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NOTES

AN L^p-CONVERGENCE THEOREM

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Recently Pyke and Root [4] strengthened a theorem of Marcinkiewicz proving that if f_n is a sequence of independent, identically distributed rv's with $\int |f_1|^p < \infty, 0 < p < 1$ and $\int f_1 = 0$ if $1 \leqq p < 2$ then $n^{-1/p}\{f_1 + \cdots + f_n\} \to 0$ a.s. and in L^p. The strengthening consisted in proving L^p-convergence. The purpose of this paper is to prove a similarly strengthened version of a theorem for martingales which is a generalization of the above-mentioned Marcinkiewicz's theorem. The a.s. convergence version of this theorem is in Loève [2], pp. 387. Our theorem contains that of Pyke and Root and is proved by using a tightened form of Minkowski's inequality due to Esseen and Von Bahr [1] which is stated as Lemma 1 and proved in a simple direct way.

LEMMA 1. *If* $E(f_j \mid f_1 + \cdots + f_{j-1}) = 0$ *(in particular if* f_j *is a martingale-difference sequence) for* $2 \leqq j \leqq n$ *and* $f_j \varepsilon L^p$, $1 \leqq p \leqq 2$ *then*

$$\int |f_1 + \cdots + f_n|^p \leqq \alpha \sum_{j=1}^{n} \int |f_j|^p$$

where $\alpha \leqq 2^{2-p} < 2$. *(The actual value of* α *will be immaterial in the proof of the theorem.)*

The cases $p = 1, 2$ being trivial, consider $1 < p < 2$. Here use the elementary inequality $|a + b|^p \leqq |a|^p + p|a|^{p-1} \cdot s(a)b + \alpha|b|^p$ for real numbers a, b ($s(a) =$ sign of a). This inequality follows easily from the fact that

$$\alpha = \sup_x \{|1 + x|^p - 1 - px\}/|x|^p$$

is finite. An elementary but tedious argument shows that $\alpha \leqq 2^{2-p} < 2$. Note also that $\alpha > 1$. Integrating the inequality we get $\int |f_1 + f_2|^p \leqq \int |f_1|^p + \alpha \int |f_2|^p$. Now apply induction.

THEOREM. *Let* f_n, $n \geqq 1$, *and* f *be measurable functions such that either* $f \varepsilon L^p$, $0 < p < 2$, $p \neq 1$ *and* $P(|f_n| \geqq x) \leqq P(|f| \geqq x)$, $0 \leqq x < \infty$ *or* $f \varepsilon L^1$ *and* $P(|f_n| \geqq x | f_1 \cdots f_{n-1}) \leqq P(|f| \geqq x \mid f_1 \cdots f_{n-1})$ a.s. *Then*

$$\lim_n n^{-1/p} \sum_{k=1}^{n} (f_k - \alpha_k) = 0 \quad \text{a.s.} \quad \textit{and in } L^p$$

where $\alpha_k = 0$ *if* $0 < p < 1$ *and* $\alpha_k = E(f_k \mid f_1 \cdots f_{k-1})$ *if* $1 \leqq p < 2$.

PROOF. The condition $P(|f_n| \geqq x) \leqq P(|f| \geqq x)$ with $f \varepsilon L^p$ implies that $f_n \varepsilon L^p$, $\sup_n \int |f_n|^p \leqq \int |f|^p$ and

(a) $$\sum_{n=1}^{\infty} P(A_n) < \infty \quad \text{where} \quad A_n = \{|f_n| \geqq n^{1/p}\};$$

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(b) $\sum_n n^{-1/p} \int_{B_n} |f_n| < \infty$ for $0 < p < 1$, $B_n = \{|f_n| < n^{1/p}\}$,

$\sum_n n^{-1/p} \int_{A_n} |f_n| < \infty$ for $1 < p < 2$,

$\lim_{n\to\infty} \int_{A_n} |f_n| = 0$ for $p = 1$;

(c) $\sum_n n^{-2/p} \int_{B_n} |f_n|^2 < \infty$.

The proof of (a)–(c) is standard (cf. [3], pp. 154) and depends only on the stochastic domination of $|f|$ on $|f_n|$. Put $g_n = f_n \cdot 1_{B_n}$, $h_n = f_n - g_n$. Consider first the case $0 < p < 1$. In the identity

$$\sum n^{-1/p} f_n = \sum n^{-1/p}(g_n - \beta_n) + \sum n^{-1/p} h_n + \sum n^{-1/p} \beta_n$$

where $\beta_n = E(g_n \mid f_1 \cdots f_{n-1})$, the first term on the right hand side converges a.s. and in L^2 (and hence in L^p, $p < 2$) because of (c) and a martingale theorem. The second term converges a.s. since by (a) $h_n = 0$ for n sufficiently large a.s. The third term converges a.s. and in L^1 (and hence in L^p, $p < 1$) because

$$\sum n^{-1/p} \int |\beta_n| \leqq \sum n^{-1/p} \int |g_n| = \sum n^{-1/p} \int_{B_n} |f_n| < \infty \quad \text{by (b)}.$$

Using now the identity

$$n^{-1/p} \sum_1^n f_k = n^{-1/p} \sum_1^n (g_k - \beta_k) + n^{-1/p} \sum_1^n h_k + n^{-1/p} \sum_1^n \beta_k$$

and the so-called Kronecker's lemma we see that the proof will be complete if we can show that the second term on the right converges in L^p i.e.

$$n^{-1} \int \left|\sum_1^n h_k\right|^p \to 0 \quad \text{as} \quad n \to \infty.$$

But for $0 < p < 1$

$$n^{-1} \int \left|\sum_1^n h_k\right|^p \leqq n^{-1} \sum_1^n \int |h_k|^p \to 0 \quad \text{as} \quad n \to \infty$$

since $\lim_{k\to\infty} \int |h_k|^p = \lim_{k\to\infty} \int_{A_k} |f_k|^p \leqq \lim_{k\to\infty} \int_{\{|f| \geqq k^{1/p}\}} |f|^p = 0$. Consider now the case $1 < p < 2$. In the identity

$$\sum n^{-1/p}(f_n - \alpha_n) = \sum n^{-1/p}(g_n - \beta_n) + \sum n^{-1/p}(h_n + \beta_n - \alpha_n)$$

the first term on the right converges a.s. and in L^2 as before and the second converges a.s. and in L^1 since $\beta_n - \alpha_n = E(-h_n \mid f_1 \cdots f_{n-1})$ and

$$\sum n^{-1/p} \int |h_n + \beta_n - \alpha_n| \leqq 2 \sum n^{-1/p} \int |h_n| = 2 \sum n^{-1/p} \int_{A_n} |f_n| < \infty$$

by (b). Arguing as before the proof will be completed by showing that

$$n^{-1/p} \sum_1^n (h_k + \beta_k - \alpha_k) \to 0 \quad \text{in} \quad L^p$$

i.e.

$$n^{-1} \int \left|\sum_1^n (h_k + \beta_k - \alpha_k)\right|^p \to 0 \quad \text{as} \quad n \to \infty.$$

Here we use the Esseen-Von Bahr inequality of Lemma 1 which we can since

$h_k + \beta_k - \alpha_k$ is a martingale-difference sequence. So

$$n^{-1} \int |\sum_1^n (h_k + \beta_k - \alpha_k)|^p \leqq 2n^{-1} \sum_1^n \int |h_k + \beta_k - \alpha_k|^p$$
$$\leqq 2^p n^{-1} \sum_1^n \{\int |h_k|^p + \int |\beta_k - \alpha_k|^p\}$$
$$\leqq 2^{p+1} n^{-1} \sum_1^n \int |h_k|^p \to 0$$

as $n \to \infty$ since $\lim_{k\to\infty} \int |h_k|^p \leqq \lim_{k\to\infty} \int_{\{|f| \geqq k^{1/p}\}} |f|^p = 0$.

The proof for $p = 1$ is as before except for one small detail. In the identity

$$n^{-1} \sum_1^n (f_k - \alpha_k) = n^{-1} \sum_1^n (g_k - \beta_k) + n^{-1} \sum_1^n (h_k + \beta_k - \alpha_k)$$

the first term on the right converges a.s. and in L^2 to 0 as before and the second converges to 0 in L^1 since $\int |h_k + \beta_k - \alpha_k| \leqq 2 \int |h_k| \to 0$ by (b). We simply have to ensure the a.s. convergence of the second term. Since $h_k \to 0$ a.s. by (a), it will be enough to show that $\lim (\beta_k - \alpha_k) = 0$ a.s. It is here that we use the stronger hypothesis made for the case $p = 1$. We shall show that if

$$\delta_k = E(|h_k| \mid f_1 \cdots f_{k-1})$$

then $\delta_k \to 0$ a.s. which is certainly sufficient. A simple calculation shows that

$$\delta_k \leqq 2E(X_k \mid f_1 \cdots f_{k-1})$$

where $X_k = |f| \cdot 1_{\{|f| \geqq k\}}$. Using the fact that $X_{n+1} \leqq X_n$ we see that $E(X_k \mid f_1 \cdots f_{k-1})$ is a positive super-martingale. Indeed

$$E(X_n \mid f_1 \cdots f_{n-1}) \geqq E(X_{n+1} \mid f_1 \cdots f_{n-1})$$
$$= E(E(X_{n+1} \mid f_1 \cdots f_n) \mid f_1 \cdots f_{n-1})$$

Since every positive super-martingale converges a.s. $\lim_{k\to\infty} E(X_k \mid f_1 \cdots f_{k-1}) = X$ exists a.s. But $\int X \leqq \lim_{k\to\infty} \int X_k = \lim_{k\to\infty} \int_{\{|f| \geqq k\}} |f| = 0$ so that X being non-negative must be zero a.s. Hence $\lim \delta_k = 0$ a.s. The theorem is thus completely proved.

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