The Annals of Mathematical Statistics
1969, Vol. 40, No. 3, 1042–1067

ADMISSIBILITY OF THE USUAL CONFIDENCE SETS FOR THE MEAN OF A UNIVARIATE OR BIVARIATE NORMAL POPULATION

By V. M. Joshi[1]

University of North Carolina, Chapel Hill

1. Introduction. Let X be an m-dimensional vector distributed normally with mean vector θ and covariance matrix equal to the $m \times m$ identity matrix. A non-randomized confidence procedure C is a procedure, which assigns to each possible point x, a Lebesgue measurable subset $C(x, \cdot)$ of the parameter space within which θ is estimated to lie. Let $vC(x, \cdot)$ denote the Lebesgue measure of the set $C(x, \cdot)$. The usual procedure C_0 is a procedure in which the confidence sets $C_0(x, \cdot)$ are spheres of fixed volume, centered at the observed sample mean. C_0 has the property that amongst the class of confidence procedures with lower confidence level $(1 - \alpha)$, C_0 minimizes the maximum expected measure of the confidence sets viz.

$$\sup_\theta E_\theta vC(x, \cdot). \tag{1}$$

Stein (1962) raised the question whether the usual procedure is unique in having this property and conjectured that it is probably unique for $m = 1$, probably not so for $m \geqq 3$, the case $m = 2$ being doubtful. For the case $m \geqq 3$, the conjecture has already been shown to be true in a previous paper (Joshi (1967)). In this paper we now investigate the remaining cases $m = 1$ and $m = 2$.

A connected question is that of the admissibility of the usual procedure. Using the definition of admissibility of confidence sets formulated by Godambe (1961) and subsequently slightly revised by the author (1966) it is here shown that if apart from measurability there is no restriction on the form of the confidence sets, then no unique minimax or even admissible procedure can exist, as given any procedure another one uniformly superior to it can always be constructed. All the procedures so constructed however form a class called equivalence class such that for any two procedures in the class, for almost all x, the confidence sets differ from each other at most by null subsets of the parameter space. Admissibility or uniqueness of the minimax property can thus only pertain to the equivalence class which contains the usual procedure. Alternatively a unique minimax or admissible procedure can exist in the restricted class of confidence procedures for which the confidence sets are all convex sets or all open sets.

In the following remarks, therefore, the uniqueness or admissibility of the usual confidence procedure means the uniqueness or admissibility of the equivalence class which contains the usual procedure or alternatively its uniqueness

Received 10 August 1966; revised 17 September 1968.

[1] Now Secretary, Maharashtra Government, Bombay.

or admissibility in the restricted class of confidence procedures with the restriction on the form of the confidence sets that they are all open sets or alternatively are all convex sets. Subject to this qualification, it is shown in this paper that for $m = 1$ and for $m = 2$, the usual confidence procedure is uniquely minimax and admissible. The uniqueness and the admissibility are actually proved for a wider class of randomized confidence procedures, with a corresponding generalization of the definition of an equivalence class. Also the admissibility of the usual procedure is proved on the basis of a certain loss function and this admissibility is of a stronger type than that implied by Godambe's definition (1961) as revised by the author.

The result proved in the previous paper (1967) means that for $m \geqq 3$, the usual confidence procedure is inadmissible. The results are thus exactly parallel to Stein's (1956) results regarding point estimation of the population mean.

2. Notation. In this paper we prove the results for $m = 1$ and for $m = 2$. For the sake of clarity we shall give the notation for the case $m = 2$ only. The modifications required for the case $m = 1$ will be obvious. Let then $X = (X_1, X_2)$ be a random vector distributed normally, with unknown mean $\theta = (\theta_1, \theta_2)$ and the 2×2 identity matrix as the covariance matrix. In the general case confidence sets will be based on n observations of X. However by the *principle of sufficiency* the result if true for $n = 1$ is true for all n. Hence as this will avoid considerable unnecessary detail in our computations, we shall state and prove our result for the case $n = 1$ only.

Therefore, let $x = (x_1, x_2)$ denote the observed value of X. x is a point in the sample space R, and θ a point in the parameter space Ω. R and Ω are two dimensional Euclidian spaces. On R, Ω and the product space $R \times \Omega$ is defined the Lebesgue measure, all sets considered being Lebesgue measurable. The Lebesgue measure of a set D of Ω is denoted by vD.

Next following Wallace (1959) we define a confidence procedure C as a Lebesgue measurable subset of the product space $R \times \Omega$; $C(x, \cdot)$ and $C(\cdot, \theta)$ denote the cross sections of C for given x and θ respectively, $C(x, \cdot)$ being the confidence sets. We define equivalent procedures as

DEFINITION 2.1. Confidence procedures C_1 and C_2 are equivalent if the set differences $(C_1 - C_1 \cdot C_2)$ and $(C_2 - C_1 \cdot C_2)$ are null subsets of $R \times \Omega$.

By Fubini's theorem it follows from Definition 2.1 that if C_1 and C_2 are equivalent, then for almost all x, the confidence sets $C_1(x, \cdot)$ and $C_2(x, \cdot)$ differ at most by null subsets of Ω, and conversely for almost all θ, the sections $C_1(\cdot, \theta)$ and $C_2(\cdot, \theta)$ differ by null subsets of R.

The definition of admissibility of confidence sets, formulated by Godambe (1961) and subsequently slightly modified by the author is as follows:

DEFINITION 2.2. A confidence procedure C_0 is admissible, if there exists no alternative procedure C_1 such that

$$\text{(i) } P_\theta[C_1(\cdot, \theta)] \geqq P_\theta[C_0(\cdot, \theta)] \qquad \text{for all } \theta \in \Omega,$$

and

(ii) $vC_1(x, \cdot) \leqq vC_0(x, \cdot)$ for almost all $x \varepsilon R$,

and the strict inequality holds either in (i) for some $\theta \varepsilon \Omega$, or in (ii) on a subset of R with positive measure.

As the discussion in Section 3 shows, an admissible procedure can exist only up to the equivalence in Definition 2.1. Subject to this qualification, in the following we prove the admissibility of the usual procedure according to a stronger definition (Definition 7.1 in Section 7) which includes the admissibility according to Definition 2.2.

3. Necessity of the restriction regarding equivalent class. It is obvious that without this restriction no admissible procedure can at all exist. For given any any procedure C we obtain a uniformly superior procedure C_1 as follows:

Take any isolated point $\theta = \theta_0$ in Ω. For all x for which $C(x, \cdot) \ni \theta_0$, we take $C_1(x, \cdot) = C(x, \cdot)$ and for all x for which $C(x, \cdot) \not\ni \theta_0$ we put $C_1(x, \cdot) = C(x, \cdot)$ + the point θ_0. Then clearly for all x, $vC_1(x, \cdot) = vC(x, \cdot)$ and for all $\theta \neq \theta_0$, $C_1(\cdot, \theta) = C(\cdot, \theta)$ while for $\theta = \theta_0$, $C_1(\cdot, \theta) = R$. Excluding the trivial case of $C(\cdot, \theta) = R$ for all $\theta \varepsilon \Omega$, θ_0 can always be so selected that the inclusion probability of C at θ_0 is < 1. Then C_1 has the same inclusion probability as C for all $\theta \neq \theta_0$ and higher inclusion probability at $\theta = \theta_0$ and is therefore uniformly superior to C. Thus there is no upper bound to the inclusion probabilities and hence no admissible procedure can exist. The uniformly superior procedures constructed by the method indicated above are however equivalent according to Definition 2.1 and hence an admissible procedure may exist upto the equivalence class.

Alternatively we may place a restriction on the geometrical form of the confidence sets, the restriction being such as to exclude the possibility of adding null subsets of Ω to the confidence sets. A suitable restriction of this type is that the confidence sets $C(x, \cdot)$ should be open sets, or alternatively, convex sets. In practice, confidence sets which are not convex are seldom, if ever, used. In such a restricted class of confidence procedures then, optimum procedures may exist and it follows from the main result of this paper that in this restricted class, in the cases $m = 1$ and $m = 2$ the usual procedure C_0 is unique in having the minimax property.

4. Randomized confidence procedures. A randomized procedure is a procedure in which the confidence set for each point x, instead of being a fixed set, is selected by a random process. Thus, for instance to each point x, we may assign k confidence sets $C_i(x, \cdot)$, $i = 1, 2, \cdots, k$; one of the sets being selected when x is the observed value, by an independent random process, with probability of selection $p_i(x)$ for the set $C_i(x, \cdot)$; $p_i(x)$ are measurable functions on R such that $\sum_{i=1}^{k} p_i(x) = 1$; k itself may be a measurable integral function of x. Clearly such a procedure determines a function $\phi(x, \theta)$ on $R \times \Omega$, which satisfies

(a) ϕ is a measurable function on the product space $R \times \Omega$;

(b) for all $(x, \theta) \varepsilon R \times \Omega$,

$$0 \leqq \phi(x, \theta) \geqq 1; \tag{2}$$

(c) for each x,

(3) $\phi(x, \theta)$ = probability that the point θ is included in the confidence set when x is the observed value;

(d) the expected measure of the confidence sets which we denote by $v\phi(x, \cdot)$ is given by

$$v\phi(x, \cdot) = \int_\Omega \phi(x, \theta)\, d\theta \tag{4}$$

where $d\theta$ is short for $d\theta_1\, d\theta_2$; and

(e) the expected inclusion probability at the point θ, which we denote by $P_\theta[\phi(\cdot, \theta)]$ is given by

$$P_\theta[\phi(\cdot, \theta)] = \int_R \phi(x, \theta) p(x, \theta)\, dx, \tag{5}$$

where dx is short for $dx_1\, dx_2$ and $p(x, \theta)$ is the probability density of X on R for given θ.

Therefore we take as our decision space the space defined by

$$\mathfrak{D} = \{\phi(x, \theta), \phi \text{ jointly measurable in } x \text{ and } \theta, 0 \leqq \phi(x, \theta) \leqq 1\}. \tag{6}$$

Every $\phi \varepsilon \mathfrak{D}$ may not represent a randomized confidence procedure. But it is easily seen that every $\phi \varepsilon \mathfrak{D}$, which is a simple or elementary function, determines a randomized confidence procedure and every other $\phi \varepsilon \mathfrak{D}$, being the limit of a non-decreasing sequence of simple functions, represents the limit of a corresponding sequence of randomized confidence procedures.

It is easily seen from (3) that any non-randomized procedure defined by a set C is obtained by putting

$$\phi(x, \theta) = 1 \quad \text{if} \quad \theta \varepsilon C(x, \cdot)$$
$$= 0 \quad \text{if} \quad \theta \not\varepsilon C(x, \cdot),$$

i.e., by taking ϕ to be the indicator function of the set C.

For this extended class of confidence procedures ϕ, we now generalize the Definition 2.1 of equivalent procedures.

Definition 4.1. Two procedures ϕ_1 and ϕ_2 are equivalent if

$$\phi_1(x, \theta) = \phi_2(x, \theta) \qquad \text{for almost all} \quad (x, \theta) \varepsilon R \times \Omega. \tag{7}$$

This definition is clearly consistent with Definition 2.1; i.e. an equivalence class under the latter definition is a subclass of an equivalence class under the Definition 4.1.

Similarly in place of Definition 2.2 for the extended class of procedures we define admissibility, by

Definition 4.2. A confidence procedure ϕ_0 is admissible if there exists no alternative procedure ϕ_1 such that

(8) (i) $P_\theta[\phi_1(\cdot, \theta)] \geqq P_\theta[\phi_0(\cdot, \theta)]$ for all $\theta \varepsilon \Omega$,

and (ii) $v\phi_1(x, \cdot) \leqq v\phi_0(x, \cdot)$ for almost all $x \varepsilon R$

and the strict inequality holds either in (i) for some $\theta \varepsilon \Omega$ or in (ii) for a subset of R with positive measure.

5. Preliminary results. We revert to the two dimensional case. Let the usual procedure ϕ_0 consist of confidence circles of fixed radius h and centered at x. Hence by (3)

$$\phi_0(x, \theta) = 1 \quad \text{if} \quad |x - \theta| \leqq h, \qquad = 0 \quad \text{otherwise}. \tag{9}$$

Here, as usual $|x - \theta|^2 = (x_1 - \theta_1)^2 + (x_2 - \theta_2)^2$. Let v_0 be the fixed area of the confidence circles and $(1 - \alpha)$ the fixed confidence level of ϕ_0. Then by (4) and (5)

$$v\phi_0(x, \cdot) = v_0 = \pi h^2 \tag{10}$$

and $$P_\theta[\phi_0(\cdot, \theta)] = 1 - \alpha = 1 - \exp(-h^2/2).$$

Next, following the method of Blyth (1951), we define a loss function $L_\phi(x, \theta)$ for any procedure ϕ by

$$L_\phi(x, \theta) = bv\phi(x, \cdot) - \phi(x, \theta) \tag{11}$$

where $v\phi(x, \cdot) = \int_\Omega \phi(x, \theta)\, d\theta$ as in (4) and $b = (2\pi)^{-1} \exp(-h^2/2)$. Hence the expected loss at θ, of the procedure ϕ, is

$$E_\theta L_\phi(x, \theta) = \int_R L_\phi(x, \theta)\cdot p(x, \theta)\, dx = b\cdot E_\theta v\phi(x, \cdot) - P_\theta[\phi(\cdot, \theta)] \tag{12}$$

where $P_\theta[\phi(\cdot, \theta)] = \int_R \phi(x, \theta)p(x, \theta)\, dx$ as in (5). Here $p(x, \theta)$ is the probability density of X on R, i.e.

$$p(x, \theta) = (2\pi)^{-1} \exp[-\tfrac{1}{2}|x - \theta|^2]. \tag{13}$$

We shall now state our result in the form of the following theorem:

THEOREM 5.1. *ϕ_0 being the usual procedure defined by* (9), *if ϕ_1 is any other procedure such that*

$$E_\theta L_{\phi_1}(x, \theta) \leqq E_\theta L_{\phi_0}(x, \theta) \quad \textit{for all} \quad \theta \varepsilon \Omega, \tag{14}$$

then ϕ_1 is equivalent to ϕ_0, i.e.

$$\phi_1(x, \theta) = \phi_0(x, \theta) \quad \textit{for almost all} \quad (x, \theta) \varepsilon (R \times \Omega).$$

NOTE 1. We note that the uniqueness up to the equivalent class of the minimax property of ϕ_0 follows immediately from the theorem. For if ϕ_1 is a procedure with lower confidence level $(1 - \alpha)$, such that

$$\sup_\theta E_\theta v\phi_1(x, \cdot) \leqq v_0 = E_\theta v\phi_0(x, \cdot),$$

then $E_\theta v\phi_1(x, \cdot) \leqq E_\theta v\phi_0(x, \cdot)$ and

$$P_\theta[\phi_1(\cdot, \theta)] \geqq 1 - \alpha = P_\theta[\phi_0(\cdot, \theta)] \quad \text{for all} \quad \theta \varepsilon \Omega.$$

Hence by (12), ϕ_1 satisfies (14) and hence must be equivalent to ϕ_0. The admissibility up to the equivalence of ϕ_0, according to Definition 4.2, similarly follows from the theorem. For if ϕ_1 is an alternative procedure satisfying (i) and (ii) of Definition 4.2, we have

$$v\phi_1(x, \cdot) \leqq v\phi_0(x, \cdot) \quad \text{for almost all} \quad x \varepsilon R$$

and $$P_\theta[\phi_1(\cdot, \theta)] \geqq P_\theta[\phi_0(\cdot, \theta)] \quad \text{for all} \quad \theta \varepsilon \Omega$$

so that ϕ_1 again satisfies (14) and hence must be equivalent to ϕ_0.

We revert to the main theorem, Theorem 5.1. Before proceeding to its proof it is necessary to obtain certain preliminary results. We first determine the Bayes procedure with respect to a prior density on Ω given by

$$\xi_\tau(\theta) = (2\pi\tau^2)^{-1} \exp(-\tfrac{1}{2}|\theta|^2\tau^{-2}) \tag{15}$$

where $|\theta|^2 = \theta_1^2 + \theta_2^2$ and τ be any arbitrary positive number. We state the result in the form of the following:

LEMMA 5.1. *The Bayes procedure ϕ_τ with respect to the distribution on Ω, given by* (15), *is a non-randomized procedure in which the confidence circles are centered at the point*

$$\theta' = xg^{-1}, \quad \textit{where} \quad g = 1 + \tau^{-2} \tag{16}$$

and with fixed radius c, where

$$c^2 = h^2g^{-1} + g^{-1}(2 \log g). \tag{17}$$

PROOF. Let E_τ denote expectation with respect to the prior density defined in (15). For brevity we put the loss function

$$L_{\phi_\tau}(x, \theta) = L_\tau(x, \theta) \quad \text{and} \quad v\phi_\tau(x, \cdot) = v_\tau(x). \tag{18}$$

Then we have by (11) and (15),

$$E_\tau L_\tau(x, \theta) = (2\pi\tau^2)^{-1} \int_\Omega \exp(-|\theta|^2/2\tau^2)\, d\theta \int_R [bv_\tau(x) - \phi_\tau(x, \theta)]p(x, \theta)\, dx \tag{19}$$

where $d\theta$ is short for $d\theta_1, d\theta_2$ and dx for dx_1, dx_2.

The integrand in the right hand side of (19) is seen to be integrable on $R \times \Omega$. For ϕ_τ being the Bayes procedure its expected risk must be less than that of ϕ_0 and hence is bounded from above. In the right hand side of (19), since the term $\phi_\tau(x, \theta)$ lies between 0 and 1, the integral arising from it also lies between 0 and 1. Hence the integral arising from the term involving $v_\tau(x)$ must also be finite. Thus the integrand in the right hand side of (19) is the difference of two integrable functions and is therefore itself integrable. Hence, by Fubini's theorem we may interchange the order of integration. We thus get from (19),

$$\begin{aligned} (20) \quad & E_\tau L_\tau(x, \theta) \\ & = (2\pi\tau^2)^{-1} \int_R dx \int_\Omega [bv_\tau(x) - \phi_\tau(x, \theta)]p(x, \theta) \exp(-|\theta|^2/2\tau^2)\, d\theta. \end{aligned}$$

Now from (13) after a little reduction we get

$$(21)\quad (2\pi\tau^2)^{-1}p(x,\theta)\exp(-|\theta|^2/2\tau^2) = (2\pi g\tau^2)^{-1}\exp(-|x|^2/2g\tau^2)g(2\pi)^{-1}\exp[-\tfrac{1}{2}g|\theta - xg^{-1}|^2]$$

where g has the value given by (16). Substituting (21) in (20), we have

$$(22)\quad \begin{aligned} E_\tau L_\tau(x,\theta) &= (2\pi g\tau^2)^{-1}\textstyle\int_R \exp(-|x|^2/2g\tau^2)\,dx\, g(2\pi)^{-1}\int_\Omega [bv_\tau(x) - \phi_\tau(x,\theta)] \\ &\quad\cdot \exp[-\tfrac{1}{2}g|\theta - xg^{-1}|^2]\,d\theta \\ &= (2\pi g\tau^2)^{-1}\textstyle\int_R \exp(-|x|^2/2g\tau^2)\,dx \\ &\quad\cdot\{bv_\tau(x) - g(2\pi)^{-1}\textstyle\int_\Omega \phi_\tau(x,\theta)\exp[-\tfrac{1}{2}g|\theta - xg^{-1}|^2]\,d\theta\}. \end{aligned}$$

By (18) and (4)

$$(23)\quad v_\tau(x) = \textstyle\int_\Omega \phi_\tau(x,\theta)\,d\theta.$$

Substituting (23) in (22), we get

$$(24)\quad E_\tau L_\tau(x,\theta) = (2\pi g\tau^2)^{-1}\textstyle\int_R \exp(-|x|^2/2g\tau^2)\,dx \cdot\int_\Omega\{b - g(2\pi)^{-1}\exp[-\tfrac{1}{2}g|\theta - xg^{-1}|^2]\}\phi_\tau(x,\theta)\,d\theta.$$

We obtain the Bayes procedure by choosing $\phi_\tau(x,\theta)$, $0 \leqq \phi_\tau(x,\theta) \leqq 1$, so as to minimize the right hand side and hence the inner integral on the right hand side of (24). Clearly the solution is given by taking

$$(25)\quad \begin{aligned} \phi_\tau(x,\theta) &= 0 \quad \text{if} \quad b > g(2\pi)^{-1}\exp[-\tfrac{1}{2}g|\theta - xg^{-1}|^2] \\ &= 1 \quad \text{if} \quad b \leqq g(2\pi)^{-1}\exp[-\tfrac{1}{2}g|\theta - xg^{-1}|^2]. \end{aligned}$$

Substituting in (25) the value of b by (11), and taking logarithms, it is seen that (25) is equivalent to

$$(26)\quad \begin{aligned} \phi_\tau(x,\theta) &= 1 \quad \text{if} \quad |\theta - xg^{-1}| \leqq c, \\ &= 0 \quad \text{otherwise,} \end{aligned}$$

where c is as in (17).

Remembering the meaning of $\phi_\tau(x,\theta)$ as given in (3), it is seen that (26) implies that the Bayes procedure ϕ_τ is as stated in the Lemma 5.1.

We next determine the risk of the Bayes procedure and prove

LEMMA 5.2. *The improvement in risk of the Bayes procedure ϕ_τ over the risk of the procedure ϕ_0 in (9) is for every τ bounded by $(bv_0 + \alpha)\tau^{-2}$.*

PROOF. Since in the Bayes procedure ϕ_τ, the confidence circles are of fixed area, we have in the right hand side of (22)

$$(27)\quad bv_\tau(x) = b\pi c^2 = bv_0 g^{-1} + g^{-1}(2\pi b\log g) \qquad \text{by (17) and (10).}$$

Also substituting for $\phi_\tau(x)$ by (26), we get in the right hand side of (22),

$$g(2\pi)^{-1}\int_\Omega \exp[-\tfrac{1}{2}g|\theta - xg^{-1}|^2]\phi_\tau(x, \theta)\,d\theta$$

(28) $$= g(2\pi)^{-1}\int_{|\theta - xg^{-1}|\leqq c}\exp[-\tfrac{1}{2}g|\theta - xg^{-1}|^2]\,d\theta = g\int_0^c \exp(-\tfrac{1}{2}gr^2)r\,dr$$

$$= 1 - \exp(-\tfrac{1}{2}gc^2) = 1 - g^{-1}\exp(-\tfrac{1}{2}h^2) \quad \text{by (17)}$$

$$= 1 - \alpha g^{-1} \quad \text{from (9) and (10).}$$

It is easily seen that

(29) $$1 - \alpha = P_\theta[\phi_0(\cdot, \theta)] = 1 - \exp(-\tfrac{1}{2}h^2).$$

Using (27) and (28), we have from (22),

(30) $$E_\tau L_\tau(x, \theta) = bv_0 g^{-1} + 2\pi b g^{-1}\log g - (1 - \alpha g^{-1}).$$

Also from (10) and (12) we get, writing $L_0(x, \theta)$ in place of $L_{\phi_0}(x, \theta)$,

(31) $$E_\theta L_0(x, \theta) = bv_0 - (1 - \alpha) \quad \text{for every} \quad \theta \varepsilon \Omega,$$

and hence

(32) $$E_\tau L_0(x, \theta) = bv_0(1 - \alpha).$$

Combining (30) and (32), we get

$$E_\tau L_0(x, \theta) - E_\tau L_\tau(x, \theta)$$

(33) $$= bv_0(1 - g^{-1}) + \alpha(1 - g^{-1}) - 2\pi b g^{-1}(\log g) < (bv_0 + \alpha)(1 - g^{-1})$$

$$< (bv_0 + \alpha)\tau^{-2}$$

substituting the value of g by (16). Thus Lemma 5.2 is proved.

We next define two functions on R, by

(34) $$U_0(x) = bv_0(x) - \int_\Omega \phi_0(x, \theta)p(x, \theta)\,d\theta$$

and $$U_1(x) = bv_1(x) - \int_\Omega \phi_1(x, \theta)p(x, \theta)\,d\theta$$

where $\phi_1(x, \theta)$ is the alternative confidence procedure in Theorem 5.1 and

(35) $$v_1(x) = v\phi_1(x) = \int_\Omega \phi_1(x, \theta)\,d\theta.$$

By (35),

$$U_1(x) = \int_\Omega [b - p(x, \theta)]\phi_1(x, \theta)\,d\theta.$$

Since by (6), $0 \leqq \phi_1(x) \leqq 1$, $U_1(x)$ is minimized by taking $\phi_1(x)$ to be such that

$$\phi_1(x, \theta) = 0 \quad \text{if} \quad b > p(x, \theta)$$
$$= 1 \quad \text{if} \quad b \leqq p(x, \theta)$$

which noting the value of b in (11) and of $p(x, \theta)$ in (13) is equivalent to

(36) $$\phi_1(x, \theta) = 0 \quad \text{if} \quad |x - \theta| > h$$
$$= 1 \quad \text{if} \quad |x - \theta| \leqq h.$$

But by (9) ϕ_0 is the procedure which satisfies (36).

Hence we have

$$U_1(x) \geqq U_0(x) \qquad \text{for all} \quad x \varepsilon R. \tag{37}$$

Now there are two possibilities, viz. that in (37) (I) the sign of equality holds for almost all $x \varepsilon R$ or (II) the sign of inequality holds on some subset S of R with positive measure.

Suppose alternative (II) is true. We now prove the following:

LEMMA 5.3. *If alternative* (II) *under* (37) *applies for the procedures ϕ_0 and ϕ_1 as described in the statement of Theorem* 5.1, *then the functions $U_1(x)$ and $U_0(x)$ defined by* (34) *satisfy the condition that the integral of $\{U_1(x) - U_0(x)\}$ with respect to x on R is finite and positive,* i.e. *putting*

$$M = \int_R [U_1(x) - U_0(x)]\, dx, \qquad 0 < M < \infty. \tag{38}$$

PROOF. For any positive number a we define a subset T_a of R by

$$x \varepsilon T_a \quad \text{if, and only if,} \quad |x| \leqq a \quad \text{where} \quad |x|^2 = x_1^2 + x_2^2. \tag{39}$$

Alternative (II) implies that there exists a positive number k ($k > 0$) such that for some a

$$\int_{T_a} [U_1(x) - U_0(x)]\, dx = k. \tag{40}$$

Let T_a^c be the complement of the set T_a. We then have from (11), writing $L_1(x, \theta)$ for $L_{\phi_1}(x, \theta)$ and $v_1(x)$ for $v\phi_1(x, \cdot)$,

$$\begin{aligned} (41)\quad & E_\tau L_1(x, \theta) \\ &= (2\pi\tau^2)^{-1} \int_\Omega \exp\,(-|\theta|^2/2\tau^2)\, d\theta \int_R [bv_1(x) - \phi_1(x, \theta)]p(x, \theta)\, dx. \end{aligned}$$

The integrand in the right hand side of (41) is the difference of two expressions each of which is integrable on $R \times \Omega$, and hence is itself integrable. Therefore by Fubini's theorem we can interchange the order of integration with respect to x and θ and thus have from (41), using (21),

$$\begin{aligned} & E_\tau L_1(x, \theta) \\ &= \int_R dx \int_\Omega [bv_1(x) - \phi_1(x, \theta)] \cdot (2\pi\tau^2)^{-1} \exp\,(-|\theta|^2/2\tau^2) p(x, \theta)\, d\theta \\ &= (2\pi g\tau^2)^{-1} \int_R \exp\,(-|x|^2/2g\tau^2)\, dx \\ &\quad \cdot \int_\Omega [bv_1(x) - \phi_1(x, \theta)] g(2\pi)^{-1} \exp\,[-\tfrac{1}{2}g|\theta - xg^{-1}|^2]\, d\theta \\ (42)\quad &= (2\pi g\tau^2)^{-1} \int_R \exp\,(-|x|^2/2g\tau^2)\, dx \\ &\quad \cdot \{bv_1(x) - \int_\Omega g(2\pi)^{-1} \cdot \phi_1(x, \theta) \exp\,[-\tfrac{1}{2}g|\theta - xg^{-1}|^2]\, d\theta\} \\ &= (2\pi g\tau^2)^{-1} \int_{T_a} \exp\,(-|x|^2/2g\tau^2)\, dx \\ &\quad \cdot \{bv_1(x) - g(2\pi)^{-1} \int_\Omega \exp\,[-\tfrac{1}{2}g|\theta - xg^{-1}|^2] \phi_1(x, \theta)\, d\theta\} \\ &\quad + (2\pi g\tau^2)^{-1} \int_{T_a^c} \exp\,(-|x|^2/2g\tau^2)\, dx \\ &\quad \cdot \{bv_1(x) - g(2\pi)^{-1} \int_\Omega \exp\,[-\tfrac{1}{2}g|\theta - xg^{-1}|^2] \phi_1(x, \theta)\, d\theta\}. \end{aligned}$$

We now write down the similar expression for $E_\tau L_0(x, \theta)$ and combine the two expressions. Putting

$$G_\tau(x) = \exp(-|x|^2/2g\tau^2) \cdot \{[bv_1(x) - g(2\pi)^{-1} \int_\Omega \exp(-\tfrac{1}{2}g|\theta - xg^{-1}|^2)\phi_1(x, \theta)\, d\theta] - [bv_0 - g(2\pi)^{-1} \int_\Omega \exp(-\tfrac{1}{2}g|\theta - xg^{-1}|^2)\phi_0(x, \theta)\, d\theta]\} \tag{43}$$

we have

$$E_\tau L_1(x, \theta) - E_\tau L_0(x, \theta) = (2\pi g\tau^2)^{-1} \int_{T_a} G_\tau(x)\, dx + (2\pi g\tau^2)^{-1} \int_{T_a^c} G_{\tau_a}(x)\, dx. \tag{44}$$

Now as $\tau \to \infty$, $g = 1 + \tau^{-2} \to 1$, and hence the probability density on Ω,

$$g(2\pi)^{-1} \exp(-\tfrac{1}{2}g|\theta - xg^{-1}|^2) \to (2\pi)^{-1} \exp(-\tfrac{1}{2}|\theta - x|^2) = p(x, \theta). \tag{45}$$

Hence, since $\phi_1(x, \theta)$ and $\phi_0(x, \theta)$ are bounded in absolute magnitude by 1, we have by the Helley-Bray theorem, in the right hand side of (43), as $\tau \to \infty$,

$$g(2\pi)^{-1} \int_\Omega \exp(-\tfrac{1}{2}g|\theta - xg^{-1}|)^2\phi_1(x, \theta) \to \int_\Omega \phi_1(x, \theta)p(x, \theta)\, d\theta, \tag{46}$$

and $\quad g(2\pi)^{-1} \int_\Omega \exp(-\tfrac{1}{2}g|\theta - xg^{-1}|)^2\phi_0(x, \theta) \to \int_\Omega \phi_0(x, \theta)p(x, \theta)\, d\theta.$

Using (46) in (43), and comparing with (34) it is seen that

$$\lim_{\tau\to\infty} G_\tau(x) = U_1(x) - U_0(x) = \text{integrand in the left hand side of (40)}. \tag{47}$$

We shall next show that in the first integral in the right hand side of (44), the limit can be taken under the integral sign. The function $G_\tau(x)$ is bounded in absolute magnitude uniformly in τ by the function

$$G(x) = bv_1(x) + 1 + bv_0 + 1. \tag{48}$$

By the definition of the set T_a in (39)

$$\int_{T_a} (bv_0 + 2)\, dx = (bv_0 + 2) \int_{|x| \leq a} dx_1\, dx_2 = (bv_0 + 2)\cdot \pi a^2. \tag{49}$$

Denote the probability density $p(x, \theta)$ when $\theta = 0$, by $p_0(x)$, i.e.

$$p_0(x) = (2\pi)^{-1} \exp(-\tfrac{1}{2}|x|^2). \tag{50}$$

As $p_0(x)$ decreases as $|x|$ increases, we have,

$$\begin{aligned} \int_{T_a} bv_1(x)\, dx &\leqq 2\pi \cdot \exp(a^2/2) \int_{T_a} bv_1(x)p_0(x)\, dx \\ &\leqq 2\pi \cdot \exp(a^2/2) \int_R bv_1(x)p_0(x)\, dx \\ &= 2\pi \cdot \exp(a^2/2) \cdot bE_{\theta=0}v_1(x). \end{aligned} \tag{51}$$

Now (14) combined (12) and (10) gives

$$bE_\theta v_1(x) \leqq bv_0 - (1 - \alpha) + P_\theta[\phi_1(\cdot, \theta)] \leqq bv_0 + \alpha, \quad \text{for all} \quad \theta \in \Omega, \tag{52}$$

since the inclusion probability always satisfies $P_\theta[\phi_1(\cdot, \theta)] \leqq 1$. (48), (49),

(51) and (52) combined give

(53) $$\int_{T_a} G(x)\,dx < \infty.$$

Hence by the dominated convergence theorem and using (47), we have

(54) $$\lim_{\tau\to\infty} \int_{T_a} G_\tau(x)\,dx = \int_{T_a} [U_1(x) - U_0(x)]\,dx = k \qquad \text{by (40).}$$

(54) implies that given any arbitrarily small positive number $\epsilon > 0$, we can find τ_0 such that for all $\tau \geqq \tau_0$

(55) $$\int_{T_a} G_\tau(x)\,dx \geqq k - \epsilon.$$

Next consider the second term in the right hand side of (44). By the property of the Bayes procedure, the value of $G_\tau(x)$ becomes reduced if the term in the first square bracket in the right hand side of (43) is replaced by the posterior risk

$$bv_\tau(x) - g(2\pi)^{-1} \int_\Omega \exp\left(-\tfrac{1}{2}g|\theta - xg^{-1}|^2\right)\phi_\tau(x, \theta)\,d\theta.$$

Again by the Bayes property the resulting integrand is non-positive for all x, and hence the integration can be extended from the set T_a^c to the space R. We thus have

(56)
$$\begin{aligned}
&(2\pi g\tau^2)^{-1} \int_{T_a^c} G_\tau(x)\,dx \\
&\quad \geqq (2\pi g\tau^2)^{-1} \int_{T_a^c} \exp\left(-|x|^2/2g\tau^2\right)dx \\
&\qquad \cdot \{[bv_\tau(x) - \tfrac{1}{2}g \textstyle\int_\Omega \exp\left(-\tfrac{1}{2}g\,|\theta - xg^{-1}|^2\right)\phi_\tau(x, \theta)\,d\theta] \\
&\qquad - [bv_0 - \tfrac{1}{2}g \textstyle\int_\Omega \exp\left(-\tfrac{1}{2}g\,|\theta - xg^{-1}|^2\right)\phi_0(x, \theta)\,d\theta]\} \\
&\quad \geqq (2\pi g\tau^2)^{-1} \int_R \exp\left(-|x|^2/2g\tau^2\right)dx \\
&\qquad \cdot \{[bv_\tau(x) - \tfrac{1}{2}g \textstyle\int_\Omega \exp\left(-\tfrac{1}{2}g\,|\theta - xg^{-1}|^2\right)\phi_\tau(x, \theta)\,d\theta] \\
&\qquad - [bv_0(x) - \tfrac{1}{2}g \textstyle\int_\Omega \exp\left(-\tfrac{1}{2}g\,|\theta - xg^{-1}|^2\right)\phi_0(x, \theta)\,d\theta]\} \\
&\quad = E_\tau L_\tau(x, \theta) - E_\tau L_0(x, \theta) \qquad \text{by (22)} \\
&\quad \geqq -(bv_0 + \alpha)\tau^{-2} \qquad \text{by (33).}
\end{aligned}$$

Combining (55) and (56) with (44), we get

(57)
$$\begin{aligned}
&E_\tau L_1(x, \theta) - E_\tau L_0(x, \theta) \\
&\quad \geqq (k - \epsilon)(2\pi g\tau^2)^{-1} - (bv_0 + \alpha)\tau^{-2} \qquad \text{for all } \tau \geqq \tau_0.
\end{aligned}$$

But by (14), we have

(58) $$E_\tau L_1(x, \theta) - E_\tau L_0(x, \theta) \leqq 0.$$

Hence (57) implies that

(59) $$k \leqq 2\pi g(bv_0 + \alpha) + \epsilon \qquad \text{for all } \tau \geqq \tau_0.$$

Since ϵ can be made arbitrarily small and $g \leqq 2$ for $\tau \geqq 1$, it follows from (59) that

$$k \leqq 4\pi(bv_0 + \alpha). \tag{60}$$

As the integrand in the left hand side of (40) is non-negative, the integral is non-decreasing as a increases. It follows from (60) that as $a \to \infty$, so that $T_a \to R$, the integral converges to a finite limit $M \geqq k > 0$. This completes the proof of Lemma 5.3.

6. Main result. We shall now show that our main theorem, Theorem 5.1, follows from (38).

[Explanatory note: As the following argument is rather long, we shall give its brief outline. We consider the improvement in the expected risk of the procedure ϕ_1 over that of ϕ_0, viz. $E_\tau L_1(x, \theta) - E_\tau L_0(x, \theta)$. Expressing each expectation as in the right hand side of (42), we combine the two expressions. It is then shown that the worsening of the expected risk of ϕ_1 over that of ϕ_0 on the set T_a, can be made arbitrarily close to M by taking a sufficiently large. This worsening has to be offset by the improvement in risk on the complementary set T_a^c. But it is shown that the latter, for any fixed a, can be made arbitrarily small by making τ sufficiently large. Hence M must be $= 0$. The theorem follows from this.]

It is necessary first to introduce some new notation. We define for each $x \varepsilon R$, subsets A_x, H_x and K_x of Ω by

(61) $\theta \varepsilon A_x$, if and only if, $|\theta - x| \leqq h$;

$\theta \varepsilon H_x$, if and only if, $h < |\theta - x| \leqq (h + d)$;

and $\theta \varepsilon K_x$, if and only if, $|\theta - x| > (h + d)$.

Here $d > 0$ is a constant whose value will be suitably fixed later. Then

$$A_x + H_x + K_x = \Omega \qquad \text{for all} \quad x \varepsilon R. \tag{62}$$

Now in (34), using (4),

$$U_1(x) = \int_\Omega [b - p(x, \theta)]\phi_1(x, \theta)\, d\theta$$

and

$$U_0(x) = \int_\Omega [b - p(x, \theta)]\phi_0(x, \theta)\, d\theta$$

Hence,

$$U_1(x) - U_0(x) = \int_\Omega [b - p(x, \theta)][\phi_1(x, \theta) - \phi_0(x, \theta)]\, d\theta. \tag{63}$$

Substituting (63) in the right hand side of (38), using (62) and substituting for $\phi_0(x, \theta)$ by (9), we get

$$M = M_1 + M_2 + M_3, \tag{64}$$

where

$$M_1 = \int_R dx \int_{A_x} [p(x, \theta) - b][1 - \phi_1(x, \theta)]\, d\theta, \tag{65}$$

$$M_2 = \int_R dx \int_{H_x} [b - p(x, \theta)]\phi_1(x, \theta)\, d\theta, \tag{66}$$

$$M_3 = \int_R dx \int_{K_x} [b - p(x, \theta)]\phi_1(x, \theta)\, d\theta. \tag{67}$$

Noting the value of b in (11) and of $p(x, \theta)$ in (13), we have in (64),

(68) $$M_1 \geqq 0, \quad M_2 \geqq 0 \quad \text{and} \quad M_3 \geqq 0.$$

Next we put for each $x \varepsilon R$,

(69)
$$v_A(x) = \int_{A_x} [1 - \phi_1(x, \theta)]\, d\theta,$$
$$v_H(x) = \int_{H_x} \phi_1(x, \theta)\, d\theta,$$
$$v_K(x) = \int_{K_x} \phi_1(x, \theta)\, d\theta.$$

Next, we prove

LEMMA 6.1. *The relations* (65), (66) *and* (67) *respectively imply that the functions* $v_A(x)$, $v_H(x)$ *and* $v_K(x)$ *defined in* (69) *are such that*

(70) $$\int_R v_A^2(x)\, dx < \infty,$$

(71) $$\int_R v_H^2(x)\, dx < \infty$$

provided the constant d is sufficiently small and

(72) $$\int_R v_K(x)\, dx < \infty.$$

PROOF. By (61), A_x is a circle of radius h, and by (6), $0 \leqq \phi_1(x, \theta) \leqq 1$. Hence in (69) $v_A(x) \leqq \pi h^2$.

Let $h_1 \leqq h$, be such that the concentric circles in Ω, centered at the point $\theta' = x$, and with radii h and h_1 enclose on area $v_A(x)$. Hence

(73) $$v_A(x) = \int_{A_x} [1 - \phi_1(x, \theta)]\, d\theta = \pi(h^2 - h_1^2).$$

Then as $p(x, \theta)$ is a decreasing function of $|x - \theta|$, we have in (65),

(74)
$$\begin{aligned}\int_{A_x} p(x, \theta)[1 - \phi_1(x, \theta)]\, d\theta &\geqq \int_{h_1 \leqq |x-\theta| \leqq h} p(x, \theta)\, d\theta \\ &= (2\pi)^{-1} \int_{h_1}^{h} 2\pi r \cdot \exp(-r^2/2)\, dr \qquad \text{by (13)} \\ &= \exp(-h^2/2)[\exp((h^2 - h_1^2)/2) - 1] \\ &\geqq \exp(-h^2/2)[\tfrac{1}{2}(h^2 - h_1^2) + \tfrac{1}{8}(h^2 - h_1^2)^2] \\ &= \exp(-h^2/2)[(2\pi)^{-1} v_A(x) + (8\pi^2)^{-1} v_A^2(x)] \qquad \text{by (73).}\end{aligned}$$

Also by (69),

(75) $$\int_{A_x} b[1 - \phi_1(x, \theta)]\, d\theta = b v_A(x) = (2\pi)^{-1} \exp(-h^2/2) v_A(x) \qquad \text{by (11).}$$

Combining (74) and (75) with (65), we get

(76) $$(8\pi^2)^{-1} \exp(-h^2/2) \int_R v_A^2(x)\, dx \leqq M_1.$$

This proves (70).

Next let $h_2 \geqq h$, be such that concentric circles centered at the point θ' and with radii h and h_2 enclose an area equal to $v_H(x)$, i.e.

$$(77)\qquad v_H(x) = \int_{H_x} \phi_1(x, \theta)\, d\theta = \pi(h_2^2 - h^2).$$

Then again by the property of $p(x, \theta)$ of decreasing with $|x - \theta|$, we have

$$\begin{aligned}(78)\qquad &\int_{H_x} p(x, \theta)\phi_1(x, \theta)\, d\theta \\ &\leqq \int_{h \leqq |x-\theta| \leqq h_2} p(x, \theta)\, d\theta \\ &= \exp(-h^2/2)\{1 - \exp[-\tfrac{1}{2}(h_2^2 - h^2)]\} \\ &\leqq \exp(-h^2/2)\{\tfrac{1}{2}(h_2^2 - h^2) - \tfrac{1}{8}(h_2^2 - h^2)^2 + (1/48)(h_2^2 - h^2)^3\},\end{aligned}$$

since $e^{-t} \geqq 1 - t/1! + t^2/2! - t^3/3!$ for all t. Hence using (77), we get from (78),

$$(79)\qquad \int_{H_x} p(x, \theta)\phi_1(x, \theta)\, d\theta \leqq \exp(-h^2/2) \cdot\{(2\pi)^{-1}v_H(x) - (8\pi^2)^{-1}v_H^2(x) + (48\pi^3)^{-1}v_H^3(x)\}.$$

Also

$$\begin{aligned}(80)\qquad \int_{H_x} b\phi_1(x, \theta)\, d\theta &= bv_H(x) \qquad \text{by (69)} \\ &= \exp(-h^2/2)(2\pi)^{-1}v_H(x) \qquad \text{by (11)}.\end{aligned}$$

Combining (79) and (80), we have

$$(81)\qquad \int_{H_x} [b - p(x, \theta)]\phi_1(x, \theta)\, d\theta \geqq \exp(-h^2/2)\{(8\pi^2)^{-1}v_H^2(x) - (48\pi^3)^{-1}v_H^3(x)\}.$$

Now since by (61), H_x is the area between two concentric circles of radii h and $h + d$, and since by (6) $0 \leqq \phi_1 \leqq 1$, we have in (69)

$$(82)\qquad v_H(x) \leqq \pi(2h\,d + d^2).$$

We take d to be sufficiently small, so that

$$(83)\qquad (2hd + d^2) \leqq 3.$$

We then have, from (81),

$$(84)\qquad \int_{H_x} [b - p(x, \theta)]\phi_1(x, \theta)\, d\theta \geqq (16\pi^2)^{-1}\exp(-h^2/2)v_H^2(x),$$

so that by (66),

$$M_2 \geqq (16\pi^2)^{-1}\exp(-h^2/2)\int_R v_H^2(x)\, dx.$$

Thus (71) is proved.

Lastly, since for $x \varepsilon K_x$

$$|x - \theta| > (h + d) \qquad \text{by (61)},$$

$$\begin{aligned}\int_{K_x} p(x, \theta)\phi_1(x, \theta)\, d\theta &\leqq (2\pi)^{-1}\exp[-\tfrac{1}{2}(h + d)^2]\int_{K_x}\phi_1(x, \theta)\, d\theta \\ &= (2\pi)^{-1}\exp[-\tfrac{1}{2}(h + d)^2]v_K(x) \qquad \text{by (69)}.\end{aligned}$$

Hence by (69) and (11),

$$\int_{K_x} [b - p(x, \theta)]\phi_1(x, \theta)\, d\theta \geqq (2\pi)^{-1}\{\exp(-h^2/2) - \exp[-\tfrac{1}{2}(h + d)^2]\}v_K(x),$$

from which and (67), (72) follows. This completes the proof of Lemma 6.1.

Now let $\epsilon > 0$ by any given arbitrarily small number. The relations (65) to (67), (70) and (71), imply that T_a and T_a^c being the sets defined by (39), we can find a_0, such that for all $a \geqq a_0$, all the following relations hold, viz.

$$\begin{aligned}
&\text{(i)} && \int_{T_a} dx \int_{A_x} [p(x, \theta) - b][1 - \phi_1(x, \theta)]\, d\theta \geqq M_1 - \epsilon,\\
&\text{(ii)} && \int_{T_a} dx \int_{H_x} [b - p(x, \theta)]\phi_1(x, \theta)\, d\theta \geqq M_2 - \epsilon,\\
(85)\quad &\text{(iii)} && \int_{T_a} dx \int_{K_x} [b - p(x, \theta)]\phi_1(x, \theta)\, d\theta \geqq M_3 - \epsilon\\
&\text{(iv)} && \int_{T_a^c} v_A^2(x)\, dx \leqq \epsilon^2,\\
&\text{(v)} && \int_{T_a^c} v_H^2(x)\, dx \leqq \epsilon^2.
\end{aligned}$$

We select any particular $a \geqq a_0$, which we now keep fixed. We next prove the following

Lemma 6.2. *The relation* (85) (iv) *implies that for any fixed* $a > 0$ *and* $\epsilon > 0$, *as* $\tau \to \infty$, *each of the following relations hold*, viz.

$$\begin{aligned}
&\text{(i)} && (2\pi g\tau^2)^{-1} \int_{T_a^c} \exp(-|x|^2/2g\tau^2)v_A(x)\, dx = O(\epsilon/\tau),\\
(86)\quad &\text{(ii)} && (2\pi g\tau^2)^{-1} \int_{T_a^c} \exp(-|x|^2/2g\tau^2)|x|\, \tau^{-1}v_A(x)\, dx = O(\epsilon/\tau),\\
&\text{(iii)} && (2\pi g\tau^2)^{-1} \int_{T_a^c} \exp(-|x|^2/2g\tau^2)|x|^2\, \tau^{-2}v_A(x)\, dx = O(\epsilon/\tau).
\end{aligned}$$

Proof. All the three relations are proved by a common method. First let

$$(87)\qquad (2\pi g\tau^2)^{-1} \int_{T_a^c} \exp(-|x|^2/2g\tau^2)\, dx = k_1,$$

and $\qquad (2\pi g\tau^2)^{-1} \int_{T_a^c} \exp(-|x|^2/2g\tau^2)v_A(x)\, dx = k_1u_1 \qquad$ say.

Then putting $v_A(x) = u_1 + \Delta_1$, where $\Delta_1 = \Delta_1(x)$,

$$(88)\qquad (2\pi g\tau^2)^{-1} \int_{T_a^c} \exp(-|x|^2/2g\tau^2)\Delta_1\, dx = 0.$$

Hence

$$\begin{aligned}
&(2\pi g\tau^2)^{-1} \int_{T_a^c} \exp(-|x|^2/2g\tau^2)v_A^2(x)\, dx\\
&\quad = (2\pi g\tau^2)^{-1} \int_{T_a^c} \exp(-|x|^2/2g\tau^2)[u_1^2 + 2u_1\cdot\Delta_1 + \Delta_1^2]\, dx\\
(89)\qquad &\quad = (2\pi g\tau^2)^{-1} \int_{T_a^c} \exp(-|x|^2/2g\tau^2)[u_1^2 + \Delta_1^2]\, dx \qquad \text{by (88)}\\
&\quad \geqq (2\pi g\tau^2)^{-1} \int_{T_a^c} \exp(-|x|^2/2g\tau^2)\cdot u_1^2\, dx\\
&\quad = k_1u_1^2 \qquad \text{by (87)}.
\end{aligned}$$

Also by (85)(iv),

$$(90)\qquad \text{left hand side of (89)} \leqq \epsilon^2(2\pi g\tau^2)^{-1}.$$

By (89) and (90), and since $g > 1$, by (16),

$$u_1^2 \leqq \epsilon^2(2\pi g\tau^2k_1)^{-1} < \epsilon^2(2\pi\tau^2k_1)^{-1}.$$

Hence,

(91) $$k_1^2 u_1^2 < k_1 \epsilon^2 (2\pi\tau^2)^{-1}.$$

Also from (87)

(92) $$k_1 = (2\pi g\tau^2)^{-1} \int_a^\infty \exp(-r^2/2g\tau^2) 2\pi r \, dr = \exp(-a^2/2g\tau^2) < 1.$$

Substituting for k_1 by (92) in the right hand side of (91) and taking square roots, we obtain

(93) $$k_1 u_1 < \epsilon\tau^{-1}(2\pi)^{-\frac{1}{2}}$$

thus proving (i) of the lemma.

Next put,

(94) $$(2\pi g\tau^2)^{-1} \int_{T_a^c} \exp(-|x|^2/2g\tau^2)|x| \, \tau^{-1} \, dx = k_2,$$

and $$(2\pi g\tau^2)^{-1} \int_{T_a^c} \exp(-|x|^2/2g\tau^2)|x| \, \tau^{-1} v_A(x) \, dx = k_2 u_2.$$

Again putting, $v_A(x) = u_2 + \Delta_2$, where $\Delta_2 = \Delta_2(x)$,

$$(2\pi g\tau^2)^{-1} \int_{T_a^c} \exp(-|x|^2/2g\tau^2)|x| \, \tau^{-1} \Delta_2 \, dx = 0.$$

Hence as before,

(95) $$(2\pi g\tau^2)^{-1} \int_{T_a^c} \exp(-|x|^2/2g\tau^2)|x| \, \tau^{-1} v_A^2(x) \, dx \geqq k_2 u_2^2.$$

In the integrand in (95), the factor $\exp(-|x|^2/2g\tau^2)|x| \, \tau^{-1}$ is maximized when $|x| \, \tau^{-1} = g^{\frac{1}{2}}$, and hence

$$\exp(-|x|^2/2g\tau^2)|x| \, \tau^{-1} \leqq g^{\frac{1}{2}} \exp(-\tfrac{1}{2})$$
$$\leqq 1 \qquad \text{for} \quad \tau \geqq \tau_0$$

for sufficiently large τ_0. Hence by (85) (iv),

(96) $$\text{left hand side of (95)} \leqq \epsilon^2 (2\pi g\tau^2)^{-1} < \epsilon^2 (2\pi\tau^2)^{-1}$$

and therefore

(97) $$k_2^2 u_2^2 < k_2 \epsilon^2 (2\pi\tau^2)^{-1} \qquad \text{for all} \quad \tau \geqq \tau_0.$$

Also from (94),

(98) $$k_2 = (2\pi g\tau^2)^{-1} \int_a^\infty \exp(-r^2/2g\tau^2) \cdot r\tau^{-1} \cdot 2\pi r \, dr$$
$$= g^{-1} \int_{a\tau^{-1}}^\infty \exp(-\rho^2/2g)\rho^2 \, d\rho, \qquad \text{by putting} \quad \rho = r\tau^{-1}.$$

In (98), for $\tau > 1$, $g < 2$, so that the integrand is uniformly bounded by $\exp(-\rho^2/4) \cdot \rho^2$ which is integrable. Hence by the dominated convergence theorem, as $\tau \rightarrow \infty$,

$$k_2 \rightarrow k' = \int_0^\infty \exp(-\rho^2/2)\rho^2 \, d\rho = \tfrac{1}{2}(2\pi)^{\frac{1}{2}}.$$

Hence for sufficiently large τ_0, we have for all $\tau \geqq \tau_0$, $k_2 \leqq 4$ say, and hence from (97),

$$(99) \qquad k_2 u_2 < \epsilon\tau^{-1} \cdot 2(2\pi)^{-\frac{1}{2}} \qquad \text{for all} \quad \tau \geqq \tau_0 ,$$

thus proving (ii) of Lemma 6.2.

Lastly put,

$$(100) \qquad (2\pi g\tau^2)^{-1} \int_{T_a^c} \exp\, (-|x|^2/2g\tau^2)|x|\, \tau^{-2}\, dx = k_3 ,$$

and $\quad (2\pi g\tau^2)^{-1} \int_{T_a^c} \exp\, (-|x|^2/2g\tau^2)|x|^2\, \tau^{-2} v_A(x)\, dx = k_3 \cdot u_3 .$

Proceeding as before we get,

$$(101) \qquad (2\pi g\tau^2)^{-1} \int_{T_a^c} \exp\, (-|x|^2/2g\tau^2)|x|^2\, \tau^{-2} v_A(x)\, dx \geqq k_3 u_3^2 .$$

Now, $\exp\, (-|x|^2/2g\tau^2) \cdot |x|^2\, \tau^{-2}$ is maximized for $|x|\, \tau^{-1} = (2g)^{\frac{1}{2}}$. Hence

$$\begin{aligned} \exp\, (-|x|^2/2g\tau^2) \cdot |x|^2\, \tau^{-2} &\leqq 2g \exp\, (-1) \\ &\leqq 1 \qquad \text{for all} \quad \tau \geqq \tau_0 \qquad \text{sufficiently large.} \end{aligned}$$

Hence by (85) (iv),

$$\text{left hand side of (101)} \quad \leqq \epsilon^2 (2\pi g\tau^2)^{-1} < \epsilon^2 (2\pi\tau^2)^{-1} \qquad \text{by (16).}$$

Hence,

$$(102) \qquad k_3^2 u_3^2 < k_3 \epsilon^2 (2\pi\tau^2)^{-1} \qquad \text{for all} \quad \tau \geqq \tau_0 .$$

Also as $\tau \to \infty$, $k_3 \to k_3' = \int_0^\infty \exp\, (-t^2/2) t^3\, dt = 2$. Hence taking τ_0 sufficiently large, $k_3 \leqq 4$ say, for all $\tau \geqq \tau_0$, so that from (102)

$$(103) \qquad k_3 u_3 \leqq \epsilon\tau^{-1} \cdot 2(2\pi)^{-\frac{1}{2}}.$$

This completes the proof of Lemma 6.2

We can now proceed to the proof of our main theorem.

Proof of Theorem 5.1. From (44), we have

$$\begin{aligned} (104) \quad & E_\tau L_0(x, \theta) - E_\tau L_1(x, \theta) \\ & \qquad = -(2\pi g\tau^2)^{-1} \int_{T_a} G_\tau(x)\, dx - (2\pi g\tau^2)^{-1} \int_{T_a^c} G_\tau(x)\, dx \end{aligned}$$

where $G_\tau(x)$ is given by (43).

Now in $-G_\tau(x)$, in substituting for $v_1(x)$ and v_0 by (4), we have

$$\begin{aligned} & [bv_0(x) - g(2\pi)^{-1} \int_\Omega \exp\, (-\tfrac{1}{2}g\, |\theta - xg^{-1}|^2)\phi_0(x, \theta)\, d\theta] \\ & \qquad - [bv_1 - g(2\pi)^{-1} \int_\Omega \exp\, (-\tfrac{1}{2}g\, |\theta - xg^{-1}|^2)\phi_1(x, \theta)\, d\theta] \\ (105) \qquad & = \int_\Omega [b - g(2\pi)^{-1} \exp\, (-\tfrac{1}{2}g\, |\theta - xg^{-1}|^2)][\phi_0(x, \theta) - \phi_1(x, \theta)]\, d\theta \\ & = \text{integral on the set } A_x + \text{integral on the set } H_x + \text{integral on} \\ & \qquad \text{the set } K_x , \qquad \text{by (62).} \end{aligned}$$

Further, since by (9) and (61),

$$\begin{aligned} \phi_0(x, \theta) &= 1 \quad \text{for} \quad \theta \,\varepsilon\, A_x \\ &= 0 \quad \text{for} \quad \theta \,\varepsilon\, (H_x + K_x), \end{aligned}$$

the extreme right hand side of (105)

$$\begin{aligned} &= \textstyle\int_{A_x} [b - g(2\pi)^{-1} \exp(-\frac{1}{2}g\,|\theta - xg^{-1}|^2)]\cdot[1 - \phi_1(x_1, \theta)]\, d\theta \\ &\quad + \textstyle\int_{H_x} [g(2\pi)^{-1} \exp(-\frac{1}{2}g\,|\theta - xg^{-1}|^2) - b]\phi_1(x, \theta)\, d\theta \\ &\quad + \textstyle\int_{K_x} [g(2\pi)^{-1} \exp(-\frac{1}{2}g\,|\theta - xg^{-1}|^2 - b]\phi_1(x, \theta)\, d\theta \\ &= [bv_A(x) - g(2\pi)^{-1} \textstyle\int_{A_x} \exp(-\frac{1}{2}g\,|\theta - xg^{-1}|^2)\cdot[1 - \phi_1(x, \theta)]\, d\theta] \qquad (106) \\ &\quad + [g(2\pi)^{-1} \textstyle\int_{H_x} \exp(-\frac{1}{2}g\,|\theta - xg^{-1}|^2)\phi_1(x, \theta)\, d\theta - bv_{H_x}] \\ &\quad + [g(2\pi)^{-1} \textstyle\int_{K_x} \exp(-\frac{1}{2}g\,|\theta - xg^{-1}|^2)\phi_1(x, \theta)\, d\theta - bv_{K_x}] \qquad \text{by (69).} \end{aligned}$$

Then substituting by (43), (105) and (106) in the second integral in (104), we get

$$\begin{aligned} &E_\tau L_0(x, \theta) - E_\tau L_1(x, \theta) \\ &\quad = -(2\pi g\tau^2)^{-1} \textstyle\int_{T_a} G_\tau(x)\, dx \\ &\qquad + (2\pi g\tau^2)^{-1} \textstyle\int_{T_a^c} \exp(-|x|/2g\tau^2)\, dx \\ &\qquad\quad \cdot\{bv_A(x) - g(2\pi)^{-1} \textstyle\int_{A_x} \exp(-\frac{1}{2}g\,|\theta - xg^{-1}|^2)[1 - \phi_1(x, \theta)]\, d\theta\} \\ &\qquad + (2\pi g\tau^2)^{-1} \textstyle\int_{T_a^c} \exp(-|x|^2/2g\tau^2)\, dx \qquad (107) \\ &\qquad\quad \cdot\{g(2\pi)^{-1} \textstyle\int_{H_x} \exp(-\frac{1}{2}g\,|\theta - xg^{-1}|^2)\phi_1(x, \theta)\, d\theta - bv_H(x)\} \\ &\qquad + (2\pi g\tau^2)^{-1} \textstyle\int_{T_a^c} \exp(-|x^2|/2g\tau^2)\, dx \\ &\qquad\quad \cdot\{g(2\pi)^{-1} \textstyle\int_{K_x} \exp(-\frac{1}{2}g\,|\theta - xg^{-1}|^2)\phi_1(x, \theta)\, d\theta - bv_K(x)\} \\ &\quad = (2\pi g\tau^2)^{-1}\{-I_1 + I_2 + I_3 + I_4\} \qquad \text{say,} \end{aligned}$$

where I_1, I_2, I_3 and I_4, respectively, denote the first to the fourth integrals on the right hand side.

We now prove the theorem by showing that as $\tau \to \infty$, $I_1 \geqq M - 4\epsilon$, while I_2, I_3 and I_4 become small.

First consider I_1. From (54) and (63), and partitioning the integral on Ω in (63) into integrals on A_x, H_x, and K_x by (62), and putting $\phi_0(x, \theta) = 1$ on A_x and $\phi_0(x, \theta) = 0$ on $(H_x + K_x)$ we get,

$$\begin{aligned} \lim_{\tau\to\infty} I_1 &= \textstyle\int_{T_a} dx \int_\Omega [b - p(x, \theta)][\phi_1(x, \theta) - \phi_0(x, \theta)]\, d\theta \\ &= \textstyle\int_{T_a} dx \int_{A_x} [p(x, \theta) - b][1 - \phi_1(x, \theta)]\, d\theta \end{aligned}$$

(108) $$+ \int_{T_a} dx \int_{H_x} [b - p(x, \theta)]\phi_1(x, \theta)\, d\theta$$
$$+ \int_{T_a} dx \int_{K_x} [b - p(x, \theta)]\phi_1(x, \theta)\, d\theta$$
$$\geqq M - 3\epsilon \qquad \text{by (85) and (64).}$$

Hence we can take τ_0 sufficiently large so that

(109) $$I_1 \geqq M - 4\epsilon \qquad \text{for all} \quad \tau \geqq \tau_0 .$$

Next consider the integral I_4 in (107). Writing

$$\int_{K_x} \phi_1(x)\, dx \qquad \text{for} \quad v_K(x) \qquad \text{by (69)}$$

(110) $$I_4 = \int_{T_a^c} \exp(-|x|^2/2g\tau^2)\, dx \cdot \int_{K_x} [g(2\pi)^{-1} \exp(-\tfrac{1}{2}g\,|\theta - xg^{-1}|^2) - b]\phi_1(x, \theta)\, d\theta.$$

In the right hand side of (110), the integrand is bounded in absolute magnitude on the product set $R \times K_x$, uniformly for all $\tau \geqq 1$, by $(b + \pi^{-1})\phi_1(x, \theta)$, since $g = 1 + \tau^{-2} \leqq 2$, and

$$\int_R dx \int_{K_x} (b + \pi^{-1})\phi_1(x, \theta)\, d\theta = (b + \pi^{-1}) \int_R v_K(x)\, dx < \infty \qquad \text{by (69) and (72).}$$

Hence by the dominated convergence theorem, we can take the limit as $\tau \to \infty$ under the integral sign. Since as $\tau \to \infty$, $g \to 1$ by (16), we get noting the value of (b) in (11),

(111) integrand in the right hand side of (110)
$$\to \{[(2\pi)^{-1} \exp(-\tfrac{1}{2}\,|\theta - x|^2) - (2\pi)^{-1} \exp(-\tfrac{1}{2}h^2)]\phi_1(x, \theta)\} \leqq 0,$$
$$\text{as by (61),} \qquad |\theta - x| > h \quad \text{for} \quad \theta \,\varepsilon\, K_x .$$

From (111) it follows that

(112) $$\lim_{\tau \to \infty} I_4 \leqq 0.$$

Hence we can take τ_0 sufficiently large, so that

(113) $$I_4 \leqq \epsilon \qquad \text{for all} \quad \tau \geqq \tau_0 .$$

Next consider the integral I_2 in (107). The calculation of its limit is somewhat more involved. For all $\theta \,\varepsilon\, A_x$, since by (61),

$$|\theta - x| \leqq h,$$

(114) $$|\theta - xg^{-1}| \leqq h + |x| - |x|\, g^{-1}$$
$$= h + |x|\,(g\tau^2)^{-1} \qquad \text{by (16)}$$
$$= h + f$$

where we write

(115) $$f = f(x) = |x|\,(g\tau^2)^{-1} \qquad \text{for brevity.}$$

Let $h_3 = h_3(x) \leqq h + f$ be such that the concentric circles in Ω, centered at $\theta' = xg^{-1}$ and with radii h_3, and $(h + f)$ enclose an area equal to $v_A(x)$, i.e.

$$\pi[(h + f)^2 - h_3^2] = v_A(x). \tag{116}$$

Since $\exp(-\frac{1}{2}|\theta - xg^{-1}|^2)$ is a decreasing function of $|\theta - xg^{-1}|$, subject to

$$\int_{A_x} [1 - \phi_1(x, \theta)]\, dx = v_A(x) \qquad \text{by (69)}$$

and subject to (114), $\int_{A_x} \exp[-\frac{1}{2}g|\theta - xg^{-1}|^2][1 - \phi_1(x, \theta)]\, d\theta$ is minimized by taking A_x to be the area contained between the concentric circles with center $\theta' = xg^{-1}$ and radii h_3 and $(h + f)$ and putting $\phi_1(x, \theta) = 0$ in this area. Hence

$$\begin{aligned}
&\int_{A_x} \exp(-\tfrac{1}{2}g|\theta - xg^{-1}|^2)[1 - \phi_1(x, \theta)]\, d\theta \\
&\quad\geqq \int_{h_3 \leqq |\theta - xg^{-1}| \leqq h+f} \exp(-\tfrac{1}{2}g|\theta - xg^{-1}|^2)\, d\theta \\
&\quad= \int_{h_3}^{h+f} \exp(-gr^2/2) 2\pi r\, dr \\
&\quad= 2\pi g^{-1}\{\exp(-gh_3^2/2) - \exp[-\tfrac{1}{2}g(h + f)^2]\} \qquad (117)\\
&\quad= 2\pi g^{-1} \exp[-\tfrac{1}{2}g(h + f)^2]\cdot\{\exp[\tfrac{1}{2}g[(h + f)^2 - h_3^2]] - 1\} \\
&\quad= 2\pi g^{-1} \exp[-\tfrac{1}{2}g(h + f)^2]\cdot\{\exp[g(2\pi)^{-1}v_A(x)] - 1\} \qquad \text{by (116)} \\
&\quad\geqq 2\pi g^{-1} \exp[-\tfrac{1}{2}g(h + f)^2]\cdot g(2\pi)^{-1}v_A(x).
\end{aligned}$$

Substituting by (117) and repeatedly using the inequality $e^{-z} \geqq 1 - z$, for all z, we get, in the integrand for I_2 in (107),

$$\begin{aligned}
&bv_A(x) - g(2\pi)^{-1} \int_{A_x} \exp(-\tfrac{1}{2}g|\theta - xg^{-1}|^2)[1 - \phi_1(x, \theta)]\, d\theta \\
&\quad\leqq v_A(x)\{b - g(2\pi)^{-1} \exp[-\tfrac{1}{2}g(h^2 + 2hf + f^2)]\} \\
&\quad\leqq v_A(x)\{(2\pi)^{-1} \exp(-h^2/2) - g(2\pi)^{-1} \exp(-gh^2/2)[1 - ghf - gf^2/2]\} \\
&\quad= v_A(x)\{(2\pi)^{-1} \exp(-h^2/2)[1 - g\exp(-h^2/2\tau^2)] \qquad (118)\\
&\qquad + g(2\pi)^{-1} \exp(-gh^2/2)\cdot(ghf + gf^2/2)\} \\
&\quad\leqq v_A(x)\{b(1 - g + gh^2/2\tau^2) \\
&\qquad + g(2\pi)^{-1} \exp(-gh^2/2)(ghf + gf^2/2)\} \\
&\quad< v_A(x)\{bgh^2(2\tau^2)^{-1} + g(2\pi)^{-1} \exp(-gh^2/2)(ghf + gf^2/2)\}.
\end{aligned}$$

We recall that by (115) $gf = |x|\tau^{-2}$. To obtain an upper bound for I_2, we substitute the extreme right hand side of (118) in the integrand of I_2 in (107), and use the upper bound for the integrals on T_a^c obtained in Lemma 6.2 in (93), (99) and (103). We thus get,

$$\begin{aligned}
&I_2(2\pi g\tau^2)^{-1} \qquad (119)\\
&\quad\leqq \epsilon(2\pi)^{-\frac{1}{2}}\tau^{-1}\{bgh^2(2\tau^2)^{-1} + g(2\pi)^{-1} \exp(-gh^2/2)(2h\tau^{-1} + (g\tau^2)^{-1})\}
\end{aligned}$$

so that

$$I_2 \leqq (2\pi)^{\frac{1}{2}}\cdot g\epsilon\{bgh^2(2\tau)^{-1} + g(2\pi)^{-1} \exp(-gh^2/2)\cdot(2h + (g\tau)^{-1})\}. \tag{120}$$

As $\tau \to \infty$,

$$\text{the right hand side of } (120) \to \{(2\pi^{-1})^{\frac{1}{2}} \exp(-h^2/2)\epsilon h\}.$$

Hence by taking τ_0 sufficiently large, we have

$$(121) \qquad I_2 \leqq \epsilon h \qquad \text{for all} \quad \tau \geqq \tau_0 .$$

It remains only to obtain an upper bound for the term I_3 in (107). For all $\theta \, \varepsilon \, H_x$, since by (61),

$$|\theta - x| > h,$$

$$(122) \qquad |\theta - xg^{-1}| \geqq h - (|x| - |x|g^{-1})$$

$$= h - |x|(g\tau^2)^{-1} = h - f, \quad \text{by (115)}.$$

We now have to distinguish between the cases $f = |x|(g\tau^2)^{-1} \leqq h$ and $f > h$. Let W_τ be the set on which $f \leqq h$, i.e.

$$(123) \qquad x \, \varepsilon \, W_\tau, \qquad \text{if and only if,} \qquad |x| \leqq hg\tau^2.$$

Let W_τ^c be the complementary set of W_τ. We split up I_3 into parts I_3' and I_3'' arising respectively from integrations over the sets $T_a^c \cdot W_\tau$ and $T_a^c \cdot W_\tau^c$. Consider first I_3'.

Let $h_4 \geqq h - f$ be such that the concentric circles in Ω, centered at $\theta' = xg^{-1}$ and with radii $(h - f)$ and h_4 enclose an area equal to $v_H(x)$, i.e.

$$(124) \qquad \pi[h_4^2 - (h - f)^2] = v_H(x).$$

Then since by (69), $\int_{H_x} \phi_1(x, \theta)\, d\theta = v_H(x)$ and $\exp(-\frac{1}{2}g\,|\theta - xg^{-1}|^2)$ is a decreasing function of $|\theta - xg^{-1}|$, by an argument similar to that below (116), it follows from (122) that

$$\int_{H_x} \exp(-\tfrac{1}{2}g\,|\theta - xg^{-1}|^2)\phi_1(x, \theta)\, d\theta$$

$$\leqq \int_{(h-f) \leqq |\theta - xg^{-1}| \leqq h_4} \exp(-\tfrac{1}{2}g\,|\theta - xg^{-1}|^2)\, d\theta$$

$$= \int_{h-f}^{h_4} \exp(-\tfrac{1}{2}gr^2) \cdot 2\pi r\, dr$$

$$(125) \qquad = 2\pi g^{-1}\{\exp[-\tfrac{1}{2}g(h - f)^2] - \exp(-gh_4^2/2)\}$$

$$= 2\pi g^{-1} \exp[-\tfrac{1}{2}g(h - f)^2]\{1 - \exp[-\tfrac{1}{2}g[h_4^2 - (h - f)^2]]\}$$

$$= 2\pi g^{-1} \exp[-\tfrac{1}{2}g(h - f)^2]\{1 - \exp[-g(2\pi)^{-1}v_H(x)]\}$$

$$\leqq 2\pi g^{-1} \exp[-\tfrac{1}{2}g(h - f)^2]g(2\pi)^{-1}v_H(x) \qquad \text{by (124)}.$$

By substituting by (125) and using the value of b in (11), we get in the integrand of I_3 in (107),

$$
\begin{aligned}
g(2\pi)^{-1} \textstyle\int_{H_x} & \exp(-\tfrac{1}{2}g\,|\theta - xg^{-1}|^2)\,d\theta - bv_H(x) \\
&\leqq v_H(x)(2\pi)^{-1}\{g\exp[-\tfrac{1}{2}g(h^2 - 2hf + f^2)] - \exp(-h^2/2)\} \\
&\leqq v_H(x)(2\pi)^{-1}\{g\exp(-gh^2/2)\exp(ghf) - \exp(-h^2/2)\} \\
&\leqq v_H(x)(2\pi)^{-1}\{g\exp(-gh^2/2)[1 + ghf/1! + g^2h^2f^2/2! + \cdots] \\
&\qquad - \exp(-h^2)2)\} \qquad (126)\\
&< v_H(x)(2\pi)^{-1}\{(1 + \tau^{-2})\exp(-gh^2/2) - \exp(-h^2/2)\} \\
&\qquad + v_H(x)g(2\pi)^{-1}\{ghf/1! + g^2h^2f^2/2! + \cdots\} \qquad \text{by (16)} \\
&< v_H(x)(2\pi)^{-1}\cdot\tau^{-2} + v_H(x)g(2\pi)^{-1}\{ghf/1! + g^2h^2f^2/2! + \cdots\} \\
&< v_H(x)g(2\pi)^{-1}\{\tau^{-2} + ghf/1! + g^2h^2f^2/2! + \cdots\}.
\end{aligned}
$$

We now substitute the extreme right hand side of (126) in the integrand of I_3 in (107), and thus get

$$
\begin{aligned}
(127)\quad I_3'(2\pi g\tau^2)^{-1} < g(2\pi)^{-1}\cdot(2\pi g\tau^2)^{-1} \textstyle\int_{T_a^c\cdot W_\tau} & \exp(-|x|^2/2g\tau^2)v_H(x)\,dx \\
& \cdot\{\tau^{-2} + hgf/1! + g^2h^2f^2/2! + \textstyle\sum_{r=3}^{\infty} g^rh^rf^r/r!\}.
\end{aligned}
$$

The series in the right hand side of (127) being of non-negative terms, can be integrated term by term. Also the upper bounds obtained in Lemma 6.2 in (93), (99) and (103), remain valid on substituting $v_H(x)$ for $v_A(x)$ because of (85) (v). Using these upper bounds for the integrals of the first three terms in the series in the right hand side of (127), we get, recalling that $gf = |x|\,\tau^{-2}$ by (115),

$$
\begin{aligned}
I_3'(2\pi g\tau^2)^{-1} &< g(2\pi)^{-1}\epsilon(2\pi)^{-\frac{1}{2}}\tau^{-1}\{\tau^{-2} + 2h\tau^{-1} + h^2\tau^{-2}\} \\
(128)\qquad &+ g(2\pi)^{-1}(2\pi g\tau^2)^{-1}\textstyle\int_{T_a^c\cdot W}\exp(-|x|^2/2g\tau^2)v_H(x)\{\textstyle\sum_{r=3}^{\infty} g^rh^rf^r/r!\}\,dx \\
&= t_1 + t_2 \qquad \text{say.}
\end{aligned}
$$

In (128),

$$(129)\qquad t_1\cdot 2\pi g\tau^2 = g^2\epsilon(2\pi)^{-\frac{1}{2}}\{2h + \tau^{-1} + h^2\tau^{-1}\}.$$

As $\tau \to \infty$, $g \to 1$ by (16) and hence the right hand side of (129) $\to$ $(2\pi^{-1})^{\frac{1}{2}}\epsilon h$. Hence by taking τ_0 sufficiently large we have

$$(130)\qquad t_1\cdot 2\pi g\tau^2 \leqq \epsilon h \qquad \text{for all} \quad \tau \geqq \tau_0.$$

Next in the expression for t_2 in (128),

$$(131)\qquad v_H(x) \leqq \pi(2h\,d + d^2) \qquad \text{as in (82),}$$

and by (115), for $r \geqq 3$,

$$(132)\qquad g^rf^r = |x|^r\tau^{-r}\cdot\tau^{-r} < |x|^r\tau^{-r}\cdot\tau^{-3},$$

assuming that $\tau \geqq \tau_0 > 1$. Therefore in (128), using (131) and (132),

$$
\begin{aligned}
t_2 \cdot 2\pi g\tau^2 \\
&< g^2\tau^{-1}(2h\,d + d^2)\cdot\pi(2\pi g\tau^2)^{-1}\int_{T_a^c\cdot W_\tau} \exp(-|x|^2/2g\tau^2) \\
&\qquad\qquad \cdot \exp(h|x|/\tau)\,dx \\
&< g^2\tau^{-1}(2h\,d + d^2)\cdot\pi(2\pi g\tau^2)^{-1}\int_R \exp(-|x|^2(2g\tau^2)^{-1} + h|x|\tau^{-1})\,dx
\end{aligned}
$$

(133)
$$
\begin{aligned}
&= g^2\tau^{-1}(2h\,d + d^2)\pi(2\pi g\tau^2)^{-1}\int_0^\infty \exp(-r^2(2g\tau^2)^{-1} + hr\tau^{-1})\,2\pi r\,dr \\
&= g\tau^{-1}(2h\,d + d^2)\pi\int_0^\infty \exp(-\rho^2(2g)^{-1} + h\rho)\rho\,d\rho \qquad \text{putting } r = \rho\tau, \\
&< K\tau^{-1} \qquad \text{say,}
\end{aligned}
$$

where

$$K = 2(2h\,d + d^2)\cdot\pi\int_0^\infty \exp(-\tfrac{1}{4}\rho^2 + h\rho)\rho\,d\rho < \infty.$$

We use here the fact that, since by assumption in (132), $\tau > 1$, $g = 1 + \tau^{-2} < 2$. (133) implies that by taking τ_0 sufficiently large, we have

(134) $$t_2\cdot 2\pi g\tau^2 \leqq \epsilon \qquad \text{for all} \quad \tau \geqq \tau_0.$$

Combining (130) and (134) with (128), we get

(135) $$I_3' \leqq \epsilon(1 + h) \qquad \text{for all} \quad \tau \geqq \tau_0.$$

Lastly,

(136)
$$
\begin{aligned}
I_3''(2\pi g\tau^2)^{-1} = {} & (2\pi g\tau^2)^{-1}\int_{W_\tau^c\cdot T_a^c} \exp(-|x|^2/2g\tau^2) \\
& \cdot\{g(2\pi)^{-1}\int_{H_x} \exp(-\tfrac{1}{2}g|\theta - xg^{-1}|)^2\phi_1(x,\theta)\,d\theta \\
& - bv_H(x)\}\,dx.
\end{aligned}
$$

Using (131), it is seen that in (136) the term in curly brackets is bounded in absolute magnitude by $\pi(2h\,d + d^2)(b + g(2\pi)^{-1})$. Hence, since by (123), for $x \,\varepsilon\, W_\tau^c$, $|x| > hg\tau^2$, we have from (136),

$$I_3''(2\pi g\tau^2)^{-1}$$

(137)
$$
\begin{aligned}
&< \pi(2h\,d + d^2)(b + g(2\pi)^{-1})(2\pi g\tau^2)^{-1}\int_{hg\tau^2}^\infty \exp(-r^2/2g\tau^2)\,2\pi r\,dr \\
&= \pi(2h\,d + d^2)(b + g(2\pi)^{-1})g^{-1}\int_{hg\tau}^\infty \exp(-\rho^2/2g)\rho\,d\rho, \quad \text{by putting } \rho = r\tau^{-1} \\
&= \pi(2h\,d + d^2)(b + g(2\pi)^{-1})\exp(-h^2g\tau^2/2).
\end{aligned}
$$

Since, $\tau^2\cdot\exp(-h^2g\tau^2/2) \to 0$ as $\tau \to \infty$, (137) implies that by taking τ_0 sufficiently large, we have

(138) $$I_3'' \leqq \epsilon \qquad \text{for all} \quad \tau \geqq \tau_0.$$

Combining (135) and (138), we have

$$I_3 = I_3' + I_3'' \leqq \epsilon(2 + h). \tag{139}$$

Adding up (109), (113), (121) and (139), we have from (107),

$$E_\tau L_0(x, \theta) - E_\tau L_1(x, \theta) \leqq (2\pi g\tau^2)^{-1}[-M + \epsilon(7 + 2h)]. \tag{140}$$

But (14) implies that the left hand side of (140) $\geqq 0$. Hence from (140) we have

$$M \leqq \epsilon(7 + h). \tag{141}$$

Since ϵ can be taken arbitrarily small, it follows that

$$M = 0. \tag{142}$$

(142) combined with Lemma 5.3, shows that alternative II under the inequality (37) cannot be true and hence in (37), we have

$$U_1(x) = U_0(x) \qquad \text{for almost all} \quad x \,\varepsilon\, R. \tag{143}$$

But by (63) and (62), and substituting $\phi_0(x, \theta) = 1$, for $\theta \,\varepsilon\, A_x$ and $\phi_0(x, \theta) = 0$ for $\theta \,\varepsilon\, (H_x + K_x)$, we have

$$\begin{aligned} &U_1(x) - U_0(x) \\ &= \int_{A_x} [p(x, \theta) - b][1 - \phi_1(x, \theta)]\, d\theta + \int_{H_x} [b - p(x, \theta)]\phi_1(x, \theta)\, d\theta \\ &\quad + \int_{K_x} [b - p(x, \theta)]\phi_1(x, \theta)\, d\theta. \end{aligned} \tag{144}$$

Noting the values of b and $p(x, \theta)$ in (11) and (13), the integrand of each term in the right hand side of (144) is seen to be non-negative. Hence (143) implies that for almost all $x \,\varepsilon\, R$, we have

$$\phi_1(x, \theta) = 1 \qquad \text{for almost all} \quad \theta \,\varepsilon\, A_x$$

and

$$\phi_1(x, \theta) = 0 \qquad \text{for almost all } \theta \,\varepsilon\, (H_x + K_x).$$

Hence by (9) and (61) and Fubini's theorem

$$\phi_1(x, \theta) = \phi_0(x, \theta) \qquad \text{for almost all} \quad (x, \theta) \,\varepsilon\, R \times \Omega.$$

This completes the proof of Theorem 5.1.

7. Strong admissibility. As stated in the note below Theorem 5.1, that theorem implies the admissibility of the usual confidence sets according to the Definition 2.1. A stricter definition of admissibility called strong admissibility may be formulated as follows:

DEFINITION 7.1. A confidence procedure C_0 is strongly admissible if there exists no confidence procedure C_1 such that for all $\theta \,\varepsilon\, \Omega$, (i) $P_\theta(C_1(\cdot, \theta)) \geqq P_\theta(C_0(\cdot, \theta))$ and (ii) $E_\theta v C_1(x, \cdot) \leqq E_\theta v C_0(x, \cdot)$ with the strict inequality holding in either (i) or (ii) for at least one $\theta \,\varepsilon\, \Omega$. Here C_1 and C_0 denote subsets of the product space $R \times \Omega$, which define non-randomized confidence procedures. It is obvious that the strong admissibility implies the admissibility according to Definition 2.2 but not conversely. If a confidence procedure C_0 is only weakly admissible, then there exists a procedure C_1 with the same or higher inclusion probabilities, such that C_1, on the average locates θ more closely for at least one value of θ, and at least as closely as C_0 for other values of θ. Hence it would be reasonable to use procedure C_1 in preference to C_0. It follows that procedures which are strongly admissible should be preferred over those which are only weakly admissible, i.e. admissible according to Definition 2.2. Thus for example, the symmetrical confidence intervals based on the t-statistic which were shown to be admissible by the author (1966) are only weakly admissible.

It follows from Theorem 5.1 of this paper that for $m = 1$ or $m = 2$ the usual confidence procedures are strongly admissible up to the equivalence in Definition 2.1 or in the restricted class of confidence procedures with open or convex sets discussed in Section 3.

8. Case $m = 1$. In this case the proof is much simpler, and it suffices to indicate its broad outline. The usual confidence sets are now confidence intervals of fixed length $2h$ centered at $\theta = x$. The Bayes procedure in the class of randomized procedure is found to consist of intervals centered at the point $\theta' = xg^{-1}$, and of length $2c$ where c is now given in place of (17), by

$$c^2 = h^2 g^{-1} + g^{-1}(\log g).$$

In place of (11), we now have

$$b = (2\pi)^{-\frac{1}{2}} \exp(-h^2/2).$$

Then in place of (33) we get the reduction in expected loss due to Bayes procedure as

$$\begin{aligned} &E_\tau L_0(x, \theta) - E_\tau L_\tau(x, \theta) \\ &\quad = 2b(h - c) + 2(2\pi)^{-\frac{1}{2}} \textstyle\int_0^{cg^{\frac{1}{2}}} \exp(-t^2/2)\, dt - 2(2\pi)^{-\frac{1}{2}} \int_0^h \exp(-t^2/2)\, dt \\ &\quad < 2bh(1 - g^{-\frac{1}{2}}) + 2b(cg^{\frac{1}{2}} - h). \end{aligned}$$

Now

$$\begin{aligned} 1 - g^{-\frac{1}{2}} &= (g^{\frac{1}{2}} - 1)g^{-\frac{1}{2}} = (g - 1)g^{-\frac{1}{2}}(g^{\frac{1}{2}} + 1)^{-1} < \tfrac{1}{2}(g - 1) = (2\tau^2)^{-1}, \\ cg^{\frac{1}{2}} - h &= (c^2 g - h^2)(cg^{\frac{1}{2}} + h)^{-1} < (2h)^{-1}(\log g) < (2h\tau^2)^{-1}. \end{aligned}$$

Hence $E_\tau L_0 - E_\tau L_\tau < bh\tau^{-2}(1 + h^{-2})$. Then in place of (57), we get

$$E_\tau L_1(x, \theta) - E_\tau L_0(x, \theta) \geqq (k - \epsilon)(2\pi g)^{-\frac{1}{2}}\tau^{-1} - \tfrac{1}{2}bh(1 + h^{-2}). \tag{145}$$

Since the left hand side of (145) is non-positive, we have

$$k \leqq \epsilon + 2\pi(1 + h^{-2})bh\tau^{-1}. \tag{146}$$

Since τ can be made arbitrarily large, and ϵ arbitrarily small, we must have

$k = 0$. Hence alternative (II) under the relation (37), cannot be true, and the equivalence of ϕ_1 and ϕ_0 follows from alternative (I) as before.

Acknowledgment. I am deeply grateful to the referee for his many valuable suggestions and in particular for suggesting the use of Blyth's method without which the result in the bivariate case could not be proved.

I have also to thank Mr. P. M. Joshi for doing the repeated typing work with great patience.

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