The Annals of Mathematical Statistics
1969, Vol. 40, No. 3, 1071–1073

UNIFORM CONVERGENCE OF FAMILIES OF MARTINGALES[1]

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It is known [3] that under bounded conditions on the expected values of a martingale sequence the martingale will converge almost surely to a random variable and if the rth powers of the absolute values of the random variables in the martingale are uniformly integrable then the martingale will converge in L_r to a random variable with finite rth absolute moment. In this note we consider the case of a family of martingales each adapted to the same increasing family of σ-fields and give a condition on the family which will assure under bounded conditions on the martingales that the convergence given by the martingale convergence theorem is uniform in the family. We obtain uniform L_1 convergence for arbitrary families and uniform a.s. convergence for countable families. The a.s. convergence was proven for a slightly different case in [4] and the L_1-convergence is obtained from a suggestion made by the referee of that paper.

Throughout we will be working in a fixed probability space $(\Omega, \mathcal{B}, P)$ and all σ-fields will be sub σ-fields of $\mathcal{B}$. If A and B are sets $A \triangle B$ will denote the symmetric difference of A and B, i.e., $A \triangle B = (A - B) \cup (B - A)$. The expected value of a random variable will be denoted by E, the conditional expectation given a σ-field $\mathcal{C}$ by $E(\cdot \mid \mathcal{C})$ and the condition probability given $\mathcal{C}$ by $P(\cdot \mid \mathcal{C})$. The conditional entropy of a σ-field $\mathcal{A}$ given a σ-field $\mathcal{C}$ is denoted by $H(\mathcal{A} \mid \mathcal{C})$ and is defined to be

$$\sup \{E[-\textstyle\sum_{F \varepsilon \mathfrak{F}} P(F \mid \mathcal{C}) \log P(F \mid \mathcal{C})]\}$$

where the supremum is taken over all finite partitions $\mathfrak{F}$ of Ω into sets from $\mathcal{A}$. For properties of $H(\mathcal{A} \mid \mathcal{C})$ one may consult Jacobs [2] or Billingsley [1].

DEFINITION 1. Let I be an index set and for each i in I let $\{X_n^i : n \geqq 0\}$ be a sequence of random variables. We say that $\{X_n^i\}$ L_r-converges uniformly in i to X^i provided that for every $\epsilon > 0$ there is an $N(\epsilon)$ such that for all $n \geqq N(\epsilon)$ $\sup_i E\{|X_n^i - X^i|^r\} < \epsilon$. We say that $\{X_n^i\}$ a.s. converges uniformly in i to X^i provided that there exists a set Z of probability zero such that for every $\epsilon > 0$ and $w \not\varepsilon Z$, there exists an integer $N(\epsilon, w)$ such that $\sup_i |X_n^i(w) - X^i(w)| < \epsilon$. We shall denote these types of convergences respectively by $X_n^i \to X^i$ [L_r unif i] and $X_n^i \to X^i$ [a.s. unif i].

LEMMA 1. *Let $\{\mathcal{A}_n\}$ denote an increasing sequence of σ-fields and $\mathcal{A}$ denote the σ-field generated by $\bigcup_n \mathcal{A}_n$. If for some n, $H(\mathcal{A} \mid \mathcal{A}_n) < \infty$ then for every $\epsilon > 0$ there exists an integer N such that for all $A \varepsilon \mathcal{A}$, there exists an event $B \varepsilon \mathcal{A}_N$ such that $P(A \triangle B) < \epsilon$.*

PROOF. Since $H(\mathcal{A} \mid \mathcal{A}_n) < \infty$ for some n, $\lim_n H(\mathcal{A} \mid \mathcal{A}_n) = H(\mathcal{A} \mid \mathcal{A}) = 0$

Received 12 August 1968.
[1] Research supported in part by ARO Grant 662.

and for $\epsilon > 0$ given, there exists an integer N such that $H(\mathfrak{A} \mid \mathfrak{A}_N) < \epsilon^2/4$. We will show that if $A \varepsilon \mathfrak{A}$ and B is defined to be the $\mathfrak{A}_N$-event $[P(A \mid \mathfrak{A}_N) > 1 - \epsilon/2]$ then $P(A \bigtriangleup B) < \epsilon$.

Let A be an $\mathfrak{A}$ event. From the definition and properties of $H(\mathfrak{A} \mid \mathfrak{A}_N)$ by considering the finite $\mathfrak{A}$ partition $\{A, \Omega - A\}$ of Ω we obtain $0 \leqq -\int_A \log P(A \mid \mathfrak{A}_N)\, dP < \epsilon^2/4$ and hence $P(A \mid \mathfrak{A}_N)$ is positive almost surely on A. Then

$$0 \leqq \int_A [1 - P(A \mid \mathfrak{A}_N)]\, dP \leqq \int_A - \log P(A \mid \mathfrak{A}_N)\, dP < \epsilon^2/4.$$

Hence

$$\epsilon^2/4 > \int_{A-B} [1 - P(A \mid \mathfrak{A}_N)]\, dP \geqq (\epsilon/2)P(A - B)$$

and one obtains that $P(A - B) < \epsilon/2$. Also, since $B \varepsilon \mathfrak{A}_N$,

$$P(A \cap B) = \int_B P(A \mid \mathfrak{A}_N)\, dP > (1 - \epsilon/2)P(B).$$

so that $P(B - A) < \epsilon/2$ and it follows that $P(A \bigtriangleup B) < \epsilon$.

THEOREM 1. *Let $\{\mathfrak{A}_n\}$ be an increasing sequence of σ-fields and $\mathfrak{A}$ be the σ-field generated by $\bigcup_n \mathfrak{A}_n$. Let $\{g^{(i)}: i \varepsilon I\}$ be a uniformly bounded family of $\mathfrak{A}$ measurable random variables. If $H(\mathfrak{A} \mid \mathfrak{A}_n) < \infty$ for some n then $E(g^{(i)} \mid \mathfrak{A}_n) \rightarrow g^{(i)}$* [$L_1$ unif i].

PROOF. Suppose that $\{A^j: j \varepsilon J\}$ is an indexed family of events from $\mathfrak{A}$, and consider the indicator functions $I(A^i)$ of these sets. Let $\epsilon > 0$ be given. By Lemma 1, there exists $N(\epsilon)$ such that for every $j \varepsilon J$, there is a set $B^j \varepsilon \mathfrak{A}_{N(\epsilon)}$ such that $P(A^j \bigtriangleup B^j) < \epsilon$.

Since $I(A^i)$ is square summable and $E(\cdot \mid \mathfrak{A}_N)$ is projection of $L_2(\mathfrak{A})$ onto $L_2(\mathfrak{A}_N)$ and $I(B^j) \varepsilon L_2(\mathfrak{A}_N)$ for all $n \geqq N(\epsilon)$ we have that

$$E\{|P(A^j \mid \mathfrak{A}_n) - I(A^j)|^2\} \leqq E\{|I(B^j) - I(A^j)|^2\}$$

so that $E\{|P(A^j \mid \mathfrak{A}_N) - I(A^j)|^2\} < \epsilon$ for all $j \varepsilon J$ and all $n \geqq N(\epsilon)$. Thus $E(I(A^j) \mid \mathfrak{A}_n) \rightarrow I(A^j)$ [L_2 unif j]. But $P(A^j \mid \mathfrak{A}_n) - I(A^j) \varepsilon L_1$ so $E(I(A^j) \mid \mathfrak{A}_n) \rightarrow I(A^j)$ [L_1 unif j].

Now let $\{g^i: i \varepsilon I\}$ be a uniformly bounded family of $\mathfrak{A}$ measurable random variables, say $|g^i| \leqq M$ for all i, where M is an integer. Define g_k^i, $k = 1, 2, \cdots$; $i \varepsilon I$ as follows:

$$g_k^i = \sum_{m=-Mk}^{Mk} mk^{-1} I(A_{m,k}^i)$$

where $A_{m,k}^i = [mk^{-1} \leqq g^i < (m + 1)k^{-1}]$. Let $\epsilon > 0$ be given and select $K > 0$ so large that $K^{-1} < \epsilon/3$. Then $|g_K^i - g^i| < \epsilon/3$ for all i and we have $E(|g_K^i - g^i| \mid \mathfrak{A}_n) < \epsilon/3$ for all i and n.

Also

$$|E(g_K^i \mid \mathfrak{A}_n) - g_K^i| \leqq \sum_{M=-MK}^{MK-1} |m|\, K^{-1}\, |P(A_{m,K}^i \mid \mathfrak{A}_n) - I(A_{m,K}^i)|$$

so that

$$\sup_i E\{|E(g_K^i \mid \mathfrak{A}_n) - g_K^i|\} \leqq 2M \sup_{m,i} E\{|P(A_{m,K}^i \mid \mathfrak{A}_n) - I(A_{m,K}^i)|\}.$$

From the first part of the proof there exists N such that for all $n \geqq N$, $\sup_{m,i} E\{|P(A^i_{m,K} \mid \mathcal{A}_n) - I(A^i_{m,K})|\} < \epsilon/6M$.

Thus if $n \geqq N$,

$$E\{|E(g^i \mid \mathcal{A}_n) - g^i|\} \leqq 2E\{|g_K^i - g^i|\} + E\{|E(g_K^i \mid \mathcal{A}_n) - g_K^i|\} < \epsilon$$

for all i and it follows that $E(g^i \mid \mathcal{A}_n) \to g^i$ [L_1 unif i].

Corollary. *Let $\{\mathcal{A}_n\}$ be an increasing sequence of σ-fields and $\mathcal{A}$ be the σ-field generated by $\bigcup_n \mathcal{A}_n$. For each i in an index set I let $\{X_n^i: n \geqq 0\}$ be a martingale adapted to $\{\mathcal{A}_n\}$. If $H(\mathcal{A} \mid \mathcal{A}_n) < \infty$ for some n and if $|X_n^i| \leqq M$ for all i and n then there exist random variables X^i such that $X_n^i \to X^i$* [L_1 unif i].

Proof. Since $\{X_n^i: n \geqq 0\}$ is uniformly bounded it is uniformly integrable for each i and hence there exist L_1-functions X^i such that $X_n^i \to X^i$ [L_1]. Moreover $X_n^i \to X^i$ [a.s.] so $|X^i| \leqq M$ for all i and $E(X^i \mid \mathcal{A}_n) \to X^i$ [L_1 unif i]. Since X^i is a right closing random variable for $\{X_n^i: n \geqq 0\}$, $E(X^i \mid \mathcal{A}_n) = X_n^i$ and the result follows.

Theorem 2. *Let $\{\mathcal{A}_n\}$ be an increasing sequence of σ-fields with $\mathcal{A}$ being the σ-field generated by $\bigcup_n \mathcal{A}_n$, and let $\{g^i: i \varepsilon I\}$ be a countable uniformly bounded family of $\mathcal{A}$-measurable random variables. If $H(\mathcal{A} \mid \mathcal{A}_n) < \infty$ for some n then $E(g^i \mid \mathcal{A}_n) \to g^i$* [a.s. unif i].

Proof. This follows from Lemma 1 and a slight modification of Theorem 4.4 in [4].

Corollary. *Let $\{\mathcal{A}_n\}$ be an increasing sequence of σ-fields and $\mathcal{A}$ be the σ-field generated by $\bigcup_n \mathcal{A}_n$. For each i in a countable index set I let $\{X_n^i: n \geqq 0\}$ be a martingale adapted to $\{\mathcal{A}_n\}$. If $H(\mathcal{A} \mid \mathcal{A}_n) < \infty$ for some n and if $|X_n^i| \leqq M$ for all i and n then there exist random variables X^i such that $X_n^i \to X^i$* [a.s. unif i].

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