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AN INEQUALITY AND ALMOST SURE CONVERGENCE

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1. Introduction and main result. In this paper we prove an inequality similar to Kolmogorov's but without the assumption of independence. Our main result is given in the following:

THEOREM 1. *Let* $X_1, X_2, \cdots,$ *be a sequence of random variables such that* $E|X_i|^r = v_i < \infty$ *for some* $0 < r \leqq 1$ *and all* $i = 1, 2, \cdots$. *If* $c_1, c_2, \cdots,$ *is a non-increasing sequence of positive constants, then for any positive integers,* m, n *with* $m < n$ *and arbitrary* $e > 0$.

$$(1)\quad P(\max_{m\leqq k\leqq n} c_k|X_1 + X_2 + \cdots + X_k| \geqq e) \leqq ((c_m^r \textstyle\sum_{i=1}^m E|X_i|^r + \sum_{i=m+1}^n c_i^r E|X_i|^r)/e^r).$$

PROOF. Let $S_i = X_1 + \cdots + X_i$ and A_i $(i = m, m+1, \cdots, n)$ be the event $(c_m|S_m| < e, \cdots, c_{i-1}|S_{i-1}| < e, c_i|S_i| \geqq e)$, then $A_i \cap A_j = \emptyset$ for $i \neq j$ and $A = \bigcup_{i=m}^n A_i$ where $A = (\max_{m\leqq i\leqq n} c_i|S_i| \geqq e)$.

Now consider the random variable

$$Z = c_n^r|S_n|^r + \textstyle\sum_{k=m}^{n-1} |S_k|^r(c_k^r - c_{k+1}^r) + \sum_{k=m}^{n-1} I_k(c_k^r|S_k|^r - c_n^r|S_n|^r - \sum_{i=k}^{n-1} S_i^r(c_i^r - c_{i+1}^r))$$

where I_k is the indicator random variable of the event A_k, then $Z \geqq 0$ everywhere and $Z \geqq e^r$ in A. Hence, if $F(x_1, \cdots, x_n)$ is the joint distribution of $(X_1, \cdots, X_n) = X$, we have

$$P(\max_{m\leqq i\leqq n} c_i|S_i| \geqq e = P(X \varepsilon A) = \textstyle\int_A dF \leqq (\int_A Z\, dF)/e^r \leqq (EZ)/e^r.$$

It is easy to see that

$$(2)\quad Z = c_m^r|S_m|^r + \textstyle\sum_{k=m+1}^n c_k^r(|S_k|^r - |S_{k-1}|^r)(1 - I_{k-1} - \cdots - I_m)$$

and since the events A_i are disjoint, $I_m + \cdots + I_n \leqq 1$. Thus applying the c_r-inequality $|S_k|^r \leqq |S_{k-1}|^r + |X_k|^r$ which holds for $0 < r \leqq 1$, obtain

$$Z \leqq c_m^r \textstyle\sum_{k=1}^m |X_k|^r + \sum_{k=m+1}^n c_k^r|X_k|^r.$$

Therefore,

$$EZ \leqq c_m^r \textstyle\sum_{k=1}^m v_k + \sum_{k=m+1}^n c_k^r v_k$$

which proves the theorem. Q.E.D.

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It should be noted that if we make the additional assumptions

(i) $E(X_k \mid X_1, \cdots, X_{k-1}) = 0$, all $k = m, \cdots, n$,

(ii) $EX_i^2 = \sigma_i^2 < \infty$ all $i = 1, 2, \cdots, n$,

then we obtain stronger results by writing in (2)

$$(S_k^2 - S_{k-1}^2)(1 - I_{k-1} - \cdots - I_m) = (X_k^2 + 2X_kS_{k-1})(1 - I_{k-1} - \cdots - I_m) \leqq X_k^2 + 2X_kS_{k-1}(1 - I_{k-1} - \cdots - I_m).$$

Hence,

$$EZ \leqq c_m^2 \sum_{i=1}^m \sigma_i^2 + \sum_{i=m+1}^n c_i^2\sigma_i^2$$

which gives the well known Hájek-Rényi; inequality.

THEOREM 2. *If* $E|X_i|^r < \infty$ *for some* $r \geqq 1$ *and all* $i = 1, 2, \cdots, n$, *and the constants* $c_1, \cdots, c_n, m, n, e$ *are as in Theorem* 1, *then*

$$(3) \quad P(\max_{m \leqq k \leqq n} c_k|X_1 + \cdots + X_k| \geqq e) \leqq (c_m \sum_{i=1}^m E^{1/r}|X_i|^r + \sum_{i=m+1}^n c_iE^{1/r}|X_i|^r)^r/e^r.$$

PROOF. As in the previous theorem the random variable

$$Z = c_m|S_m| + \sum_{k=m+1}^n c_k(|S_k| - |S_{k-1}|)(1 - I_{k-1} - \cdots - I_m) \leqq c_m \sum_{k=1}^m |X_k| + \sum_{k=m+1}^n c_k|X_k|$$

is non-negative everywhere and $Z \geqq e$ in A. Then

$$P(\max_{m \leqq i \leqq n} c_i|S_i| \geqq e) = \int_A dF \leqq \int_A (Z/e)^r dF \leqq (EZ^r)/e^r \leqq (E(c_m \sum_{k=1}^m |X_k| + \sum_{k=m+1}^n c_k|X_k|)^r)/e^r.$$

By application of Minkowski's inequality we obtain the desired result. Q.E.D.

2. Applications. The above theorems are intimately related to almost sure convergence as it becomes clear in the following:

COROLLARY 1. *If* $b_n \uparrow \infty$ *and either*

$$\sum_{n=1}^\infty (E|X_n|^r/b_n^r) < \infty \text{ for } 0 < r \leqq 1, \text{ or } \sum_{n=1}^\infty (E^{1/r}|X_n|^r/b_n) < \infty \quad \text{for } 1 \leqq r,$$

then

$$(\sum_{k=1}^n X_n)/b_n \rightarrow_{\text{a.s.}} 0 \qquad \text{as} \quad n \rightarrow \infty.$$

PROOF. Take $c_i = (1/b_i) \downarrow 0$ and apply the previous theorems, then for $0 < r \leqq 1$ since

$$\lim_{m\to\infty} \sum_{k=m+1}^\infty (E|X_k|^r)/b_k^r = 0$$

as the tail of a convergent sequence, and $\lim_{m\to\infty} (\sum_{k=1}^m E|X_k|^r)/b_m^r = 0$ by Kronecker's lemma ([2], page 238), we have

$$\lim_{m\to\infty} P(\max_{m \leqq n} |X_1 + \cdots + X_n|/b_n \geqq e) = 0.$$

The proof for $1 \leqq r$ is similar. Q.E.D.

EXAMPLE. If $P(X_i = 1) = 1/i^q$ and $P(X_i = 0) = 1 - 1/i^q$, with $q > 0$, then $E|X_i| = 1/i^q$ and $(S_n/n) \rightarrow_{\text{a.s.}} 0$. Note that X_i might be either dependent or independent.

COROLLARY 2. *If* $E|X_i| = v$ *for all* $i = 1, \cdots$ *then*

$$(X_1 + X_2 + \cdots + X_n)/n^{1+q} \rightarrow_{\text{a.s.}} 0 \qquad \text{as} \quad n \rightarrow \infty,$$

where $q > 0$.

PROOF. This is a consequence of Corollary 2, for $b_n = n^{1+q}$ and $r = 1$. Q.E.D.

It is interesting to note that if we want the above result to hold for $q = 0$, we should make strong additional assumptions; i.e., X_i should be independent and identically distributed with $EX_i = 0$.

In the particular case $c_1 = c_2 = \cdots = c_n = 1$, from (2) and (3), we obtain,

$$(4) \qquad P(\max_{m \leqq i \leqq n} |X_1 + \cdots + X_i| \geqq e) \leqq (\textstyle\sum_{i=1}^{n} E^{1/s}|X_i|^r)^s/e^r$$

where $s = 1$ if $0 < r \leqq 1$ and $s = r$ if $r \geqq 1$, which leads to:

COROLLARY 3. *If* $\sum_{i=1}^{\infty} E^{1/s}|X_i|^r < \infty$, *then* $S_n = \sum_{k=1}^{n} X_k$ *converges almost surely.*

PROOF. From (4),

$$P(\max_{k \geqq 1} |S_{m+k} - S_m| \geqq e) \leqq (\textstyle\sum_{i=m+1} E^{1/s}|X_i|^r)^s/e^r.$$

Hence,

$$\lim_{m \to \infty} P(\max_{k \geqq 1} |S_{m+k} - S_m| \geqq e) = 0,$$

but the mutual almost sure convergence of S_n implies the almost sure convergence of S_n ([2], page 113). Q.E.D.

EXAMPLE. Let $P(X_i = 0) = 1 - 1/i^{1+q}$ and $P(X_i = 1) = 1/i^{1+q}$, $q > 0$, then $E|X_i| = 1/i^{1+q}$ and hence, $\sum_{i=1}^{n} X_i$ converges almost surely.

3. Related work. Hájek and Rényi [1] proved (1) for $r = 2$ and independent random variables; Loève [2] proved Corollaries 1, 2, 3 for $0 < r \leqq 2$ and independent random variables.

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