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CAUCHY-DISTRIBUTED FUNCTIONS AND A CHARACTERIZATION OF THE CAUCHY DISTRIBUTION

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1. Summary. A simple representation of a class of functions of Cauchy variates whose members are also Cauchy variates is given. This leads to a characterization of the Cauchy distribution.

2. Notation and preliminary results. It is convenient to express μ and σ, the location and scale parameters of the general Cauchy variate, in terms of 'polar parameters' θ, ρ defined by

$$\mu = \rho \sin \theta, \qquad 0 < \rho < \infty,$$
$$\sigma = \rho \cos \theta, \qquad |\theta| < \tfrac{1}{2}\pi.$$

The density of the general Cauchy distribution then becomes

$$f(x) = (\rho \cos \theta)/[\pi(\rho^2 - 2\rho x \sin \theta + x^2)].$$

The expression $C(\theta, \rho)$ will stand for "Cauchy-distributed (with parameters θ, ρ)", and the symbol $\sim$ for "is distributed like".

We note that, if X is $C(\theta, \rho)$, then X/ρ and ρ/X are both $C(\theta, 1)$, a result demonstrated in less illuminating notation by Menon (1962), and corresponding to the familiar result that if U is $C(0, 1)$, $U \sim 1/U$.

Since any real number may be expressed as the tangent of an angle in the interval $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$, we shall represent all angles by their 'residues' reduced modulo π to this interval.

3. Cauchy-distributed functions. Pitman and Williams (1967) describe a class of functions of Cauchy variates that are C. Their main result may be expressed succinctly as follows: If the a_n possess no finite limit-point, and if $w_n \geqq 0$, $\sum w_n = 1$, then

$$\sum w_n[(1 + a_n U)/(a_n - U)] \sim U, \tag{1}$$

where U is $C(0, 1)$. Clearly, provided $U \neq a_n$ for any n, the sum (1) is bounded and represents a well-defined random variable; the points $U = a_n$ form a set of measure zero.

Note that $a_n = \pm\infty$ gives a term in U, and $a_n = 0$ gives a term in $-U^{-1}$, so that all the terms in Pitman and Williams' expression are included in (1).

The result (1) may also be written in the form

$$U + (1 + U^2) \sum [w_n/(a_n - U)] \sim U. \tag{1'}$$

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A more interesting alternative form is obtained as follows: Put $U = \tan Z$, $a_n = \cot \alpha_n$, and write the residue of $Z + \alpha_n$ as Z_n. With the convention mentioned above, Z and Z_n are uniformly distributed in $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$. Then the coefficient of w_n in the sum (1) is simply $\tan (Z + \alpha_n) = \tan Z_n$, so that we have

$$(2) \qquad \sum w_n \tan Z_n \sim \tan Z.$$

In words, given a set of angles whose differences are a fixed set and which are uniformly distributed, the weighted average of their tangents is distributed independently of the weights. Incidentally, the same is trivially true of the angles themselves, provided residues modulo π are taken of the weighted averages and not of the individual components.

The distribution of such functions of general Cauchy variates is readily found. Pitman and Williams showed that, if U is $C(0, 1)$, and $G(U)$ is C, then

$$G(U) \sim G_1 + G_2 U, \qquad \text{where} \quad G(i) = G_1 + G_2 i.$$

Similarly, if X is $C(\theta, \rho)$, then $X \sim \rho(U \cos \theta + \sin \theta)$. Hence if $H(X)$ is C, we find its parameters by replacing X by $i\rho e^{-i\theta}$. Thus $H(X) \sim H_1 + H_2 U$, where $H(i\rho e^{-i\theta}) = H_1 + H_2 i$, or $(H(X) - H_1)/H_2 \sim U$.

4. Characterization of the Cauchy distribution. Menon (1962) has given a characterization of the Cauchy distribution which requires that a variate X be stable, that X^{-1} be distributed as $g(x)$, where $g(x) = ax + O(1)$ as $x \to \infty$, and that a condition be imposed on $g'(x)$.

The question arises of whether result (1) is peculiar to the Cauchy distribution. It is easily shown that, with certain restrictions on the a_n, this is so. We show that a simple case of (1) establishes the characterization.

THEOREM. *If, for some real a that is not the tangent of a rational multiple of* π,

$$[(1 + aX)/(a - X)] \sim X,$$

then X is $C(0, 1)$.

PROOF. Putting $X = \tan Z_0$, $a = \cot \alpha$, we have $\tan (Z_0 + \alpha) \sim \tan Z_0$, where α is not a rational multiple of π, and we have to prove that Z_0 is uniformly distributed in $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$.

Let z_n denote the residue of $z_0 + n\alpha$, with similar notation for random variables Z_n, so that the condition given is equivalent to $Z_0 \sim Z_1$.

Consider the residues $z_1, z_2, \cdots, z_n$ corresponding to any z_0; as n increases without limit, the number of such residues in any interval contained in $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$ tends to proportionality with the length of the interval (see Feller, p. 262). This fact may be expressed in a probabilistic guise as

$$\text{Lt}_{n\to\infty} n^{-1} \textstyle\sum_{r=1}^{n} \Pr (c < Z_r \leqq b \mid Z_0 = z_0) = (b - c)/\pi, \quad -\frac{1}{2}\pi < c < b \leqq \frac{1}{2}\pi,$$

where the conditional probabilities in the sum are in fact indicators. But the limit is independent of z_0; hence the conditional probabilities may be replaced by unconditional probabilities, and we have

$$\text{Lt}_{n\to\infty} n^{-1} \textstyle\sum_{r=1}^{n} \Pr (c < Z_r \leqq b) = (b - c)/\pi.$$

Now by induction, $Z_0 \sim Z_1 \sim Z_2 \sim \cdots$, so that the unconditional probabilities in the sum are all identical. Hence each Z_r, and in particular Z_0, is uniformly distrubuted in $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$.

It then follows that X is $C(0, 1)$.

For a class of general Cauchy variates we have a similar but more rudimentary result: If

$$[\rho^2/(2\rho \sin\theta - X)] \sim X$$

where θ is not a rational multiple of π, then X is $C(\theta, \rho)$. Putting

$$X = \rho(\sin\theta + \tan Z \cos\theta),$$

we require to prove that $\tan Z$ is $C(0, 1)$, or that Z is uniformly distributed. The condition becomes

$$[1/(\sin\theta - \tan Z \cos\theta)] \sim \sin\theta + \tan Z \cos\theta,$$

or

$$[(\cot\theta + \tan Z)/(1 - \tan Z \cot\theta)] = \tan(Z + \tfrac{1}{2}\pi - \theta) \sim \tan Z.$$

The remainder of the proof follows the lines of the theorem, and shows that X is indeed $C(\theta, \rho)$.

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