of which is proportional to

$$|W|^{-\frac{1}{2}} \exp - \tfrac{1}{2} \operatorname{tr} W. \tag{4}$$

Suppose now that X is an observed matrix and that Σ is unknown. Since the distribution of W does not depend on Σ, it is natural to assume that the *a posteriori* distribution of W is the same Wishart distribution. *A posteriori*, X and X' are constant matrices, and (3) is a transformation between the symmetric matrices W and Σ^{-1}, the Jacobian of which is, according to Theorem 3.7 of [1],

$$dW/d\Sigma^{-1} = |X|^{p+1}. \tag{5}$$

Hence the *a posteriori* density of Σ^{-1} is, up to a constant factor,

$$|\Sigma|^{\frac{1}{2}} \exp - \tfrac{1}{2} \operatorname{tr} \Sigma^{-1}XX'. \tag{6}$$

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A simple comparison between (1) and (6) shows that the *a priori* density of Σ^{-1} is

$$\pi(\Sigma^{-1}) = |\Sigma|^{(p+1)/2}. \tag{7}$$

This *a priori* distribution was derived from an invariance argument by Jeffreys (1961) for the case $p = 2$; it was considered, for arbitrary values of p, by Geisser and Cornfield (1963) and it was used by Tiao and Zellner (1964) and by Geisser (1965) to develop a Bayesian multivariate theory. The same *a posteriori* distribution has been obtained recently by Fraser and Haq (see Fraser (1968)) using an interesting new approach.

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