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SELECTION PROCEDURES FOR RESTRICTED FAMILIES OF PROBABILITY DISTRIBUTIONS[1]

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1. Introduction and summary. Let $\Pi_1, \Pi_2, \cdots, \Pi_k$ be k populations. The random variable X_i associated with Π_i has a continuous distribution F_i, $i = 1, 2, \cdots, k$. We are primarily interested in selecting a subset such that the probability is at least P^* that the selected subset includes the population with the largest (smallest) quantile of a given order α $(0 < \alpha < 1)$. We assume each F_i has a unique α-quantile, $\xi_{\alpha i}$. Let $F_{[i]}(x) = F_{[x]}$ denote the cumulative distribution function of the population with the ith smallest α-quantile. In the following, we consider families of distributions ordered in a certain sense with respect to a specified continuous distribution G and propose and study a selection procedure which is different from the non-parametric procedure of Rizvi and Sobel (1967). We assume

(a) $F_{[i]}(x) \geqq F_{[k]}(x)$, $i = 1, 2, \cdots, k$ and all x.

(b) $\exists$ a continuous distribution $G \ni F_{[i]} \precsim G$, $\forall i = 1, 2, \cdots, k$,

where $\precsim$ denotes a partial ordering relation on the space of distributions.

A relation $\precsim$ on the space of distributions is a *partial ordering* if

$$F \precsim F \qquad \forall \text{ distributions } F$$

$$F \precsim G, \qquad G \precsim H \text{ implies } F \precsim H.$$

Note that $F \precsim G$ and $G \precsim F$ do *not* necessarily imply $F \equiv G$.

Various special cases in addition to stochastic ordering are:

(i) $F <_* G$ iff $F(0) = G(0) = 0$ and $G^{-1}F(x)/x$ is nondecreasing in $x \geqq 0$ on the support of F.

(ii) $F <_c G$ iff $G^{-1}F(x)$ is convex on the support of F.

(iii) $F <_r G$ iff $F(0) = G(0) = \frac{1}{2}$ and $G^{-1}F(x)/x$ is increasing (decreasing) for x positive (negative) on the support of F.

(iv) $F <_s G$ iff $F(0) = G(0) = \frac{1}{2}$ and $G^{-1}F$ is concave-convex about the origin, on the support of F; i.e., $\{x \mid 0 < F(x) < 1\}$.

If $G(x) = 1 - e^{-x}$ for $x \geqq 0$, then (i) defines the class of IFRA distributions studied by Birnbaum, Esary and Marshall (1966) while (ii) defines the class of IFR distributions studied by Barlow, Marshall and Proschan (1963). For any distribution G, $F <_* G$ iff $F(x)$ crosses $G(\theta x)$ at most once and from below if at all as a function of x for all $\theta > 0$. If $G(x) = 1 - \exp(-x^\lambda)$ for $x \geqq 0$ and $\lambda > 0$,

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then $F <_* G$ implies that F is "sharper" than the family of Weibull distributions with shape parameter λ. Implications of orderings defined by (iii) were studied by Lawrence (1966). Van Zwet (1964) studied orderings defined by both (ii) and (iv). Clearly $<_c$ ordering implies $<_*$ ordering and $<_s$ ordering implies $<_r$ ordering.

If $\mathbf{X}_i = (X_{i1}, X_{i2}, \cdots, X_{in})$ is the observed sample from the ith population, then we restrict ourselves to the class of statistics $T_i = T(\mathbf{X}_i)$ that preserve both ordering relations (a) and (b), i.e.,

(a′) $P_{F_{[i]}}\{T(\mathbf{X}) \leqq x\} \geqq P_{F_{[k]}}\{T(\mathbf{X}) \leqq x\}$ for all x and $i = 1, 2, \cdots, k$.

(b′) $F_{T(\mathbf{X}_i)} \precsim G_{T(\mathbf{Y})}$, $i = 1, 2, \cdots, k$, where $F_{T(\mathbf{X}_i)}$ represents the cdf of $T(\mathbf{X}_i)$ under $F_{[i]}$ and $G_{T(\mathbf{Y})}$ is the cdf of $T(\mathbf{Y})$ under G, $\mathbf{Y} = (Y_1, Y_2, \cdots, Y_n)$ being a random sample from G.

In Section 2 of this paper, we propose and study procedures R (R') for selecting the population with the largest (smallest) α-quantile for distributions which are $<_*$ ordered with respect to a specified distribution G. The infimum of the probability of a correct selection is obtained in Theorem 2.1 and asymptotic evaluation is given in Theorem 2.2. Section 3 deals with quantile selection procedures for the class of IFRA distributions. In Section 4, we study the efficiency of procedure R with respect to a procedure studied by Rizvi and Sobel (1967) under scale type slippage configurations. Asymptotic relative efficiency of R with respect to a selection procedure for the gamma populations proposed by Gupta (1963) is also investigated. Section 5 deals with selection procedures for the median for distributions that are $<_r$ ordered with respect to a specified G. In Section 6 we propose a selection procedure with respect to the means for distributions that are $<_c$ ordered with respect to $G(x) = 1 - e^{-x}$. Application to the selection of gamma populations is also given in Section 6.

2. Quantile selection rules for distributions $<_*$ ordered with respect to G. We are given a sample of size n from each of the k populations Π_i, $i = 1, 2, \cdots, k$. The distributions $F_{[i]}$ and the specified distribution G satisfy the assumptions (a) and (b) of Section 1. The distributions F_i are, otherwise, unspecified. Of course, the correct pairing of the unordered and ordered F_i's is not known. We denote the k-tuples $(F_1, F_2, \cdots, F_k)$ by Ω. Let $T_{j,i}$ denote the jth order statistic from F_i where $j \leqq (n+1)\alpha < j+1$. Clearly, $T_{j,i} \rightarrow_{\text{a.s.}} \xi_{\alpha,i}$, the α-quantile as $n \rightarrow \infty$ and $j/n \rightarrow \alpha$. The rule we propose, for selecting the population with the largest α-quantile is

R: Select population Π_i iff

$$T_{j,i} \geqq c \max_{1 \leqq r \leqq k} T_{j,r}, \qquad j \leqq (n+1)\alpha < j+1, \tag{2.1}$$

where $c = c(k, P^*, n, j)$ is some number between 0 and 1 which is determined so as to satisfy the probability requirement

$$\inf_\Omega P\{\text{CS} \mid R\} = P^*, \tag{2.2}$$

where CS stands for a correct selection, i.e., the selection of any subset which contains the population $\Pi_{[k]}$ with distribution $F_{[k]}$. Before discussing the main theorem concerned with the evaluation of $P\{\text{CS} \mid R\}$, we present a known result for order statistics. Let $H_{j,i}(x)$ be the cdf of the jth order statistic from $F_{[i]}$ and let $G_j(x)$ be the cdf of the jth order statistic from G. Let us define

$$B_{j,n}(x) = [n!/(j-1)!\,(n-j)!] \int_0^x u^{j-1}(1-u)^{n-j}\,du$$

so that

$$H_{j,i}(x) = B_{j,n}(F_{[i]}(x)) \equiv B_{j,n}F_{[i]}(x). \tag{2.3}$$

Since

$$G_j^{-1}H_{j,i}(x) = [B_jG]^{-1}B_jF_{[i]}(x) = G^{-1}F_{[i]}(x), \tag{2.4}$$

we see that order statistics preserve each of the partial ordering relations (i)–(iv). For additional applications of (2.4) see van Zwet (1964).

Now we state and prove a theorem which enables us to compute the constant c which defines the procedure R.

THEOREM 2.1. *If* $F_{[i]}(0) = G(0) = 0$, $F_{[i]}(x) \geqq F_{[k]}(x)$, $x \geqq 0$, $i = 1, 2, \cdots, k$, *and* $F_{[k]} <_* G$, *then*

$$\inf_\Omega P\{\text{CS} \mid R\} = \int_0^\infty [G_j(x/c)]^{k-1}\,dG_j(x). \tag{2.5}$$

PROOF. Note that

$$P\{\text{CS} \mid R\} = \int_0^\infty \left[\prod_{i=1}^{k-1} H_{j,i}(x/c)\right] dH_{j,k}(x) \geqq \int_0^\infty [H_{j,k}(x/c)]^{k-1}\,dH_{j,k}(x).$$

We wish to bound the right hand side. Let $X_{j,r}$ $(r = 1, 2, \cdots, k)$ be iid with cdf $H_{j,k}(x)$. (Note that $X_{j,r} \equiv T_{j,r}$ when $F_{[i]} \equiv F_{[k]}$, $\forall\, i$). Let $\varphi(x) = G_j^{-1}H_{j,k}(x) = G^{-1}F_{[k]}(x)$ so that $\varphi(x)/x$ is nondecreasing in $x \geqq 0$. Then

$$\begin{aligned}\varphi(X_{j,r})/X_{j,r} &\leqq \varphi(\max_{1\leqq r\leqq k} X_{j,r})/\max_{1\leqq r\leqq k} X_{j,r} \\ &= \max_{1\leqq r\leqq k} \varphi(X_{j,r})/\max_{1\leqq r\leqq k} X_{j,r}, \qquad r = 1, 2, \cdots, k,\end{aligned} \tag{2.6}$$

so that

$$\varphi(X_{j,r})/\max_{1\leqq r\leqq k} \varphi(X_{j,r}) \leqq X_{j,r}/\max_{1\leqq r\leqq k} X_{j,r}, \qquad r = 1, 2, \cdots, k. \tag{2.7}$$

Since $Y_{j,r} = \varphi(X_{j,r})$ has distribution G_j for $r = 1, 2, \cdots, k$, we have

$$\begin{aligned}P\{\text{CS} \mid R\} &\geqq P_{H_{j,k}}\{X_{j,k}/\max_{1\leqq r\leqq k} X_{j,r} \geqq c\} \\ &\geqq P_{G_j}\{Y_{j,k}/\max_{1\leqq r\leqq k} Y_{j,r} \geqq c\} = \int_0^\infty [G_j(x/c)]^{k-1}\,dG_j(x),\end{aligned} \tag{2.8}$$

provided c is between 0 and 1. This proves Theorem 2.1.

REMARK 1. The constant $c = c(k, P^*, n, j)$ which defines the selection procedure R is determined by

$$\int_0^\infty [G_j(x/c)]^{k-1}\,dG_j(x) = P^*, \qquad (1/k < P^* < 1). \tag{2.9}$$

These constants are tabulated for $G(x) = 1 - e^{-x}$ for selected values of n, k, j and P^* in the first set of tables in the companion paper by Barlow, Gupta and Panchapakesan (1969). Clearly, c is independent of scale.

REMARK 2. *If* $G(x) = 1 - e^{-(x/\theta)^\lambda}$, for $x \geqq 0$ and $\theta, \lambda > 0$, then for $j = 1$, the values of c are independent of n. This can be seen from the fact that the distribution of the smallest order statistic involves n only as a scale parameter and that the selection procedure (2.1) is scale invariant.

REMARK 3. It should be pointed out that Theorem 2.1 requires only $F_{[k]} <_* G$; however, to apply the procedure R, we assume that $F_{[i]} <_* G, \forall_i$.

Now we discuss the asymptotic evaluation of the probability of a correct selection. We state and prove the following theorem.

THEOREM 2.2. *If* $F_{[k]}(x) <_* G$, $F_{[k]}(G)$ *has a differentiable density* $f_{[k]}(g)$ *in a neighborhood of the* α*-quantile* $\xi_\alpha(\eta_\alpha)$ *and* $f_{[k]}(\xi_\alpha) \neq 0$ $(g(\eta_\alpha) \neq 0)$, *then in our previous notation* (*see Theorem* 2.1)

$$\begin{aligned} &\lim_{n\to\infty} P_{H_{j,k}}\{X_{j,k}/\max_{1\leqq r\leqq k} X_{j,r} \geqq c\} \\ (2.10)\qquad &= \int_{-\infty}^{\infty} \Phi^{k-1}(x/c + (1-c)\xi_\alpha f_{[k]}(\xi_\alpha)n^{\frac{1}{2}}c^{-1}(\alpha\bar{\alpha})^{-\frac{1}{2}})\,d\Phi(x) \\ &\geqq \int_{-\infty}^{\infty} \Phi^{k-1}(x/c + (1-c)\eta_\alpha g(\eta_\alpha)n^{\frac{1}{2}}c^{-1}(\alpha\bar{\alpha})^{-\frac{1}{2}})\,d\Phi(x) \end{aligned}$$

where $j/n \to \alpha$ *as* $n \to \infty$, $\bar{\alpha} = 1 - \alpha$ *and* $\Phi(\cdot)$ *is the* cdf *of the standard normal variate.*

PROOF.

$$\begin{aligned} &P\{X_{j,k} \geqq c \max_{1\leqq r\leqq k-1} X_{j,r}\} \\ (2.11)\qquad &= P\{(X_{j,k} - \xi_\alpha)f_{[k]}(\xi_\alpha)n^{\frac{1}{2}}(\alpha\bar{\alpha})^{-\frac{1}{2}} \\ &\qquad \geqq c[\max_{1\leqq r\leqq k-1}(X_{j,r} - \xi_\alpha) + (c-1)c^{-1}\xi_\alpha][(\alpha\bar{\alpha})^{\frac{1}{2}}/(n^{\frac{1}{2}}f_{[k]}(\xi_\alpha))]^{-1}\} \\ &\approx \int_{-\infty}^{\infty} \Phi^{k-1}(x/c + (1-c)\xi_\alpha f_{[k]}(\xi_\alpha)n^{\frac{1}{2}}c^{-1}(\alpha\bar{\alpha})^{-\frac{1}{2}})\,d\Phi(x), \end{aligned}$$

since $X_{j,k} \sim N(\xi_\alpha, \alpha\bar{\alpha}/nf^2_{[k]}(\xi_\alpha))$. (Note: $a_n \approx b_n$ means $\lim_{n\to\infty} a_n/b_n = 1$.)

To prove the second part of (2.10), note that $F_{[k]} <_* G$ implies $G^{-1}F_{[k]}(x) - x$ changes sign at most once and from $-$ to $+$, if at all. Since $F_{[k]} <_* G$ is invariant under scale changes, we can assume $\eta_\alpha = \xi_\alpha$ so that $F(\xi_\alpha) \equiv G(\xi_\alpha)$. Either $F = G$ in a neighborhood of ξ_α or the slope of the tangent line to F at ξ_α is greater than the slope of the tangent line to G at ξ_α. In either case $f_{[k]}(\xi_\alpha) \geqq g(\xi_\alpha)$ and in general $\xi_\alpha f_{[k]}(\xi_\alpha) \geqq \eta_\alpha g(\eta_\alpha)$.

REMARK 4. Setting

$$(2.12)\qquad \int_{-\infty}^{\infty} \Phi^{k-1}(x/c + (1-c)\eta_\alpha g(\eta_\alpha)n^{\frac{1}{2}}c^{-1}(\alpha\bar{\alpha})^{-\frac{1}{2}})\,d\Phi(x) = P^*,$$

we see that

$$(2.13)\qquad c(k, P^*, n, j) \approx 1 - \omega(k, P^*, \alpha)n^{-\frac{1}{2}} \qquad \text{as} \quad n \to \infty$$

where $\omega(k, P^*, \alpha)$ is some constant independent of n.

For $k = 2$, and $g(x) = e^{-x}$, we see

$$(2.14)\qquad c(2, P^*, n, j) = 1 - 2^{\frac{1}{2}}C/n^{\frac{1}{2}} + C/n - (3/2^{\frac{1}{2}})C^3/n^{\frac{3}{2}} + 0(n^{-2})$$

where $$C = \Phi^{-1}(P^*)(\alpha\bar{\alpha})^{\frac{1}{2}}/(1-\alpha)[-\log(1-\alpha)].$$

Subset selection rule for smallest α-quantile. The rule for selecting the population with the smallest α-quantile is

R': Select population Π_i iff

$$dT_{j,i} \leqq \min_{1 \leqq r \leqq k} T_{j,r}, \qquad j \leqq (n+1)\alpha < j+1 \tag{2.15}$$

where $0 < d = d(k, P^*, n, j) < 1$ is determined so as to satisfy the basic probability requirement. If $F_{[i]}(x) \leqq F_{[1]}(x)$, $i = 1, 2, \cdots, k$, and all $x \geqq 0$ and $F_{[1]} <_* G$, then the constant d is given by the equation

$$\int_0^\infty [\bar{G}_j(x\,d)]^{k-1}\, dG_j(x) = P^* \tag{2.16}$$

where $\bar{G}(x) = 1 - G(x)$. In a manner similar to the proof of Theorem 2.1 we can show that

$$P\{\text{CS} \mid R'\} \geqq \int_0^\infty [\bar{G}_j(x\,d)]^{k-1}\, dG_j(x).$$

The values of d are tabulated for selected values of k, P^*, n and j in the companion paper by Barlow, Gupta and Panchapakesan (1969).

The rules R and R' select nonempty subsets. The size of the selected subset is a random variable which takes values $1, 2, \cdots, k$. The expected size of the selected subset is a common measure of the efficiency of the procedure (Gupta (1963)). However, it is difficult in our more general framework to set meaningful bounds on the expected size without further assumptions. If we asusme, in addition, that there exists G^+ such that $G^+ <_* F_{[i]}$ for all i, then we can obtain an upper bound on the probability of including the "worst" population in the selected subset for rule R. We consider this in more detail later for IFRA distributions.

If we assume that $F_{[i]}(x)$ is stochastically increasing with respect to i, then

$$P\{\text{select } \Pi_{[i]} \mid R\} \geqq P\{\text{select } \Pi_{[j]} \mid R\} \qquad \text{if} \quad i \geqq j. \tag{2.17}$$

The proof is similar to the one given in Gupta (1966). A result similar to (2.17) is true for R'.

3. Quantile selection procedures for the class of IFRA distributions. If $F <_* G$ where $G(x) = 1 - e^{-x}$ for $x \geqq 0$, then F is an IFRA distribution. The problem of selecting the best one of several IFRA populations has been considered by J. K. Patel (1967). He was interested in selecting that population with the smallest failure rate at a prescribed time T. His decision rule depends only on the number of observed failures in $[0, T]$ for each population and not on the times at which failure occurred.

We show how to obtain c_λ values for the Weibull distribution with shape parameter $\lambda > 0$. We remark that the class of distributions F, such that $F <_* G_\lambda$ where $G_\lambda(x) = 1 - e^{-x^\lambda/\theta}$ for $x \geqq 0$ and $\theta, \lambda > 0$ is the smallest class of continuous distributions containing the Weibull class of distributions with shape parameter λ which is closed under the formation of coherent structures and limits in distribution.[2] To select populations $<_*$ ordered with respect to G_λ, choose c cor-

[2] Private communication with James Esary and Albert Marshall.

responding to n, k, j, and P^* based on an exponential assumption and set $c_\lambda = (c)^{1/\lambda}$. To see this, let $Y_{j,i}$ denote the jth order statistic from population i (all populations having the exponential distribution). Then $Y_{j,i}^{1/\lambda}$ is distributed as the jth order statistic from G_λ and

$$\begin{aligned}(3.1)\qquad P_{G_\lambda}\{\text{CS} \mid R\} &= P\{Y_{j,i}^{1/\lambda}/\max_{1 \leqq i \leqq k} Y_{j,i}^{1/\lambda} \geqq c_\lambda\} \\ &= P\{Y_{j,i}/\max_{1 \leqq i \leqq k} Y_{j,i} \geqq (c_\lambda)^\lambda\}\end{aligned}$$

or

$$c = (c_\lambda)^\lambda.$$

If, in addition to the assumptions of Section 2 (See Theorem 2.1), we assume that (a) $F_{[1]}(x) \geqq F_{[i]}(x)$ for all $x \geqq 0$, $i = 1, 2, \cdots, k$, and (b) $G_\lambda <_* F_{[1]}$ for all $i = 1, 2, \cdots, k$, $\lambda > 1$, then we can obtain an upper bound on the probability of selecting the "worst" population, i.e.,

$$(3.2)\qquad P\{\text{Selecting } \Pi_{[1]} \mid R\} \leqq \int_0^\infty [G_j(x/c^\lambda)]^{k-1}\, dG_j(x)$$

where c is chosen so that

$$P\{\text{CS} \mid R\} \geqq \int_0^\infty [G_j(x/c)]^{k-1}\, dG_j(x) = P^*.$$

Clearly, the upper bound is an increasing function of λ for $\lambda \geqq 1$.

4. Efficiency of procedure R under slippage configurations. We consider slippage configurations $F_{[i]}(x) = F(x/\delta)$, $i = 1, 2, \cdots, k-1$, and $F_{[k]}(x) = F(x)$, $0 < \delta < 1$. We obtain asymptotic expressions for the probability of a correct selection and the expected size of the selected subset for procedure R and for two other procedures.

Using our previous notation and letting $T'_{j,i}$ (unknown) denote that $T_{j,r}$ associated with $F_{[i]}$.

$$\begin{aligned}P\{\text{CS} \mid R\} &= P\{T'_{j,k} \geqq c \max_{1 \leqq r \leqq k} T'_{j,r}\} \\ (4.1)\qquad &= P_{H_{j,k}}\{X_{j,k} \geqq c \max_{1 \leqq r \leqq k} \delta X_{j,r}\} \\ &\geqq P_{G_j}\{Y_{j,k} \geqq c\delta \max_{1 \leqq r \leqq k} Y_{j,r}\}\end{aligned}$$

where $Y_{j,r}$, $r = 1, 2, \cdots, k$, are iid with the cdf $G_j(y)$. From (2.11), we obtain

$$(4.2)\quad P\{\text{CS} \mid R\} \approx \int_{-\infty}^\infty \Phi^{k-1}(x/c\delta + (1 - c\delta)\xi_\alpha f(\xi_\alpha) n^{\frac12} c^{-1} \delta^{-1} (\alpha\bar\alpha)^{-\frac12})\, d\Phi(x).$$

Note that the probability of a correct selection is a monotone decreasing function of δ. For the slippage configuration

$$(4.3)\qquad E(S \mid R) = P\{\text{CS} \mid R\} + (k-1) P\{T'_{j,1} \geqq c \max_{i \neq 1} T'_{j,i}\},$$

$$\begin{aligned}P\{T'_{j,1} \geqq c \max_{i \neq 1} T'_{j,i}\} &= P\{T'_{j,1} - \delta\xi_\alpha \geqq c \max(\max_{2 \leqq i \leqq k-1} (T'_{j,i} - \delta\xi_\alpha), \\ (4.4)\qquad &\qquad T'_{j,k} - \xi_\alpha + \xi_\alpha - \delta\xi_\alpha) + c\delta\xi_\alpha - \delta\xi_\alpha\} \\ &\approx \int_{-\infty}^\infty \Phi(\delta x/c - f(\xi_\alpha)\xi_\alpha(1 - \delta/c) n^{\frac12} (\alpha\bar\alpha)^{-\frac12}) \\ &\qquad \cdot \Phi^{k-2}(x/c - \xi_\alpha f(\xi_\alpha)(1 - c^{-1}) n^{\frac12} (\alpha\bar\alpha)^{-\frac12})\, d\Phi(x).\end{aligned}$$

Setting $k = 2$, we have

$$(4.5)\quad E(S \mid R) - P\{\text{CS} \mid R\} \approx \Phi(-f(\xi_\alpha)\xi_\alpha(1 - \delta/c)n^{\frac{1}{2}}(\alpha\bar{\alpha})^{-\frac{1}{2}}(1 + (\delta/c)^2)^{-\frac{1}{2}}.$$

Setting the right hand side of (4.5) equal to ϵ, $-f(\xi_\alpha)\xi_\alpha(1 - \delta/c_n)n^{\frac{1}{2}} = (\alpha\bar{\alpha})^{\frac{1}{2}}[1 + (\delta/c_n)^2)]^{\frac{1}{2}}\Phi^{-1}(\epsilon)$, where we have put $c = c_n$. Now using $c_n \approx 1 - C/n^{\frac{1}{2}}$ (from (2.14))

$$(4.6)\quad -f(\xi_\alpha)\xi_\alpha(1 - \delta - 2^{\frac{1}{2}}C\delta/n^{\frac{1}{2}})n^{\frac{1}{2}} = \Phi^{-1}(\epsilon)(\alpha\bar{\alpha})^{\frac{1}{2}}[1 + \delta^2(1 + 2^{\frac{3}{2}}C/n^{\frac{1}{2}} + 2C^2/n)]^{\frac{1}{2}}$$

from which, keeping terms of order $n^{\frac{1}{2}}$, we obtain

$$(4.7)\quad n_R(\epsilon) \approx [-(\alpha\bar{\alpha})^{\frac{1}{2}}\Phi^{-1}(\epsilon)(1 + \delta^2)^{\frac{1}{2}} + 2^{\frac{1}{2}}C\delta\xi_\alpha f(\xi_\alpha)]^2[\xi_\alpha^2 f^2(\xi_\alpha)(1 - \delta)^2]^{-1}.$$

Comparison with Rizvi-Sobel Procedure. Rizvi and Sobel (1967) propose and investigate a distribution-free procedure, R_1, for the quantile selection problem.

R_1: Select population Π_i iff

$$(4.8)\quad T_{j,i} \geqq \max_{1 \leqq r \leqq k} T_{j-a,r}$$

where a is the smallest integer with $1 \leqq a \leqq j - 1$ for which

$$(4.9)\quad \inf_\Omega P\{\text{CS} \mid R_1\} \geqq P^*$$

is satisfied.

A disadvantage of this procedure is that for any given α and k a value of $a \leqq j - 1$ may not exist for some pairs (n, P^*). However if P^* is chosen not greater than some function $P_1(n, \alpha, k)$ where $1/k < P_1 < 1$, then a value of $a \leqq j - 1$ does exist that satisfies (4.9). Rizvi and Sobel compare the efficiency of this procedure relative to several competing procedures under translation configurations.

We discuss the asymptotic of a correct selection using their procedure under the scale slippage configuration.

$$\begin{aligned}(4.10)\quad P\{\text{CS} \mid R_1\} &= P\{T'_{j,k} \geqq \max_{1 \leqq r \leqq k-1} T'_{j-a,r}\} \\ &= P\{(T'_{j,k} - \xi_\alpha)n^{\frac{1}{2}}f(\xi_\alpha)(\alpha\bar{\alpha})^{-\frac{1}{2}} \geqq \\ &\qquad \max_{1 \leqq r \leqq k-1}((T'_{j-a,r} - \delta\xi_\alpha + (\delta - 1)\xi_\alpha)\delta^{-1}(\alpha\bar{\alpha})^{-\frac{1}{2}})\delta n^{\frac{1}{2}}f(\xi_\alpha)\} \\ &\approx \int_{-\infty}^{\infty} \Phi^{k-1}(x/\delta + \gamma(\alpha\bar{\alpha})^{-\frac{1}{2}} \\ &\qquad + (1 - \delta)\xi_\alpha n^{\frac{1}{2}}f(\xi_\alpha)\delta^{-1}(\alpha\bar{\alpha})^{-\frac{1}{2}})\, d\Phi_{(x)}.\end{aligned}$$

The derivation above uses the fact that

$$(T'_{j-a,r} - \delta\xi_\alpha)(\alpha\bar{\alpha})^{-\frac{1}{2}}n^{\frac{1}{2}}\delta^{-1}f(\xi_\alpha) \rightarrow_{\text{law}} N(-\gamma(\alpha\bar{\alpha})^{-\frac{1}{2}}, 1)$$

where $\gamma/n^{\frac{1}{2}} = a/(n + 1)$. (See Lemma 2 of Rizvi and Sobel (1967)).

Similarly

$$E(S \mid R_1) - P\{\mathrm{CS} \mid R_1\} \approx \int_{-\infty}^{\infty} \Phi(\delta x + \gamma(\alpha\bar{\alpha})^{-\frac{1}{2}} - (1-\delta)\xi_\alpha n^{\frac{1}{2}} f(\xi_\alpha)(\alpha\bar{\alpha})^{-\frac{1}{2}}) \cdot \Phi^{k-2}(x + \gamma(\alpha\bar{\alpha})^{-\frac{1}{2}})\, d\Phi(x). \tag{4.11}$$

Setting $k = 2$, we have from (4.11)

$$E(S \mid R_1) - P\{\mathrm{CS} \mid R_1\} \approx \Phi((\gamma - (1-\delta)\xi_\alpha f(\xi_\alpha) n^{\frac{1}{2}}) \cdot (\alpha\bar{\alpha})^{-\frac{1}{2}}(1+\delta^2)^{-\frac{1}{2}}). \tag{4.12}$$

Equating the right hand side of (4.12) to ϵ, we obtain

$$n_{R_1}(\epsilon) \approx [(\alpha\bar{\alpha})^{\frac{1}{2}}(1+\delta^2)^{\frac{1}{2}}\Phi^{-1}(\epsilon) - \gamma]^2 (1-\delta)^{-2}[\xi_\alpha f(\xi_\alpha)]^{-2}. \tag{4.13}$$

For the slippage configuration above we define the asymptotic relative efficiency ARE $(R, R_1; \delta)$ of R relative to R_1 to be the limit as $\epsilon \to 0$ of the ratio of $n_{R_1}(\epsilon)$ to $n_R(\epsilon)$.

$$\begin{aligned} \mathrm{ARE}\,(R, R_1; \delta) &= \lim_{\epsilon \downarrow 0} n_{R_1}(\epsilon)/n_R(\epsilon) \\ &= \lim_{\epsilon \downarrow 0} [(\alpha\bar{\alpha})^{\frac{1}{2}}(1+\delta^2)^{\frac{1}{2}}\Phi^{-1}(\epsilon) - \gamma]^2 \\ &\quad \cdot [-(\alpha\bar{\alpha})^{\frac{1}{2}}\Phi^{-1}(\epsilon)(1+\delta^2)^{\frac{1}{2}} + 2^{\frac{1}{2}}\delta C \xi_\alpha f(\xi_\alpha)]^{-2} \\ &= 1. \end{aligned} \tag{4.14}$$

Using the fact that $F <_* G$ implies $\xi_\alpha f(\xi_\alpha) \geqq \eta_\alpha g(\eta_\alpha)$, we see that

$$P_F\{\mathrm{CS} \mid R_1\} \geqq P_G\{\mathrm{CS} \mid R_1\} \tag{4.15}$$

and

$$EF(S \mid R_1) - PF\{\mathrm{CS} \mid R_1\} \leqq E_G(S \mid R_1) - P_G\{\mathrm{CS} \mid R_1\} \tag{4.16}$$

where both (4.15) and (4.16) are asymptotically true as $n \to \infty$ for the slippage configuration.

Now we describe the relative performance for small sample size ($n = 15$) of the two procedures R and R_1 using Monte Carlo technique. For this purpose we chose from the class of IFRA distributions the gamma and Weibull distributions with densities

$$\text{gamma: } e^{-x/\theta_i}[\theta_i \Gamma(r)]^{-1}(x/\theta_i)^{r-1}, \qquad i = 1, 2$$

$$\text{Weibull: } e^{-(x/\theta_i)^r} r\theta_i^{-1}(x/\theta_i)^{r-1}, \qquad i = 1, 2.$$

Based on 5000 simulations, we computed $P\{\mathrm{CS} \mid R\}$, $E(S \mid R)$, $P\{\mathrm{CS} \mid R_1\}$ and $E(S \mid R_1)$. These values are given in Table 1.

Comparison with Gupta procedure. Gupta (1963) gave a selection procedure for gamma populations with densities

$$(\Gamma(r)\theta_i)^{-1} \exp(-x/\theta_i)(x/\theta_i)^{r-1}, \qquad x > 0, \qquad \theta_i > 0, \qquad i = 1, 2, \cdots, k.$$

This procedure R_2, based on the means of sample size n from each of the k

TABLE 1
Monte Carlo Comparisons of R and R_1
$P^* = .90$, $k = 2$, $n = 15$

	Gamma $r = 1$ $\theta_1 = 1, \quad \theta_2 = 2$	Gamma $r = 5$ $\theta_1 = 2, \quad \theta_2 = 3$	Weibull $r = 2$ $\theta_1 = 1, \quad \theta_2 = 2$
$P\{CS \mid R\}$	.993	1.000	.985
$E(S \mid R)$	1.47	1.88	1.85
$P\{CS \mid R_1\}$	.997	1.000	.939
$E(S \mid R_1)$	1.64	1.96	1.76

populations is:

R_2 : Select the population corresponding to the observed mean $\bar{x}_i$ iff

$$\bar{x}_i \geqq b \max_{1 \leqq j \leqq k} \bar{x}_j \tag{4.17}$$

where b is the largest constant $(0 < b \leqq 1)$ chosen so that $P\{CS \mid R_2\} \geqq P^*$. Letting $\nu = 2nr$, it is shown that $\log_e b \approx -d(2/(\nu - 1))^{\frac{1}{2}}$ where d is independent of n and satisfies

$$\int_{-\infty}^{\infty} \Phi^{k-1}(x + d)\, d\Phi(x) = P^*. \tag{4.18}$$

Assume that the ranked θ_i's have the slippage configuration $\theta_{[i]} = \delta\theta_{[k]}$, $0 < \delta < 1$, $i = 1, 2, \cdots, k - 1$. Then

$$\begin{aligned} &E(S \mid R_2) - P\{CS \mid R_2\} \\ &\approx (k-1) \int_{-\infty}^{\infty} \Phi^{k-2}(x - (\log b)(2/(\nu - 1))^{-\frac{1}{2}}) \\ &\qquad \cdot \Phi(x - (\log b/\delta)(2/(\nu - 1))^{-\frac{1}{2}})\, d\Phi(x) \end{aligned} \tag{4.19}$$

so that for $k = 2$

$$E(S \mid R_2) - P(CS \mid R_2) \approx \Phi(-(\log b/\delta) 2^{-1}(\nu - 1)^{-\frac{1}{2}}). \tag{4.20}$$

Setting the right hand side of (4.20) equal to ϵ and solving for $n = n_{R_2}(\epsilon)$

$$n_{R_2}(\epsilon) \approx [2\Phi^{-1}(\epsilon) - 2^{\frac{1}{2}} d]^2 [2r(\log \delta)^2]^{-1}. \tag{4.21}$$

$$\begin{aligned} \text{ARE}(R, R_2; \delta) &= \lim_{\epsilon \downarrow 0} n_{R2}(\epsilon)/n_R(\epsilon) \\ &= 2(1-\delta)^2 [\xi_\alpha f(\xi_\alpha)]^2 [r[\log \delta]^2 \alpha\bar{\alpha}(1 + \delta^2)]^{-1} \\ &\geqq 2(1-\delta)^2 (1-\alpha)^2 [-\log(1-\alpha)]^2 \\ &\quad \cdot [r(\log \delta)^2 \alpha\bar{\alpha}(1+\delta^2)]^{-1}, \qquad r \geqq 1, \end{aligned} \tag{4.22}$$

$$\begin{aligned} \text{ARE}(R, R_2; \delta \uparrow 1) &= [\xi_\alpha f(\xi_\alpha)]^2 (r\alpha\bar{\alpha})^{-1} \\ &\geqq (1-\alpha)^2(-\log(1-\alpha))^2(\alpha(1-\alpha))^{-1}, \\ &\qquad \text{letting } r = 1, \\ &= [\log 2]^2 = .493, \qquad \alpha = \tfrac{1}{2}. \end{aligned} \tag{4.23}$$

5. Selection with respect to the median for distributions r-ordered with respect to a specified distribution G. We consider selection procedures with respect to the median for distributions F which have lighter tails than a specified distribution G. We say that F_i has a lighter tail than G if F_i centered at its median, Δ_i, is $<_r$-ordered with respect to $G(G(0) = \frac{1}{2})$ and $(d/dx)F_i(x + \Delta_i)|_{x=0} \geqq (d/dx)G(x)|_{x=0}$. Here we are following an ordering proposed by Doksum (1967). (See van Zwet (1964), Lawrence (1966).)

We wish to select a subset of the k populations containing the population with the largest median $\Delta_{[k]}$. The selection rule, we propose, is in terms of the sample medians. We use the same notation as in Section 2. The rule R_3 is:

R_3 : Select Π_i iff

$$T_{j,i} \geqq \max_{1 \leqq r \leqq k} T_{j,r} - D, \; j \leqq (n+1)/2 < j+1, \tag{5.1}$$

and D is chosen to satisfy

$$\inf_{\Omega_1} P\{\text{CS} \mid R_3\} = P^* \tag{5.2}$$

where Ω_1 is set of all k-tuples $F_1, F_2, \cdots, F_k$ satisfying assumptions given above.

Now, we state and prove a theorem related to the infimum of probability of a correct selection when rule R_3 is used. Let $F_{[i]}(x)$ denote the distributions with median $\Delta_{[i]}$, $i = 1, 2, \cdots, k$.

THEOREM 5.1. *If $F_{[i]}(x) \geqq F_{[k]}(x)$, for all x, $G(0) = \frac{1}{2}$ and $G^{-1}F_{[k]}(x + \Delta_{[k]})/x$ is nondecreasing (nonincreasing) in $x \geqq 0 (x \leqq 0)$ and*

$$(d/dx)F_{[k]}(x + \Delta_{[k]})|_{x=0} \geqq (d/dx)G(x)|_{x=0},$$

then

$$\inf_{\Omega_1} P\{\text{CS} \mid R_3\} = \int_{-\infty}^{\infty} G_j^{k-1}(t + D)\, dG_j(t)$$

where G_j is as defined before.

PROOF. By stochastic ordering of the order statistics, we have

$$\begin{aligned} P\{\text{CS} \mid R_3\} &\geqq \int_{-\infty}^{\infty} H_{j,k}^{k-1}(t + D)\, dH_{j,k}(t) \\ &= P\{X_{j,k} \geqq \max_{1 \leqq r \leqq k-1} X_{j,r} - D\} \\ &= P\{X_{j,k} - \Delta_{[k]} \geqq \max_{1 \leqq r \leqq k-1} (X_{j,r} - \Delta_{[k]}) - D\} \end{aligned} \tag{5.3}$$

where $X_{j,1}, X_{j,2}, \cdots, X_{j,k}$ are iidrv with distribution $H_{j,k}$. (Note the last part of (5.3) requiring n odd.)

Let $\varphi(x) = G^{-1}F_{[k]}(x + \Delta_{[k]}) = G_j^{-1}M(x)$ when M is the distribution of $X_{j,r} - \Delta_{[k]}$. Note that $\varphi(X_{j,r} - \Delta_{[k]}) = Y_{j,r}$ has distribution G_j. Now $\varphi(x)/x \uparrow$ in $x \geqq 0$, $\varphi(x)x \downarrow$ in $x \leqq 0$ and $\varphi'(0) > 1$ imply

$$\begin{aligned} &[\varphi(\max_{1 \leqq r \leqq k} (X_{j,r} - \Delta_{[k]}])) - \varphi(X_{j,k} - \Delta_{[k]})] \\ &\quad \cdot [\max_{1 \leqq r \leqq k} (X_{j,r} - \Delta_{[k]}) - (X_{j,k} - \Delta_{[k]})]^{-1} \geqq 1. \end{aligned} \tag{5.4}$$

Hence

(5.5) $$\max_{1 \leqq r \leqq k} Y_{j,r} - Y_{j,k} \geqq \max_{1 \leqq r \leqq k} X_{j,r} - X_{j,k}$$

implies

(5.6) $$P\{Y_{j,k} \geqq \max_{1 \leqq r \leqq k-1} Y_{j,r} - D\} \leqq P\{X_{j,k} \geqq \max_{1 \leqq r \leqq k-1} X_{j,r} - D\}$$

which proves the result.

6. Selection with respect to the means for the class of IFR distributions. Let μ_i be the mean of the distribution $F(x; \mu_i)$, $i = 1, 2, \cdots, k$, and assume

(a) $F(x; \mu_{[i]}) \geqq F(x; \mu_{[k]})$ for $i = 1, 2, \cdots, k - 1$ and all x.

(b) $F(x; \mu_{[i]}) <_c G(x) = 1 - e^{-x}$ for $i = 1, 2, \cdots, k$.

Note that by assumption (b) we are confining attention to the so-called IFR class of distributions. It will also be convenient to assume $F(0; \mu_{[i]}) = 0$ for all i.

Let $\bar{x}_i = \sum_{j=1}^{n} x_{ij}/n$, where x_{ij} is the jth observation in a random sample of size n from Π_i. Let $K_i(x) = K(x; \mu_i)$ be the distribution of $\bar{x}_i$. Then if $K_{[i]}(x) = K(x; \mu_{[i]})$

(6.1) $$K_{[i]}(x) \geqq K_{[k]}(x) \quad \text{for} \quad i = 1, 2, \cdots, k - 1 \quad \text{and all} \quad x$$

(6.2) $$K_{[i]} <_c G \quad \text{for} \quad i = 1, 2, \cdots, k.$$

(6.1) is an immediate consequence of (a) while (6.2) follows from (b) and the closure of IFR distributions under convolution (see Barlow, Marshall and Proschan (1963)).

If we are interested in selecting a subset containing the population with the largest $\mu_{[k]}$, we use the rule

R_4 : Select population Π_i iff

$$\bar{x}_i \geqq c' \max_{1 \leqq j \leqq k} \bar{x}_j .$$

It follows that

THEOREM 6.1.

$$P\{\text{CS} \mid R_4\} \geqq \int_0^\infty [G(x/c')]^{k-1} \, dG(x)$$

where $G(x) = 1 - e^{-x}$.

The proof is the same as for Theorem 2.1. The disadvantage is that the right-hand side of the inequality is independent of n. However, by restricting the class of distributions to the gamma family we can obtain a lower bound which depends on n.

Application to the selection of gamma populations. Let us consider k populations with densities

$$\lambda_i^\alpha x^{\alpha-1} e^{-\lambda_i x}/\Gamma(\alpha), \qquad x \geqq 0, \quad \lambda_i > 0, \quad i = 1, 2, \cdots, k.$$

Assume that $\alpha \geqq 1$, but otherwise unknown. This implies that the distributions are IFR. We are interested in selecting the population with the smallest (largest) value, $\lambda_{[1]}$ ($\lambda_{[k]}$), based on an independent sample of size n from each of the k

populations. Note that $\mu_{[i]} = \alpha/\lambda_{[i]}$ for $i = 1, 2, \cdots, k$. The subset selection rule based on the sample means, $\bar{x}_i$, $i = 1, 2, \cdots, k$, is R_4 as before.

Let $G^{(\alpha)}$ denote a gamma distribution with parameter α. Since $\sum_{j=1}^{n} x_{ij}$ is distributed as a gamma random variable with distribution, $G^{(n\alpha)}$ it follows from a result of van Zwet (1964) that

$$G^{(n\alpha)} <_c G^{(n)}$$

when $\alpha \geqq 1$. It follows that in this case

$$P\{\text{CS} \mid R_4\} \geqq \int_0^\infty [G^{(n)}(x/c')]^{k-1}\, dG^{(n)}(x). \tag{6.3}$$

The constant c' is determined by

$$\int_0^\infty [G^{(n)}(x/c')]^{k-1}\, dG^{(n)}(x) = P^*. \tag{6.4}$$

The values of c' are tabulated in Gupta (1963). It should be pointed out that for selecting the population with the largest λ, the rule can be modified to:

$$R_5 : \text{Select population } \Pi_i \text{ iff}$$

$$\bar{x}_i \leqq d \min_{1 \leqq j \leqq k} \bar{x}_j$$

where d is determined by

$$\int_0^\infty [1 - G^{(n)}(x/d)]^{k-1}\, dG^{(n)}(x) = P^*.$$

The values of d are tabulated in Gupta and Sobel (1962).

The shape parameter $\alpha \geqq 1$ need not be the same for all populations. It is only necessary that the distribution of the population, $\Pi_{[k]}$, with the largest mean be stochastically larger than the others.

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REFERENCES

BARLOW, R. E., GUPTA, S. S. and PANCHAPAKESAN, S. (1969). On the distribution of the maximum and minimum of ratios of order statistics. *Ann. Math. Statist.* **40** 918–934.

BARLOW, R. E., MARSHALL, A. W. and PROSCHAN, F. (1963). Properties of probability distributions with monotone hazard rate. *Ann. Math. Statist.* **34** 375–389.

BIRNBAUM, Z. W., ESARY, J. D. and MARSHALL, A. W. (1966). A stochastic characterization of wear-out for components and systems. *Ann. Math. Statist.* **37** 816–825.

DOKSUM, K. (1968). Starshaped transformations and power of rank tests (abstract). *Ann. Math. Statist.* **39** 300.

GUPTA, S. S. (1963). On a selection and ranking procedure for gamma populations. *Ann. Inst. Statist. Math.* **14** 199–216.

GUPTA, S. S. (1966). On some selection and ranking procedures for multivariate normal populations using distance functions. *Multivariate Analysis*, Ed. P. R. Krishnaiah. Academic Press, New York.

GUPTA, S. S. and SOBEL, M. (1962). On selecting a subset containing the population with the smallest variance. *Biometrika* **49** 495–507.

LAWRENCE, M. J. (1966). Inequalities and tolerance limits for s-ordered distributions. Technical Report ORC-66-37, Operations Research Center, College of Engineering, University of California, Berkeley.

PATEL, J. K. (1967). Selecting a subset containing the best one of several IFRA populations (Abstract). *Ann. Math. Statist.* **38** 957.

RIZVI, M. H. and SOBEL, M. (1967). Nonparametric procedures for selecting a subset containing the population with the largest α-quantile. *Ann. Math. Statist.* **38** 1788–1803.

VAN ZWET, W. R. (1964). *Convex Transformations of Random Variables*, Mathematical Centre, Amsterdam.