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DISTRIBUTION OF DISCRIMINANT FUNCTION WHEN COVARIANCE MATRICES ARE PROPORTIONAL[1]

BY CHIEN-PAI HAN

Iowa State University

1. Introduction. Suppose $\mathbf{X}_{p\times 1}$ is an observation to be classified into one of two populations π_1 and π_2. The individual is assigned to π_1 if the likelihood ratio $\lambda = p_1(\mathbf{X})/p_2(\mathbf{X}) > k$ and to π_2 if $\lambda < k$ where k is a constant and $p_i(\mathbf{X})$ is proportional to the probability of $\mathbf{X}$ if the individual came from π_i for $i = 1, 2$. We assume that the two populations are multivariate normal with mean vectors $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$, covariance matrices $\boldsymbol{\Sigma}$ and $\sigma^2\boldsymbol{\Sigma}$ ($\sigma^2 > 1$) respectively: Hence the covariance matrix of the second population is a constant multiplier of the covariance matrix of the first population. In this paper we shall let $\boldsymbol{\Sigma}$ and σ^2 be known but $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$ may be known or unknown. Bartlett and Please (1963) considered a special form of $\boldsymbol{\Sigma}$. They have derived the discriminant function when $\boldsymbol{\Sigma} = (1-\rho)\mathbf{I} + \rho\mathbf{ZZ}'$ and the mean difference is zero, where $\mathbf{I}$ is the identity matrix and $\mathbf{Z}$ is a $p \times 1$ vector with elements unity. The author (1968) extended this case and derived the discriminant function when the mean difference is not zero. When the covariance matrix is the same in both populations but is general and unknown Okamoto (1963) has given the asymptotic expansion for the distribution for the case $\boldsymbol{\mu}_1 \neq \boldsymbol{\mu}_2$ and $\boldsymbol{\mu}_1$, $\boldsymbol{\mu}_2$ unknown. He (1961) has also worked on the case of common mean vectors and different covariance matrices. The asymptotic expansion of the first approximation to the quadratic discriminant function is given up to the linear term. This paper will derive the distribution of the discriminant function for a general $\boldsymbol{\Sigma}$ which will lead to give the distribution of the discriminant function of the special case of Bartlett and Please.

Using the likelihood procedure, we obtain the discriminant function

$$U = (\mathbf{X} - \boldsymbol{\mu}_1)'\boldsymbol{\Sigma}^{-1}(\mathbf{X} - \boldsymbol{\mu}_1) - \sigma^{-2}(\mathbf{X} - \boldsymbol{\mu}_2)'\boldsymbol{\Sigma}^{-1}(\mathbf{X} - \boldsymbol{\mu}_2). \tag{1.1}$$

Section 2 gives the distribution of U when $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$ are known. It is shown that U has a non-central chi-square distribution when $\boldsymbol{\mu}_1 \neq \boldsymbol{\mu}_2$ and a central chi-square distribution when $\boldsymbol{\mu}_1 = \boldsymbol{\mu}_2$. Section 3 derives an asymptotic expansion for the distribution of U when $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$ are unknown. The expansion is obtained by the "studentization" method of Hartley (1938) and of Welch (1947) which was used by many authors (e.g. Ito (1956), (1960), Okamoto (1963) and Siotani (1956)) for other multivariate problems.

2. Distribution of U when $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$ are known. When $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$ are known, we may, without loss of generality, let $\boldsymbol{\mu}_1 = \mathbf{0}$ and $\boldsymbol{\mu}_2 = \boldsymbol{\delta}$. Then U becomes

$$U = \mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X} - \sigma^{-2}(\mathbf{X} - \boldsymbol{\delta})'\boldsymbol{\Sigma}^{-1}(\mathbf{X} - \boldsymbol{\delta}). \tag{2.1}$$

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With some algebra, U can be written, apart from a constant, as

$$U = (\mathbf{X} + \alpha\boldsymbol{\delta})'\boldsymbol{\Sigma}^{-1}(\mathbf{X} + \alpha\boldsymbol{\delta}) \tag{2.2}$$

where $\alpha = (\sigma^2 - 1)^{-1}$. The distribution of U in this form is easily found. If $\mathbf{X}$ comes from π_1, $\mathbf{X}$ has a $N(\mathbf{0}, \boldsymbol{\Sigma})$ distribution. Hence U is distributed as $\chi_p'^2(\alpha^2 D^2)$ where $\chi_p'^2$ denotes a non-central chi-square distribution with p degrees of freedom; $D^2 = \boldsymbol{\delta}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\delta}$ is the Mahalanobis squared distance and $\alpha^2 D^2$ is the non-centrality parameter. If X comes from π_2, $\mathbf{X}$ has a $N(\boldsymbol{\delta}, \sigma^2\boldsymbol{\Sigma})$ distribution, therefore U is distributed as $\sigma^2\chi_p'^2(\sigma^2\alpha^2D^2)$. A special form of $\boldsymbol{\Sigma}$ is that the p variates have common variances and common covariances, i.e. $\boldsymbol{\Sigma} = (1 - \rho)\mathbf{I} + \rho\mathbf{ZZ}'$. For this case the discriminant function was derived by the author (1968); it depends on the size and the shape components of Penrose (1946–1947) and on the component of sum of squares of the variates. Its distribution, when $\mathbf{X}$ comes from π_1, is then a non-central chi-square distribution with p degrees of freedom and non-centrality parameter $\alpha^2[(1 - \rho)^{-1}\sum \delta_i^2 - \rho(1 - \rho)^{-1}(1 + (p - 1)\rho)^{-1}(\sum \delta_i)^2]$, where δ_i is the ith component in $\boldsymbol{\delta}$. When $\mathbf{X}$ comes from π_2, it has a $\sigma^2\chi_p'^2$ distribution with non-centrality parameter $\sigma^2\alpha^2[(1 - \rho)^{-1}\sum \delta_i^2 - \rho(1 - \rho)^{-1}(1 + (p - 1)\rho)^{-1} \cdot (\sum \delta_i)^2]$.

When the mean difference is zero, we have $\boldsymbol{\delta} = 0$. Then the non-central chi-square distribution becomes a central chi-square distribution. In the special case, $\boldsymbol{\Sigma} = (1 - \rho)\mathbf{I} + \rho\mathbf{ZZ}'$, the discriminant function which was considered by Bartlett and Please (1963) has a χ_p^2 distribution when $\mathbf{X}$ comes from π_1 and a $\sigma^2\chi_p^2$ distribution when $\mathbf{X}$ comes from π_2.

3. Distribution of U when $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$ are unknown. When $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$ are unknown, they have to be estimated from the sample. Suppose a sample of size n_i is taken from π_i for $i = 1, 2$. The maximum likelihood estimator of $\boldsymbol{\mu}_i$ is $\bar{\mathbf{X}}_i$, $i = 1, 2$. Substituting the estimators in (1.1), we have

$$U = (\mathbf{X} - \bar{\mathbf{X}}_1)'\boldsymbol{\Sigma}^{-1}(\mathbf{X} - \bar{\mathbf{X}}_1) - \sigma^{-2}(\mathbf{X} - \bar{\mathbf{X}}_2)'\boldsymbol{\Sigma}^{-1}(\mathbf{X} - \bar{\mathbf{X}}_2). \tag{3.1}$$

It can be shown that U is invariant under any linear transformation. Hence, without loss of generality, we shall derive the distribution of U by letting $\boldsymbol{\mu}_1 = \mathbf{0}$, $\boldsymbol{\mu}_2 = \boldsymbol{\mu}_0$ and $\boldsymbol{\Sigma} = \mathbf{I}$, where the vector $\boldsymbol{\mu}_0$ has the first element nonzero and the rest all zero, i.e. $\boldsymbol{\mu}_0 = (D\ 0\ 0 \cdots 0)'$; D^2 denotes the Mahalanobis squared distance. In order to find the distribution of U, we write U in another form, with $\boldsymbol{\Sigma} = \mathbf{I}$, apart from a constant,

$$U = [(\mathbf{X} - \bar{\mathbf{X}}_1 - \alpha(\bar{\mathbf{X}}_1 - \bar{\mathbf{X}}_2))'(\mathbf{X} - \bar{\mathbf{X}}_1 - \alpha(\bar{\mathbf{X}}_1 - \bar{\mathbf{X}}_2))] - \alpha(\alpha + 1)(\bar{\mathbf{X}}_1 - \bar{\mathbf{X}}_2)'(\bar{\mathbf{X}}_1 - \bar{\mathbf{X}}_2). \tag{3.2}$$

where $\alpha = (\sigma^2 - 1)^{-1}$.

The cumulative distribution function (cdf) of U given that $\mathbf{X}$ comes from π_i is denoted by $F_i(u) = \Pr(U \leqq u/\pi_i)$ for $i = 1, 2$. Let us first derive the cdf of U when $\mathbf{X}$ comes from π_1. The characteristic function (cf) of U is $\varphi(t) = E(e^{itU}/\pi_1)$.

From a well-known property of conditional probability, we have

$$\varphi(t) = E^{\bar{\mathbf{X}}_1,\bar{\mathbf{X}}_1}\{[E[e^{itU}/\bar{X}_1, \bar{X}_2 ; \pi_1]\}, \tag{3.3}$$

where the expectation in the curled bracket is the conditional cf given $\bar{\mathbf{X}}_1$ and $\bar{\mathbf{X}}_2$. Let us denote it by $\psi(\bar{\mathbf{X}}_1, \bar{\mathbf{X}}_2)$. $\psi(\bar{\mathbf{X}}_1, \bar{\mathbf{X}}_2)$ is found to be

$$\psi(\bar{\mathbf{X}}_1, \bar{\mathbf{X}}_2) = \exp\{-it\alpha(\alpha+1)Q + (it/(1-2it))\mathbf{v}'\mathbf{v} - \tfrac{1}{2}p\log(1-2it)\} \tag{3.4}$$

where

$$Q = (\bar{\mathbf{X}}_1 - \bar{\mathbf{X}}_2)'(\bar{\mathbf{X}}_1 - \bar{\mathbf{X}}_2)$$

and

$$\mathbf{v} = -(\alpha+1)\bar{\mathbf{X}}_1 + \alpha\bar{\mathbf{X}}_2 .$$

Since the function ψ is analytic about the point $(\bar{\mathbf{X}}_1, \bar{\mathbf{X}}_2) = (\mathbf{0}, \boldsymbol{\mu}_0)$, expanding ψ into a Taylor's series, we have

$$\psi(\bar{\mathbf{X}}_1, \bar{\mathbf{X}}_2) = \exp[\textstyle\sum_{i=1}^p \bar{x}_{1i}(\partial/\partial\boldsymbol{\mu}_{1i}) + \sum_{i=1}^p (\bar{x}_{2i} - \boldsymbol{\mu}_{0i})(\partial/\partial\boldsymbol{\mu}_{2i})]\psi(\boldsymbol{\mu}_1, \boldsymbol{\mu}_2)|_0 \tag{3.5}$$

where $\bar{x}_{1i}$, $\bar{x}_{2i}$, $\boldsymbol{\mu}_{1i}$, $\boldsymbol{\mu}_{2i}$ and $\boldsymbol{\mu}_{0i}$ are components of the vectors $\bar{\mathbf{X}}_1$, $\bar{\mathbf{X}}_2$, $\boldsymbol{\mu}_1$, $\boldsymbol{\mu}_2$ and $\boldsymbol{\mu}_0$ respectively; and $|_0$ denotes that the expression is evaluated at the point $(\mathbf{0}, \boldsymbol{\mu}_0)$. Hence the cf of U is

$$\begin{aligned}\varphi(t) &= E^{\bar{\mathbf{X}}_1,\bar{\mathbf{X}}_2}\{\psi(\bar{\mathbf{X}}_1, \bar{\mathbf{X}}_2)\} \\ &= \Theta\psi(\boldsymbol{\mu}_1, \boldsymbol{\mu}_2)|_0 ,\end{aligned} \tag{3.6}$$

where Θ is the differential operator

$$\Theta = E^{\bar{\mathbf{X}}_1,\bar{\mathbf{X}}_2}\{\exp[\textstyle\sum_{i=1}^p \bar{x}_{1i}(\partial/\partial\boldsymbol{\mu}_{1i}) + \sum_{i=1}^p (\bar{x}_{2i} - \boldsymbol{\mu}_{0i})\partial/\partial\boldsymbol{\mu}_{2i}]\}. \tag{3.7}$$

Using the moment generating function of $\bar{\mathbf{X}}_1$ and $\bar{\mathbf{X}}_2$ and noting that $\bar{\mathbf{X}}_1$ and $\bar{\mathbf{X}}_2$ are independent $N(\mathbf{0}, n_1^{-1}\mathbf{I})$ and $N(\boldsymbol{\mu}_0, \sigma^{-2}n_2^{1}\mathbf{I})$ respectively, we have

$$\Theta = \exp((1/2n_1)\textstyle\sum_{i=1}^p \partial^2/\partial\boldsymbol{\mu}_{1i}^2)\exp((\sigma^2/2n_2)\sum_{i=1}^p \partial^2/\partial\boldsymbol{\mu}_{2i}^2)$$

which can be expanded into

$$\begin{aligned}\Theta = 1 &+ (1/2n_1)\textstyle\sum(\partial^2/\partial\boldsymbol{\mu}_{1i}^2) + (\sigma^2/2n_2)\sum(\partial^2/\partial\boldsymbol{\mu}_{2i}^2) \\ &+ (1/8n_1^2)\textstyle\sum_i\sum_j(\partial^4/\partial\boldsymbol{\mu}_{1i}^2\partial\boldsymbol{\mu}_{1j}^2) + \sigma^4/8n_2^2\sum_i\sum_j(\partial^4/\partial\boldsymbol{\mu}_{2i}^2\partial\boldsymbol{\mu}_{2j}^2) \\ &+ (\sigma^2/4n_1n_2)\textstyle\sum_i\sum_j(\partial^4/\partial\boldsymbol{\mu}_{1i}^2\partial\boldsymbol{\mu}_{2j}^2) + O_3 ,\end{aligned} \tag{3.8}$$

where O_3 stands for the terms of the third order with respect to (n_1^{-1}, n_2^{-1}).

Now it remains to find the individual terms in (3.6). Following Okamoto (1963), we obtain, with $\theta = -it$,

$$\begin{aligned}\varphi(t) = \{1 &+ b_1(\theta, D) + b_2(\theta, D) + \tfrac{1}{2}[b_1(\theta, D)]^2 + \tfrac{1}{2}[b_2(\theta, D)]^2 \\ &+ b_1(\theta, D)b_2(\theta, D) + c_{11}(\theta, D) + c_{22}(\theta, D) + c_{12}(\theta, D) \\ &+ O_3\}\psi(\mathbf{0}, \boldsymbol{\mu}_0),\end{aligned} \tag{3.9}$$

where
$$\begin{aligned}
b_1(\theta, D) &= n_1^{-1}\{a_1(\theta) + (a_2(\theta)/(1-2it)) + (a_3(\theta)/(1-2it)^2)\},\\
b_2(\theta, D) &= \sigma^2 n_2^{-1}\{a_1(\theta) + (a_4(\theta)/(1-2it)) + (a_5(\theta)/(1-2it)^2)\},\\
c_{11}(\theta, D) &= n_1^{-2}\{4[-c_1(\theta) + (c_1'(\theta)/(1-2it))]^2[pc_2(\theta)\\
&\quad + (c_3(\theta)/(1-2it))] + [pc_2(\theta) + (c_3(\theta)/(1-2it))]^2\},\\
c_{22}(\theta, D) &= \sigma^4 n_2^{-2}\{4[c_1(\theta) + (c_4(\theta)/(1-2it))]^2[pc_2(\theta)\\
&\quad + (c_5(\theta)/(1-2it))] + [pc_2(\theta) + (c_5(\theta)/(1-2it))]^2\},\\
c_{12}(\theta, D) &= \sigma^2 n_1^{-1} n_2^{-1}\{8[-c_1(\theta) + (c_1'(\theta)/(1-2it))][c_1(\theta)\\
&\quad + (c_4(\theta)/(1-2it))][-c_2(\theta) + (c_2(\theta)/(1-2it))]\\
&\quad + 2p(-c_2(\theta) + (c_2(\theta)/(1-2it)))^2\},
\end{aligned} \tag{3.10}$$

and
$$\begin{aligned}
a_1(\theta) &= p\alpha(\alpha+1)\theta + 2\alpha^2(\alpha+1)^2D^2\theta^2,\\
a_2(\theta) &= -p(\alpha+1)^2\theta - 4\alpha^2(\alpha+1)^2D^2\theta^2,\\
a_3(\theta) &= 2\alpha^2(\alpha+1)^2D^2\theta^2,\\
a_4(\theta) &= -p\alpha^2\theta - 4\alpha^3(\alpha+1)D^2\theta^2,\\
a_5(\theta) &= 2\alpha^4D^2\theta^2,\\
c_1(\theta) &= \alpha(\alpha+1)D\theta = c_1'(\theta),\\
c_2(\theta) &= \alpha(\alpha+1)\theta,\\
c_3(\theta) &= -p(\alpha+1)^2\theta,\\
c_4(\theta) &= -\alpha^2D^2\theta,\\
c_5(\theta) &= -p\alpha^2\theta.
\end{aligned} \tag{3.11}$$

We put $c_1(d)$ and $c_1'(d)$ in here for the convenience of the derivation of $F_2(u)$ later. It is noticed that the principal term is the characteristic function of a noncentral chi-square variate plus a constant.

To find the cdf $F_1(u)$, we have to invert the characteristic function. The technique to invert a cf of the form $(-it)^r\varphi(t)$ is well known (see Wallace (1958), p. 638) and was used by Ito (1960) for a similar problem. If $F(x)$ is the cdf of a statistic and $\varphi(t)$ is its cf, then the cdf corresponding to $(-it)^r\varphi(t)$ is $F^{(r)}(x)$ where $F^{(r)}(x)$ is the rth derivative of $F(x)$. Now let $G_p(x)$ be the cdf of a noncentral chi-square variate with cf $\psi(\mathbf{0}, \boldsymbol{\mu}_0)$, where p denotes the degrees of freedom, then the cdf of U given that $\mathbf{X}$ comes from π_1 is

$$\begin{aligned}
F_1(u) &= w_1(d, D)G_p(u) + w_2(d, D)G_{p+2}(u) + w_3(d, D)G_{p+4}(u)\\
&\quad + w_4(d, D)G_{p+6}(u) + w_5(d, D)G_{p+8}(u)
\end{aligned} \tag{3.12}$$

where d stands for the differential operator d/du and

$$
\begin{aligned}
w_1(d, D) &= 1 + (n_1^{-1} + (\sigma^2/n_2))a_1(d) + \tfrac{1}{2}(n_1^{-2} + (\sigma^4/n_2^2) \\
&\quad + (2\sigma^2/n_1n_2))[a_1(d)]^2 \\
&\quad + (n_1^{-2} + (\sigma^4/n_2^2))\{4p[c_1(d)]^2c_2(d) + p^2[c_2(d)]^2\} \\
&\quad + (\sigma^2/n_1n_2)\{2[c_2(d)]^2 + 8[c_1(d)]^2\, c_2(d)\}, \\
w_2(d, D) &= n_1^{-1}a_2(d) + (\sigma^2/n_2)a_4(d) + n_1^{-2}\{a_1(d)a_2(d) \\
&\quad - 8c_1(d)c_1'(d)c_2(d) + 4[c_1(d)]^2c_3(d) + 2pc_2(d)c_3(d)\} \\
&\quad + (\sigma^4/n_2^2)\{a_1(d)a_4(d) + 8pc_1(d)c_2(d)c_4(d) + 4[c_1(d)]^2c_5(d) \\
&\quad + 2pc_2(d)c_5(d)\} + (\sigma^2/n_1n_2)\{a_1(d)a_2(d) + a_1(d)a_4(d) \\
&\quad + 8c_1(d)c_2(d)c_4(d) - 4c_2(d)c_2(d) - 8[c_1(d)]^2c_2(d) \\
&\quad - 8[c_1(d)]^2c_2(d)\}, \qquad (3.13) \\
w_3(d, D) &= n_1^{-1}a_3(d) + (\sigma^2/n_2)a_5(d) + n_1^{-2}\{\tfrac{1}{2}[a_2(d)]^2 + a_1(d)a_3(d) \\
&\quad + 4p[c_1'(d)]^2(c_1(d)) - 8c_1(d)c_1'(d)c_3(d) + [c_3(d)]^2\} \\
&\quad + (\sigma^4/n_2^2)\{\tfrac{1}{2}[a_4(d)]^2 + a_1(d)a_5(d) + 4pc_2(d)[c_4(d)]^2 \\
&\quad + 8c_1(d)c_4(d)c_5(d) + [c_5(d)]^2\} + (\sigma^2/n_1n_2)\{a_1(d)a_3(d) \\
&\quad + a_2(d)a_4(d) + a_1(d)a_5(d) + 8[c_1(d)]^2c_2(d) + 2[c_2(d)]^2 \\
&\quad - 8c_1(d)c_2(d)c_4(d) - 8c_1(d)c_2(d)c_4(d)\}, \\
w_4(d, D) &= n_1^{-2}\{a_2(d)a_3(d) + 4[c_1'(d)]^2c_3(d)\} + (\sigma^4/n_2^2)\{a_4(d)a_5(d) \\
&\quad + 4[c_4(d)]^2c_5(d)\} + (\sigma^2/n_1n_2)\{a_3(d)a_4(d) + a_2(d)a_5(d) \\
&\quad + 8c_1(d)c_2(d)c_4(d)\}, \\
w_5(d, D) &= \tfrac{1}{2}n_1^{-2}[a_3(d)]^2 + (\sigma^4/2n_2^2)[a_5(d)]^2 + (\sigma^2/n_1n_2)a_3(d)a_5(d).
\end{aligned}
$$

Now we shall find the cdf $F_2(u)$ of U when $\mathbf{X}$ comes from π_2. A similar procedure is used. The conditional cf $\psi(\bar{\mathbf{X}}_1, \bar{\mathbf{X}}_2)$ becomes

$$\psi(\bar{\mathbf{X}}_1, \bar{\mathbf{X}}_2) = \exp[-it\alpha(\alpha + 1)Q + (it\sigma^2/(1 - 2it\sigma^2))\mathbf{n}'\mathbf{n} - \tfrac{1}{2}p \log(1 - 2it\sigma^2)]$$

where

$$Q = (\bar{\mathbf{X}}_1 - \bar{\mathbf{X}}_2)'(\bar{\mathbf{X}}_1 - \bar{\mathbf{X}}_2),$$
$$\mathbf{n} = \sigma^{-1}[\boldsymbol{\mu}_0 - (\alpha + 1)\bar{\mathbf{X}}_1 + \alpha\bar{\mathbf{X}}_2]$$

Again expanding $\psi(\bar{\mathbf{X}}_1, \bar{\mathbf{X}}_2)$ in a Taylor's series about $(\mathbf{0}, \boldsymbol{\mu}_0)$ and following the same procedure as before, we obtain the cf of U

$$
\begin{aligned}
\varphi(t) &= \{1 + B_1(\theta, D) + B_2(\theta, D) + \tfrac{1}{2}[B_1(\theta, D)]^2 + \tfrac{1}{2}[B_2(\theta, D)]^2 \\
&\quad + B_1(\theta, D)B_2(\theta, D) + C_{11}(\theta, D) + C_{22}(\theta, D) + C_{12}(\theta, D) + O_3\} \qquad (3.14) \\
&\quad \cdot\psi(\mathbf{0}, \boldsymbol{\mu}_0),
\end{aligned}
$$

where

$$
\begin{aligned}
B_1(\theta, D) &= n_1^{-1}\{A_1(\theta) + (A_2(\theta)/(1-2it\sigma^2)) + (A_3(\theta)/(1-2it\sigma^2)^2)\}, \\
B_2(\theta, D) &= \sigma^2 n_2^{-1}\{A_1(\theta) + (A_4(\theta)/(1-2it\sigma^2)) + (A_5(\theta)/(1-2it\sigma^2)^2\}), \\
C_{11}(\theta, D) &= n_1^{-2}\{4[-C_1(\theta) + (C_1'(\theta)/(1-2it\sigma^2))]^2 \\
&\qquad \cdot[pC_2(\theta) + (C_3(\theta)/(1-2it\theta))] \\
&\qquad + [pC_2(\theta) + (C_3(\theta)/(1-2it\theta))]^2\},
\end{aligned}
$$

$$
\begin{aligned}
C_{22}(\theta, D) &= \sigma^4 n_2^{-2}\{4[C_1(\theta) + (C_4(\theta)/(1-2it\sigma^2))]^2 \\
&\qquad \cdot[pC_2(\theta) + (C_5(\theta)/(1-2it\theta))] \\
&\qquad + [pC_2(\theta) + (C_5(\theta)/(1-2it\theta))]^2\}, \\
C_{12}(\theta, D) &= \sigma^2 n_1^{-1}n_2^{-1}\{8[-C_1(\theta) + (C_1'(\theta)/(1-2it\sigma^2))] \\
&\qquad \cdot[C_1(\theta) + (C_4(\theta)/(1-2it\sigma^2))] \\
&\qquad \cdot[-C_2(\theta) + (C_2(\theta)/(1-2it))] \\
&\qquad + 2p[-C_2(\theta) + (C_2(\theta)/(1-2it\sigma^2))]^2\},
\end{aligned}
\tag{3.15}
$$

and

$$
\begin{aligned}
A_1(\theta) &= p\alpha(\alpha+1)\theta + 2\alpha^2(\alpha+1)^2D^2\theta^2, \\
A_2(\theta) &= -(\alpha+1)^2p\theta - 4\alpha(\alpha+1)^3D^2\theta^2, \\
A_3(\theta) &= 2(\alpha+1)^4D^2\theta^2, \\
A_4(\theta) &= -p\alpha^2\theta - 4\alpha^2(\alpha+1)^2D^2\theta^2, \\
A_5(\theta) &= 2\alpha^2(\alpha+1)^2D^2\theta^2, \\
C_1(\theta) &= \alpha(\alpha+1)D\theta, \\
C_1'(\theta) &= (\alpha+1)^2D\theta, \\
C_2(\theta) &= \alpha(\alpha+1)\theta, \\
C_3(\theta) &= -p(\alpha+1)^2\theta, \\
C_4(\theta) &= -\alpha(\alpha+1)D\theta, \\
C_5(\theta) &= -p(\alpha+1)^2\theta.
\end{aligned}
\tag{3.16}
$$

The principal term is the cf of a constant multiplier of a non-central chi-square variate plus a constant.

Since (3.14) has the same form as that of (3.9), we obtain the cdf of U given that $\mathbf{X}$ comes from π_2 by the same computation. Hence $F_2(u)$ has a similar form as $F_1(u)$ given in (3.12) with appropriate substitutions of the functions of A's and C's for the functions of a's and c's respectively.

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