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EFFICIENT ESTIMATION OF A PROBABILITY DENSITY FUNCTION

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1. Introduction. The problem of estimating a probability density function has received a considerable amount of attention in recent years. In particular, there has been extensive interest in estimators based on weight functions. Specifically, let

$$\hat{f}_n(x) \equiv n^{-1} \sum_{j=1}^{n} \Psi_n(x - X_j), \tag{1.1}$$

where for each n, $\Psi_n(x)$ is a square integrable function, and X_j, $1 \leqq j \leqq n$, are the sample values. See, for example, Parzen [6], and Watson and Leadbetter [7]. To evaluate an estimator a "figure of merit" is needed. The usual one is I_n^2, where

$$I_n^2 \equiv 2\pi E \int_{-\infty}^{\infty} |\hat{f}_n(x) - f(x)|^2 \, dx.$$

If $f(x) \, \varepsilon \, L_2$, then I_n^2 is necessarily finite, since $\hat{f}_n(x) \, \varepsilon \, L_2$, by the assumption that the functions $\Psi_n(x) \, \varepsilon \, L_2$. By the Parseval Identity,

$$I_n^2 = E \int_{-\infty}^{\infty} |\hat{\phi}_n(t) - \phi(t)|^2 \, dt,$$

where $\hat{\phi}_n(t)$ and $\phi(t)$ are the Fourier transforms of $\hat{f}_n(x)$ and $f(x)$ respectively. For this reason in studying the asymptotic efficiency, it is convenient to work with Fourier transforms. But

$$\hat{\phi}_n(t) = A_n(t)\tilde{\phi}_n(t)$$

where

$$\tilde{\phi}_n(t) \equiv n^{-1} \sum_{j=1}^{n} e^{iX_j t}.$$

and $A_n(t)$ is the Fourier transform of $\Psi_n(x)$. We call this the empirical characteristic function. Since $\hat{\phi}_n(t)$ is not necessarily positive definite, $\hat{f}_n(x)$ is not necessarily non-negative. Observe, though, that I_n^2 can be reduced by replacing $\hat{f}_n(x)$ with 0, whenever it is negative.

If $\lim_{n\to\infty} I_n^2 = 0$, we say that the estimator sequence is consistent, or more explicitly, that it is consistent in mean square. Conditions for consistency are well known for those estimators which are commonly used. See [7].

For any given characteristic function $\phi(t)$, the function $A_n(t)$ can be so chosen as to minimize I_n^2, as shown in [7]. We define

$$J_n^2 \equiv \min I_n^2,$$

where the minimum is taken over all possible functions $A_n(t)$. This makes possible

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a definition of efficiency. Specifically

$$\text{eff.} \equiv \lim_{n\to\infty} J_n^2/I_n^2,$$

where I_n^2 is the integrated squared error of the given estimator, and J_n^2 is, of course, as defined above.

The efficiency problem is much more difficult for the problem of estimating a probability density function, than it is for the common problem of estimating a vector-valued parameter. In the latter case, it is usually possible to find estimator sequences which are simultaneously efficient for all possible true values of the parameter often excepting a subset which has Lebesgue measure zero. See, for example, Bahadur [1]. Examples are Maximum Likelihood, and BLUE estimators. In the former case, an estimator sequence, which is efficient for one density function, may very well have zero efficiency for another one. Obviously, this complicates substantially the problem of choosing an estimator. The same problem arises in estimating a spectral density function. See, for example, Parzen [4] and [5], and Lomnicki and Zaremba [3].

In Section 4, we consider a family of estimator sequences in which the choice of function $A_n(t)$ is allowed to depend upon the sample values. Its members have a weaker form of efficiency than mean-square efficiency, although this efficiency holds simultaneously for all density functions belonging to a wide class. In particular, for such a sequence, and for any characteristic function in the class,

$$\forall\, c < \infty, \qquad \lim_{n\to\infty} E \min\,(c, X_n^2/J_n^2) \leqq 1, \tag{1.2}$$

where

$$X_n^2 \equiv \int_{-\infty}^{\infty} |\hat{\phi}_n(t) - \phi(t)|^2\, dt.$$

Sections 2 and 3 contain a reformulation and extension of the study of the efficiency of estimators, wherein the function $A_n(t)$ is not dependent upon the sample values. The results are needed for the development of Section 4.

2. Minimum mean squared error. In this section we examine the asymptotic behavior of J_n^2, the infemum of all possible values of I_n^2. This is called the MISE (Mean Integrated Squared Error) by Watson and Leadbetter [7]. They have shown that

$$J_n^2 = \int_{-\infty}^{\infty} (1 - |\varphi(t)|^2)|\varphi(t)|^2\, dt/(1 + (n-1)|\varphi(t)|^2). \tag{2.1}$$

We begin by defining the following classes of characteristic functions.

Definition. The characteristic function $\varphi(t)$ is said to belong to the class C_α, $1 < \alpha \leqq \infty$, iff $\lim_{t\to\infty} t\partial \log |\varphi(t)|^2/\partial t = -\alpha$.

For finite α, the class of characteristic functions of algebraic decrease is defined in [7] to be the class of all of those characteristic functions, which are such that $|\varphi(t)|^2 \sim Ct^{-\alpha}$, as $t \to \infty$. Provided also that $\partial|\varphi(t)|^2/\partial t \sim -\alpha Ct^{-(\alpha+1)}$, as $t \to \infty$, such a characteristic function also belongs to the class C_α. On the other hand, the class C_α also contains some characteristic functions which are not of algebraic decrease. For example, $\varphi(t)\ \varepsilon\ C_\alpha$ if $|\varphi(t)|^2 \sim Ct^{-\alpha} \log t$, and

$\partial|\varphi(t)|^2/\partial t \sim -\alpha C t^{-(\alpha+1)} \log t$, as $t \to \infty$, where $0 < C < \infty$. The distinction between the two classes for a finite value of α is from a practical point of view a very slight one, however.

The class C_∞ is very general, including, for example, characteristic functions for which $|\varphi(t)|^2 \sim C_1 \exp(-C_2 t^\alpha)$, as $t \to \infty$, where $0 < C_1, C_2, \alpha < \infty$, $|\varphi(t)|^2 \sim C_1 \exp(-C_2 t^\alpha - C_3 t^\beta)$, as $t \to \infty$, where $0 < C_1, C_2, C_3, \alpha, \beta < \infty$, or $|\varphi(t)|^2 \sim C_1 t^\alpha \exp(-C_2 t^\beta)$, as $t \to \infty$, where $0 < C_1, C_2, \alpha, \beta < \infty$ and the derivatives have the obvious corresponding relations. Some of these cases might arise from convolutions.

THEOREM 2.1. *If $\varphi(t) \varepsilon C_\alpha$, $1 < \alpha \leqq \infty$, then,*

$$J_n^2 \sim n^{-1} t_n R_\alpha \quad \text{as} \quad n \to \infty, \tag{2.2}$$

where

$$R_\alpha \equiv \int_{-\infty}^{\infty} ds/(1 + |s|^\alpha), \quad 1 < \alpha < \infty, \quad R_\infty \equiv \lim_{\alpha\to\infty} R_\alpha = 2,$$

and t_n is the solution to the equation

$$|\varphi(t_n)|^2 \equiv (n-1)^{-1}. \tag{2.3}$$

Before proceeding to the proof of this theorem, two lemmas are given.

LEMMA 2.1. *If the Lebesgue measure of the set $\{t : \varphi(t) \neq 0\}$ is infinite, then, for any real finite nonnegative t_0,*

$$J_n^2 \sim 2 \int_{t_0}^{\infty} |\varphi(t)|^2 \, dt/(1 + (n-1)|\varphi(t)|^2) \quad \text{as} \quad n \to \infty.$$

PROOF. Clearly,

$$\lim_{n\to\infty} n \int_{-t_0}^{t_0} (1 - |\varphi(t)|^2)|\varphi(t)|^2 \, dt/(1 + (n-1)|\varphi(t)|^2) = \int_{-t_0}^{t_0} (1 - |\varphi(t)|^2) B(t) \, dt, \tag{2.4}$$

where $B(t) = 1$, if $\varphi(t) \neq 0$, $= 0$, if $\varphi(t) = 0$. Since the integral on the right side of (2.4) diverges as $t_0 \to \infty$, it follows that

$$\lim_{n\to\infty} n J_n^2 = \infty. \tag{2.5}$$

By (2.4), then, it clearly follows that for any real finite non-negative t_0, $J_n^2 \sim 2 \int_{t_0}^{\infty} (1 - |\varphi(t)^2)|\varphi(t)|^2 \, dt/(1 + (n-1)|\varphi(t)|^2)$, as $n \to \infty$. Let $K_n^2 \equiv \int_{-\infty}^{\infty} |\varphi(t)|^2 dt/(1 + (n-1)|\varphi(t)|^2)$. By the same reasoning, for any real, finite, non-negative t_0, $K_n^2 \sim 2 \int_{t_0}^{\infty} |\varphi(t)|^2 \, dt/(1 + (n-1)|\varphi(t)|^2)$, as $n \to \infty$. Let $\epsilon > 0$ be arbitrarily chosen. Since the distribution is absolutely continuous, it clearly follows that $\lim_{t\to\infty} \varphi(t) = 0$. So there exists a t_1 which is such that, for all $t \geqq t_1$, $1 - \epsilon \leqq 1 - |\varphi(t)|^2 \leqq 1$. It follows that $\liminf_{n\to\infty} J_n^2/K_n^2 \geqq 1 - \epsilon$, $\limsup_{n\to\infty} J_n^2/K_n^2 \leqq 1$. Since ϵ was arbitrarily chosen, the lemma is proved.

LEMMA 2.2. *Let $\varphi(t) \varepsilon C_\alpha$. If $1 < \alpha < \infty$, then, for any $\epsilon > 0$, there exists a t_0, such that if $t_2 > t_1 > t_0$, $(t_2/t_1)^{-(\alpha+\epsilon)} \leqq |\varphi(t_2)|^2/|\varphi(t_1)|^2 \leqq (t_2/t_1)^{-(\alpha-\epsilon)}$. If $\varphi(t) \varepsilon C_\infty$ then, for any $\epsilon > 0$, there exists a t_0, such that if $t_2 > t_1 > t_0$, $|\varphi(t_2)|^2/|\varphi(t_1)|^2 \leqq (t_2/t_1)^{-1/\epsilon}$.*

PROOF. First, assume $1 < \alpha < \infty$, and let $\epsilon > 0$ be arbitrarily chosen. Clearly there exists a t_0, which is such that for all $t \geqq t_0$, $-(\alpha + \epsilon)t^{-1} \leqq \partial \log |\varphi(t)|^2/\partial t \leqq -(\alpha - \epsilon)t^{-1}$. It follows, by integration, that if $t_2 > t_1 > t_0$,

$$-(\alpha + \epsilon)(\log t_2 - \log t_1) \leqq \log |\varphi(t_2)|^2 - \log |\varphi(t_1)|^2 \leqq -(\alpha - \epsilon)(\log t_2 - \log t_1),$$

or equivalently, $(t_2/t_1)^{-(\alpha+\epsilon)} \leqq |\varphi(t_2)|^2/|\varphi(t_1)|^2 \leqq (t_2/t_1)^{-(\alpha-\epsilon)}$.

Suppose $|\varphi(t)|^2 \varepsilon C_\infty$. Let $\epsilon > 0$ be arbitrarily chosen. There exists a t_0, such that if $t > t_0$, $\partial \log |\varphi(t)|^2/\partial t \leqq -\epsilon^{-1}t^{-1}$. By integration, it follows that if $t_2 > t_1 > t_0$, $\log |\varphi(t_2)|^2 - \log |\varphi(t_1)|^2 \leqq -\epsilon^{-1}(\log t_2 - \log t_1)$. Equivalently, $|\varphi(t_2)|^2/|\varphi(t_1)|^2 \leqq (t_2/t_1)^{-1/\epsilon}$. The lemma is proved.

PROOF OF THEOREM 2.1. Let

$$g(y) = y(1 + y)^{-1}. \tag{2.6}$$

Since $g'(y) = (1 + y)^{-1} - y(1 + y)^{-2} = (1 + y)^{-1} > 0$, $g(y)$ is a strictly increasing function for all y. Let $s = t/t_n$. By definition (2.3) of t_n, $(n - 1)|\varphi(t)|^2 = (n - 1)|\varphi(st_n)|^2 = |\varphi(st_n)|^2/|\varphi(t_n)|^2$. First suppose that $1 < \alpha < \infty$. Let $\epsilon > 0$ be arbitrarily chosen. By Lemma 2.2, there exists a t_0, such that if $t > t_0$ and $t_n > t_0$, $s^{-(\alpha-\epsilon)} \leqq (n - 1)|\varphi(t)|^2 \leqq s^{-(\alpha+\epsilon)}$, for $s < 1$, and $s^{-(\alpha+\epsilon)} \leqq (n - 1)|\varphi(t)|^2 \leqq s^{-(\alpha-\epsilon)}$, for $s > 1$. Recalling Lemma 2.1, and the fact that $g(y)$ given by (2.6) is strictly increasing, for all sufficiently large n,

$$2(1 - \epsilon) \textstyle\int_{t_0}^{\infty} |\varphi(t)|^2\, dt/(1 + (n - 1)|\varphi(t)|^2) \leqq J_n^2 \leqq 2(1 + \epsilon) \int_{t_0}^{\infty} |\varphi(t)|^2\, dt/(1 + (n - 1)|\varphi(t)|^2). \tag{2.7}$$

Consequently, $2(1 - \epsilon)(n - 1)^{-1}t_n(\int_{t_0/t_n}^{1} s^{-(\alpha-\epsilon)}\, ds/(1 + s^{-(\alpha-\epsilon)}) + \int_1^\infty s^{-(\alpha+\epsilon)}\, ds/(1 + s^{-(\alpha+\epsilon)})) \leqq J_n^2 \leqq 2(1 + \epsilon)(n - 1)^{-1}t_n(\int_0^1 s^{-(\alpha+\epsilon)}\, ds/(1 + s^{-(\alpha+\epsilon)}) + \int_1^\infty s^{-(\alpha-\epsilon)}\, ds/(1 + s^{-(\alpha-\epsilon)}))$. Since ϵ was arbitrarily chosen, the result (2.2) holds, when $1 < \alpha < \infty$.

Now suppose $\varphi(t) \varepsilon C_\infty$, and again let $\epsilon > 0$ be arbitrarily chosen. Clearly for all y, $g(y) \leqq 1$. So, when $s \leqq 1$, $s^{-1/\epsilon} \leqq (n - 1)|\varphi(t)|^2 \leqq \infty$. When $s \geqq 1$, $0 \leqq (n - 1)|\varphi(t)|^2 \leqq s^{-1/\epsilon}$. So, recalling Lemma 2.1,

$$2(1 - \epsilon)(n - 1)^{-1}t_n \textstyle\int_{t_0/t_n}^{1} s^{-1/\epsilon}\, ds/(1 + s^{-1/\epsilon}) \leqq J_n^2 \leqq 2(1 + \epsilon)(n - 1)^{-1}t_n(\int_0^1 ds + \int_1^\infty s^{-1/\epsilon}\, ds/(1 + s^{-1/\epsilon})).$$

Since $\epsilon > 0$ was arbitrarily chosen, the result (2.2) holds when $\alpha = \infty$. The theorem is proved.

3. A common class of estimator sequences. In this section, we consider estimators wherein

$$\Psi_n(x) \equiv a_n \Psi(a_n x), \tag{3.1}$$

$\Psi(x)$ is a square integrable function not depending upon n, and a_n is a sequence which is such that $\lim_{n\to\infty} a_n = \infty$. Then, the function

$$A_n(t) = h(t/a_n), \tag{3.2}$$

where $h(x)$ is a square integrable function symmetric about the origin. We can write $I_n^2 \equiv V_n^2 + B_n^2$, where

$$V_n^2 \equiv E \int_{-\infty}^{\infty} |\hat{\phi}_n(t) - E\hat{\phi}_n(t)|^2 \, dt$$

is the variance term, and

$$B_n^2 \equiv \int_{-\infty}^{\infty} |E\hat{\phi}_n(t) - \varphi(t)|^2 \, dt$$

is the bias term.

THEOREM 3.1. *The variance term*

$$V_n^2 \sim n^{-1} a_n \int_{-\infty}^{\infty} h^2(t) \, dt \qquad as \quad n \to \infty. \tag{3.3}$$

PROOF. Clearly $E \int_{-\infty}^{\infty} |\hat{\phi}_n(t) - E\hat{\phi}_n(t)|^2 \, dt = n^{-1} \int_{-\infty}^{\infty} h^2(t/a_n)(1 - |\varphi(t)|^2) \, dt \sim n^{-1} \int_{-\infty}^{\infty} h^2(t/a_n) \, dt$ as $n \to \infty$. Let $s = t/a_n$. The result (3.3) follows by change of variable.

THEOREM 3.2. *Let $\varphi(t) \varepsilon C_\alpha$, and assume that for any real finite positive constant t_0,*

$$\lim_{t \to \infty} \int_0^{t_0/t} (1 - h(s))^2 \, ds / |\varphi(t)|^2 = 0. \tag{3.4}$$

Then, if $1 < \alpha < \infty$,

$$\lim_{n \to \infty} B_n^2 / a_n |\varphi(a_n)|^2 = \int_{-\infty}^{\infty} (1 - h(s))^2 |s|^{-\alpha} \, ds \, (\leqq \infty). \tag{3.5}$$

If $\alpha = \infty$,

$$\lim_{n \to \infty} B_n^2 / a_n |\varphi(a_n)|^2 = \lim_{y \to \infty} y \int_{-1}^{1} (1 - h(s))^2 \, ds \, (\leqq \infty). \tag{3.6}$$

PROOF. Clearly $B_n^2 \equiv B_{1,n}^2 + B_{2,r}^2$, where $B_{1,n}^2 \equiv \int_{-t_0}^{t_0} (1 - h(t/a_n))^2 |\varphi(t)|^2 \, dt$, and $B_{2,n}^2 \equiv 2 \int_{t_0}^{\infty} (1 - h(t/a_n))^2 |\varphi(t)|^2 \, dt$, and t_0 is an arbitrary real positive finite constant. Let us first consider the term $B_{1,n}^2$. Let $s \equiv t/a_n$. Then $B_{1,n}^2 = a_n \int_{-t_0/a_n}^{t_0/a_n} (1 - h(s))^2 |\varphi(a_n s)|^2 \, ds \leqq a_n \int_{-t_0/a_n}^{t_0/a_n} (1 - h(s))^2 \, ds$. By the condition (3.4), it follows that $\lim_{n \to \infty} B_{1,n}^2 / a_n |\varphi(a_n)|^2 = 0$, for any t_0. Obviously $B_{2,n}^2 = 2a_n |\varphi(a_n)|^2 \int_{t_0/a_n}^{\infty} (1 - h(s))^2 (|\varphi(a_n s)|^2 / |\varphi(a_n)|^2) \, ds$. Assume $1 < \alpha < \infty$, and let $\epsilon > 0$ be arbitrarily chosen. By Lemma 2.2, there exists a t_0 such that if $a_n > t_0$ and $t > t_0$, then $s^{-(\alpha-\epsilon)} \leqq |\varphi(a_n s)|^2 / (\varphi(a_n))|^2 \leqq s^{-(\alpha+\epsilon)}$ provided $s \leqq 1$, and $s^{-(\alpha+\epsilon)} \leqq |\varphi(a_n s)|^2 / |\varphi(a_n)|^2 \leqq s^{-(\alpha-\epsilon)}$ provided $s \geqq 1$. For t_0, so chosen $\int_{t_0/a_n}^{1} (1 - h(s))^2 s^{-(\alpha-\epsilon)} \, ds + \int_1^{\infty} (1 - h(s))^2 s^{-(\alpha+\epsilon)} \, ds \leqq B_{2,n}^2 / 2a_n |\varphi(a_n)|^2 \leqq \int_{t_0/a_n}^{1} (1 - h(s))^2 s^{-(\alpha+\epsilon)} \, ds + \int_1^{\infty} (1 - h(s))^2 s^{-(\alpha-\epsilon)} \, ds$. Since ϵ was arbitrarily chosen, the result (3.5) follows, when $1 < \alpha < \infty$.

Now, assume $\varphi(t) \varepsilon C_\infty$, and again let $\epsilon > 0$ be arbitrarily chosen. By Lemma 2.2, $s^{-1/\epsilon} \leqq |\varphi(a_n s)|^2 / |\varphi(a_n)|^2 \leqq \infty$ provided $s \leqq 1$, and $0 \leqq |\varphi(a_n s)|^2 / |\varphi(a_n)|^2 \leqq s^{-1/\epsilon}$ provided $s \geqq 1$. So

$$\int_{t_0/a_n}^{1} (1 - h(s))^2 s^{-1/\epsilon} \, ds \leqq B_{2,n}^2 / 2a_n |\varphi(a_n)|^2$$
$$\leqq \lim_{y \to \infty} y \int_{t_0/a_n}^{1} (1 - h(s))^2 \, ds + \int_1^{\infty} (1 - h(s))^2 s^{-1/\epsilon} \, ds.$$

Since ϵ was arbitrarily chosen, the result (3.6) follows, when $\alpha = \infty$. The theorem is proved.

THEOREM 3.3. *Among all estimator sequences, for which $\Psi_n(x)$ has the form (3.1), and equivalently $A_n(t)$ has the form (3.2), the most efficient is the one, wherein*

$$A_n(t) = 1/(1 + |t/t_n|^{\alpha}), \qquad 0 < t < \infty,$$

where t_n is given by (2.3), if $\varphi(t) \varepsilon C_\alpha$, $1 < \alpha < \infty$. If $\varphi(t) \varepsilon C_\infty$,

$$A_n(t) = 1, \qquad 0 < |t| < t_n,$$
$$= 0, \qquad \text{otherwise.}$$

In both cases the efficiency is 1.

PROOF. Let

$$D_n^2 \equiv I_n^2/2n^{-1}t_n. \tag{3.7}$$

Clearly by Theorems 3.1 and 3.2, if $1 < \alpha < \infty$,

$$D_n^2 \sim \gamma_n \int_0^\infty h^2(t)\,dt + \gamma_n n|\varphi(a_n)|^2 \int_0^\infty (1 - h(t))^2 t^{-\alpha}\,dt,$$

as $n \to \infty$, where $\gamma_n \equiv a_n/t_n$, provided that the second integral is finite, and the condition (3.4) is satisfied. But $n|\varphi(a_n)|^2 \sim (n-1)|\varphi(a_n)|^2 = |\varphi(a_n)|^2/\varphi(t_n)|^2 = |\varphi(\gamma_n t_n)|^2/|\varphi(t_n)|^2 \sim \gamma_n^{-\alpha}$, as $n \to \infty$. So

$$D_n^2 \sim \gamma_n \int_0^\infty h^2(t)\,dt + \gamma_n^{1-\alpha} \int_0^\infty (1 - h(t))^2 t^{-\alpha}\,dt, \quad \text{as } n \to \infty.$$

If as $n \to \infty$, either $\gamma_n \to 0$, or $\gamma_n \to \infty$, then $D_n^2 \to \infty$, and the estimator has zero efficiency. So we can define $\gamma \equiv \lim_{n\to\infty} \gamma_n$, where γ must be positive real and finite. The problem then becomes scale invariant, since, if we define $s \equiv \gamma t$, $g(s) \equiv h(s/\gamma)$, then

$$\lim_{n\to\infty} D_n^2 = \int_0^\infty g^2(s)\,ds + \int_0^\infty (1 - g(s))^2 s^{-\alpha}\,ds.$$

To minimize this expression, for each fixed s, we differentiate the integrand with respect to $g(s)$ and set the result equal to zero. Then, we get $g(s) = s^{-\alpha}/(1 + s^{-\alpha}) = 1/(1 + s^{\alpha})$, and

$$D_{n,\min}^2 = R_\alpha/2. \tag{3.8}$$

Now recall the condition (3.4). For any ϵ no matter how small there exists a $t_0 < \infty$, and a C, $0 < C < \infty$, such that if $t > t_0$, $|\varphi(t)|^2 \leqq Ct^{-(\alpha-\epsilon)}$. But $\int_0^{t_0/t} (1 - h(s))^2\,ds = O(t^{-(2\alpha+1)})$, as $t \to \infty$. So the condition (3.4) is satisfied, and the result (3.5) holds provided that $1 < \alpha < \infty$. By Theorem 2.1, the definition (3.7) of D_n^2, and (3.8), it follows that the efficiency is 1.

Now assume $\alpha = \infty$. The reasoning is the same, except that if (3.4) is satisfied,

$$\lim_{n\to\infty} D_n^2 = \int_0^\infty g^2(s)\,ds + \lim_{y\to\infty} y \int_0^1 (1 - g(s))^2\,ds.$$

This term is finite if and only if $g(s) = 1$ for all s, $0 \leqq s \leqq 1$, and it is minimal if and only if $g(s) = 0$, for all s, $1 < s < \infty$. Then $D_{n,\min}^2 = 1$. That the condition (3.4) is satisfied is immediate. So the result holds when $\alpha = \infty$, and by Theorem 2.1, the efficiency is again 1. The theorem is proved.

The efficient estimates given in this section are not everywhere non negative, if $2 < \alpha \leqq \infty$, since the functions $g(s)$ are not positive definite.

4. The main result. Let us introduce the following subclasses of the class C_∞ of characteristic functions.

DEFINITION. The characteristic function $\varphi(t) \varepsilon D_\gamma$, $0 < \gamma < \infty$, iff

$$-\log |\varphi(t)|^2 \sim Ct^\gamma, \tag{4.1}$$

$$-\partial \log |\varphi(t)|^2/\partial t \sim C\gamma t^{\gamma-1}, \tag{4.2}$$

as $t \to \infty$, $0 < C < \infty$, and $|\varphi(t)|^2$ is monotonically decreasing from all sufficiently large t. Let $D_\infty \equiv U_{0<\gamma<\infty} D_\gamma$. In this section a class of estimator sequences is presented, such that for each estimator in the class, (1.2) holds, provided $\varphi(t) \varepsilon D_\infty$.

THEOREM 4.1. *Let $\varphi(t) \varepsilon D_\infty$. Let $\hat{t}_n$ be a sample quantity, such that*

$$\lim_{n\to\infty} P\{\hat{t}_n < t_n\} = 0, \tag{4.3}$$

and for all $\epsilon > 0$,

$$\lim_{n\to\infty} P\{\hat{t}_n > t_n(1+\epsilon)\} = 0. \tag{4.4}$$

Let

$$\begin{aligned} \hat{\phi}_n(t) &\equiv \tilde{\phi}_n(t), \qquad 0 < |t| \leqq \hat{t}_n, \\ &\equiv 0, \qquad \hat{t}_n < |t| < \infty. \end{aligned}$$

Then for all $c < \infty$, $\lim_{n\to\infty} E \min (c, X_n^2/J_n^2) \leqq 1$, where

$$X_n^2 = \int_{-\infty}^{\infty} |\hat{\phi}_n(t) - \varphi(t)|^2\, dt.$$

PROOF. Let c be any fixed real number, and let $\epsilon > 0$ be arbitrarily chosen. Let

$$\begin{aligned} M_n &\equiv 1, \qquad \text{if} \quad t_n < \hat{t}_n < t_n(1+\epsilon), \\ &= 0, \qquad \text{otherwise.} \end{aligned}$$

Clearly $E \min (c, X_n^2/J_n^2) \leqq E(X_n^2 M_n/J_n^2) + E \min (c, X_n^2(1 - M_n)/J_n^2)$. By definition, $EX_n^2 M_n \leqq E \int_{-t_n(1+\epsilon)}^{t_n(1+\epsilon)} |\tilde{\varphi}_n(t) - \varphi(t)|^2\, dt + 2 \int_{t_n}^{\infty} |\varphi(t)|^2\, dt$. But, let $\epsilon' > 0$ be arbitrarily chosen and let $s \equiv t/t_n$. By Lemma 2.2, there exists a t_0, such that if $t_n > t_0$, and $s > 1$, then $|\varphi(st_n)|^2/|\varphi(t_n)|^2 \leqq s^{-1/\epsilon'}$. So $\int_{t_n}^{\infty} |\varphi(t)|^2\, dt = t_n|\varphi(t_n)|^2 \int_1^\infty (|\varphi(st_n)|^2/|\varphi(t_n)|^2)\, ds \leqq (n-1)^{-1} t_n \int_1^\infty s^{-1/\epsilon'}\, ds$. Since ϵ' was arbitrarily chosen, $\int_{t_n}^{\infty} |\varphi(t)|^2\, dt = o((n-1)^{-1}t_n)$ as $n \to \infty$. Thus, by Theorem 2.1, $\lim \sup_{n\to\infty} EX_n^2 M_n/J_n^2 \leqq 1 + \epsilon$. Clearly $E \min (c, X_n^2(1 - M_n)/J_n^2) \leqq cP_n$, where $P_n = P\{\hat{t}_n < t_n\} + P\{\hat{t}_n > t_n(1+\epsilon)\} \to 0$, as $n \to \infty$, by assumption. But ϵ was arbitrarily chosen. The theorem is proved.

THEOREM 4.2. *Let $\varphi(t) \varepsilon D_\gamma$, $0 < \gamma < \infty$. Let $\hat{t}_n \equiv \hat{s}_n(1 + f_n)$, where $\hat{s}_n$ is the*

Lebesgue measure of the set $\{t: |\tilde{\varphi}_n(t)|^2 \geqq a_n^{-1}, 0 \leqq t \leqq b_n\}$,

$$a_n \equiv n \exp - (\log n)^{\theta}, \quad 0 < \theta < 1, \tag{4.5}$$
$$b_n \equiv \exp (\log n)^{\theta},$$

and f_n *is a sequence, such that*

$$\lim_{n\to\infty} f_n = 0, \tag{4.6}$$

$$\liminf_{n\to\infty} (\log n)^{1-\theta} f_n = \infty. \tag{4.7}$$

Then $\hat{t}_n$ *satisfies the conditions of Theorem* 4.1.

Before proceeding to the proof of this theorem, a series of lemmas is proved.

LEMMA 4.1. *If* $\varphi(t) \varepsilon D_\gamma$, *then* $t_n \sim C^{-1/\gamma}(\log n)^{1/\gamma}$, *as* $n \to \infty$.

PROOF. *Obviously* $-\log |\varphi(t_n)|^2 = \log (n - 1)$. By (2.3), for any $\epsilon > 0$, $(C - \epsilon)t_n^\gamma \leqq \log (n - 1) \leqq (C + \epsilon)t_n^\gamma$, provided n is sufficiently large. Consequently $(C + \epsilon)^{-1/\gamma}(\log (n - 1))^{1/\gamma} \leqq t_n \leqq (C - \epsilon)^{-1/\gamma}(\log (n - 1))^{1/\gamma}$. It follows, therefore, that $t_n \sim C^{-1/\gamma}(\log (n - 1))^{1/\gamma} \sim C^{-1/\gamma}(\log n)^{1/\gamma}$ as $n \to \infty$. The lemma is proved.

LEMMA 4.2. *Let* s_n *be given by the equation* $|\varphi(s_n)|^2 = a_n^{-1}$. *Then, if* $\varphi(t) \varepsilon D_\gamma$, $0 < \gamma < \infty$, $\lim_{n\to\infty} s_n/t_n = 1$ *where* t_n *is given by* (2.3).

PROOF. By analogy with Lemma 4.1, it follows that $s_n \sim C^{-1/\gamma}(\log a_n)$, as $n \to \infty$, where a_n is given by (4.5). The result is immediate.

LEMMA 4.3. *Let* ϵ_n *be a sequence such that* $\lim_{n\to\infty} \epsilon_n = 0$. *If* $\varphi(t) \varepsilon D_\gamma$, $0 < \gamma < \infty$, *then*

$$\log (|\varphi(s_n(1 - \epsilon_n))|^2/|\varphi(s_n)|^2) \sim \gamma\epsilon_n \log n, \quad \text{as} \quad n \to \infty.$$

PROOF. First, observe that

$$-s_n\epsilon_n (\inf_{s_n(1-\epsilon_n) \leqq t \leqq s_n} \partial \log |\varphi(t)|^2/\partial t) \leqq \log |\varphi(s_n(1 - \epsilon_n))|^2 - \log |\varphi(s_n)|^2$$
$$\leqq -s_n\epsilon_n (\sup_{s_n(1-\epsilon_n) \leqq t \leqq s_n} \partial \log |\varphi(t)|^2/\partial t).$$

But for any $\epsilon > 0$, and all sufficiently large n,

$$(C - \epsilon)\gamma s_n^{\gamma-1} \min (1, (1 - \epsilon_n)^{\gamma-1})$$
$$\leqq \inf_{s_n(1-\epsilon_n) \leqq t \leqq s_n} \partial \log |\varphi(t)|^2/dt \leqq \sup_{s_n(1-\epsilon_n) \leqq t \leqq s_n} \partial \log |\varphi(t)|^2/\partial t$$
$$\leqq (C + \epsilon)\gamma s_n^{\gamma-1} \max (1, (1 - \epsilon_n)^{\gamma-1}),$$

by (4.2), since the function $C\gamma t^{\gamma-1}$ is monotonically increasing or constant. Consequently $\log |\varphi(s_n(1 - \epsilon_n))|^2 - \log |\varphi(s_n)|^2 \sim C\gamma s_n^\gamma \epsilon_n = \gamma\epsilon_n \log a_n \sim \gamma\epsilon_n \log n$, as $n \to \infty$. The lemma is proved.

LEMMA 4.4. *Let* ϵ_n *be a sequence of real numbers, such that*

$$\lim_{n\to\infty} \epsilon_n = 0, \tag{4.8}$$

and for all γ, $0 < \gamma < \infty$,

$$\lim_{n\to\infty} (\gamma\epsilon_n \log n + \log \epsilon_n) = \infty. \tag{4.9}$$

Then

(4.10) $$\lim_{n\to\infty} P\{\hat{s}_n \leqq s_n(1 - \epsilon_n)\} = 0.$$

PROOF: Let t_0 be so chosen that $|\varphi(t)|^2$ is monotonic for all $t \geqq t_0$, and let ϵ_n be a sequence, satisfying (4.8) and (4.9). Let

(4.11) $$A_n^2 \equiv \int_{t_0}^{b_n} \tilde{\varphi}_n(t) - \varphi(t)|^2\, dt.$$

If $\hat{s}_n \leqq (1 - \epsilon_n)s_n$, then

(4.12) $$A_n^2 \geqq \int_{s_n(1-\epsilon_n)}^{s_n(1-\epsilon_n/2)} |a_n^{-\frac{1}{2}} - \varphi(t)|^2\, dt \geqq s_n(\epsilon_n/2)a_n^{-1}B_n,$$

where $B_n \equiv |(\varphi(s_n(1 - \epsilon_n/2))/\varphi(s_n)) - 1|^2$. The inequality is valid provided that n is so large that $s_n(1 - \epsilon_n) > t_0$. Note that

(4.13) $$EA_n^2 \leqq b_n/n.$$

By (4.13) and the Chebycheff inequality,

$$P\{\hat{s}_n \leqq (1 - \epsilon_n)s_n\} \leqq P\{A_n^2 \geqq s_n(\epsilon_n/2)a_n^{-1}B_n\}$$
$$= n^{-1}b_n/s_n(\epsilon_n/2)a_n^{-1}B_n = (a_n/n)b_n/s_nB_n(\epsilon_n/2).$$

But, by the definitions (4.5) of a_n and b_n, it clearly follows that

(4.14) $$(a_n/n)b_n = 1.$$

By the condition (4.9) and Lemma 4.3, it follows that $\lim_{n\to\infty} \epsilon_n B_n = \infty$. But, also $\lim_{n\to\infty} s_n = \infty$. The lemma is proved.

LEMMA 4.5. *Let $\epsilon > 0$ be arbitrarily chosen. Then*

$$\lim_{n\to\infty} P\{\hat{s}_n > s_n(1 + \epsilon)\} = 0.$$

PROOF. For any $\epsilon > 0$, $b_n > (1 + \epsilon)s_n$ for all sufficiently large n. Suppose $\hat{s}_n > s_n(1 + \epsilon)$. Then $A_n^2 \geqq \int_{s_n(1+\epsilon/2)}^{s_n(1+\epsilon)} |a_n^{-1/2} - \varphi(t)|^2\, dt \geqq s_n(\epsilon/2)a_n^{-1}C_n$, where $C_n \equiv |(\varphi(s_n(1 + \epsilon/2))/\varphi(s_n)) - 1|^2 \to 1$, as $n \to \infty$, and A_n^2 is given by (4.11). Again using the Chebycheff inequality, and recalling (4.13), $P\{\hat{s}_n > (1 + \epsilon)s_n\} \leqq P\{A_n^2 \geqq s_n(\epsilon/2)a_n^{-1}C_n\} \leqq (a_n/n)b_n/s_n(\epsilon/2)C_n$, which approaches 0 as $n \to \infty$, by (4.14) and the fact that $\lim_{n\to\infty} s_n = \infty$. The lemma is proved.

LEMMA 4.6. *The sequences s_n and t_n satisfy the relationship*

(4.15) $$(t_n - s_n)/s_n \sim (t_n - s_n)/t_n \sim \gamma^{-1}(\log n)^{\theta-1} \quad \text{as } n \to \infty.$$

PROOF. Clearly, for any $\epsilon > 0$,

$$(C - \epsilon)\gamma(t_n - s_n) \min(s_n^{\gamma-1}, t_n^{\gamma-1})$$
$$\leqq \log|\varphi(t_n)|^2 - \log|\varphi(s_n)|^2 = (\log n)^\theta \leqq (C + \epsilon)\gamma(t_n - s_n) \max(s_n^{\gamma-1}, t_n^{\gamma-1}),$$

for all sufficiently large n, by (4.2), and the fact that for all sufficiently large t, $\partial \log|\varphi(t)|^2/\partial t$ is monotonic. Equivalently $(\log n)^\theta/(C + \epsilon)\gamma \max(s_n^{\gamma-1}, t_n^{\gamma-1}) \leqq t_n - s_n \leqq (\log n)^\theta/(C - \epsilon)\gamma \min(s_n^{\gamma-1}, t_n^{\gamma-1})$. By Lemma 4.2, since ϵ was arbitrarily chosen, $(t_n - s_n) \sim (\log n)^\theta/C\gamma s_n^{\gamma-1}$, as $n \to \infty$, and the result (4.15) holds. The lemma is proved.

PROOF OF THEOREM 4.2. For arbitrary $\epsilon > 0$ and sufficiently large n

$$(t_n - s_n)/t_n < d_n \equiv \gamma^{-1}(\log n)^{\theta-1}(1 + \epsilon). \tag{4.16}$$

That is $1 - d_n \leqq s_n/t_n$. So $s_n \geqq t_n(1 - d_n)$. Clearly,

$$P\{\hat{t}_n < t_n\} = P\{\hat{s}_n < t_n(1 + f_n)^{-1}\}$$
$$\leqq P\{\hat{s}_n < s_n(1 - d_n)^{-1}(1 + f_n)^{-1}\} = P\{\hat{s} < s_n(1 - \epsilon_n)Q_n\},$$

where $Q_n = (1 - d_n)^{-1}(1 + f_n)^{-1}(1 - \epsilon_n)^{-1} = 1 + d_n + \epsilon_n - f_n + o(d_n + \epsilon_n - f_n)$, as $n \to \infty$. So by Lemma 4.4, the condition (4.4) is satisfied, provided that $Q_n \leqq 1$ for all sufficiently large n. The conditions (4.6) and (4.7) are satisfied if for some $\beta > 1$, $\epsilon_n = (\log \log n)^{\beta}/\log n$. In this case, clearly, $\epsilon_n = o(d_n)$, as $n \to \infty$. So $Q_n \leqq 1$, for all sufficiently large n, provided (4.4) holds, and the condition (4.3) is satisfied.

By Lemma 4.5, and the condition (4.6), the condition (4.4) is satisfied. The theorem is proved.

Now, let us investigate the properties of the present class of estimator sequences, when $\varphi(t) \varepsilon C_\alpha$, $1 < \alpha < \infty$.

LEMMA 4.7. *If $\varphi(t) \varepsilon C_\alpha$, $1 < \alpha < \infty$, then $\hat{s}_n/s_n \to 1$ in probability as $n \to \infty$.*

PROOF. Let $\epsilon > 0$ be arbitrarily chosen. Clearly, if $\hat{s}_n < s_n(1 - \epsilon)$, then $A_n^2 \geqq \int_{s_n(1-\epsilon)}^{s_n(1-\epsilon/2)} |a_n^{-\frac{1}{2}} - \varphi(t)|^2 \, dt > s_n B_n(\epsilon/2)a_n^{-1}$, where

$$B_n \equiv |(\varphi(s_n(1 - \epsilon/2))/\varphi(s_n)) - 1|^2 \geqq (1 - \epsilon)((1 - \epsilon/2)^{-\alpha} - 1) > 0,$$

for sufficiently large n. Recalling (4.13) and (4.14), by the Chebycheff inequality, $P\{\hat{s}_n < s_n(1 - \epsilon)\} \leqq P\{A_n^2 > s_n B_n(\epsilon/2)a_n^{-1}\} \leqq EA_n^2/s_n B_n(\epsilon/2)a_n^{-1} = (a_n/n)b_n/s_n B_n(\epsilon/2) \to 0$, as $n \to \infty$, since $s_n \to \infty$. To prove that for all $\epsilon > 0$, $\lim_{n\to\infty} P\{\hat{s}_n > s_n(1 + \epsilon)\} = 0$, we proceed as in Lemma 4.5, with the exception that $\liminf_{n\to\infty} C_n^2 \geqq (1 - \epsilon)(1 - (1 + \epsilon/2)^{-\alpha}) > 0$. The lemma is proved.

Let us assume that $\varphi(t)$ decreases algebraically. That is we assume that $|\varphi(t)|^2 \sim Ct^{-\alpha}$ as $t \to \infty$. Let $\epsilon > 0$ be arbitrarily chosen. Then for sufficiently large n, $(C - \epsilon)t_n^{-\alpha} \leqq |\varphi(t_n)|^2 = (n - 1)^{-1} \leqq (C + \epsilon)t_n^{-\alpha}$. Equivalently, $(C - \epsilon)^{1/\alpha}(n - 1)^{1/\alpha} \leqq t_n \leqq (C + \epsilon)^{1/\alpha}(n - 1)^{1/\alpha}$. Since $\epsilon > 0$ was arbitrarily chosen,

$$t_n \sim C^{1/\alpha}(n - 1)^{1/\alpha} \sim (Cn)^{1/\alpha} \quad \text{as} \quad n \to \infty. \tag{4.17}$$

LEMMA 4.8. *If $\varphi(t)$ decreases algebraically of order $\alpha/2$, then*

$$\lim_{n\to\infty} s_n/t_n = 0.$$

The result is immediate, following from the definitions, and from (4.17). That the efficiency of the given estimator sequence is zero follows by the reasoning in Section 2.

5. Discussion. The class D_∞ contains many density function, including the Normal, Cauchy, and double exponential ones. It does not include the Gamma density functions, unfortunately.

In a recent report by Woodroofe [8] a similar, though different, approach is taken. His method has zero efficiency, if $\varphi(t) \varepsilon D_\infty$, but it is clearly more efficient than some of those methods currently in use. It appears to be considerably easier to compute than the present one.

For the problem of estimating a spectral density function, Leppink [2] has given an estimator sequence, wherein the "truncation point" depends on the observations. His method is different from both ours and Woodroofe's. Although under very general conditions it has zero efficiency, in the opinion of the author it is probably better than most of those now generally in use.

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