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ON THE SUPERCRITICAL ONE DIMENSIONAL AGE DEPENDENT BRANCHING PROCESSES[1]

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1. Introduction and summary. Let $\{Z(t);\ t \geqq 0\}$ be a one dimensional age dependent branching process with offspring probability generating function (pgf) $h(s) \equiv \sum_{j=0}^{\infty} p_j s^j$ and lifetime distribution function $G(t)$ (see Section 2 for definitions). If $m(t) \equiv EZ(t)$ is the mean function let $Y(t) = Z(t)/m(t)$. Our objective in this paper is to study the limiting behavior of the process $\{Y(t); t \geqq 0\}$. The main result is

THEOREM 0. *Assume* $Z(0) \equiv 1$, $m = h'(1) > 1$, $G(0+) = 0$. (*Here* $\to_p$ *and* $\to_d$ *mean convergence in probability and distribution respectively*). *Then*:

$$(1) \qquad \sum_{j=2}^{\infty} j \log j p_j = \infty \qquad \text{implies} \quad Z(t)/EZ(t) \to_p 0$$

and

$$(2) \qquad \sum_{j=2}^{\infty} j \log j p_j < \infty \qquad \text{implies} \quad Z(t)/EZ(t) \to_d W$$

where W *is an nonnegative random variable such that*

(a) $EW = 1$,

(b) $\varphi(u) = E(e^{-uW})$ *for* $u \geqq 0$ *satisfies*

$$(3) \qquad \varphi(u) = \int_0^{\infty} h(\varphi(ue^{-\alpha y}))\, dG(y)$$

where α *is the unique root of the equation* $m \int_0^{\infty} e^{-\alpha y}\, dG(y) = 1$

(c) $P(W = 0) = q$ *the extinction probability*

(d) W *has an absolutely continuous distribution on the positive real axis and the density function is continuous. That is, there exists a nonnegative continuous function* $g(x)$ *defined for* $x > 0$ *such that for* $0 < x_1 < x_2 < \infty$

$$(4) \qquad P(x_1 < W < x_2) = \int_{x_1}^{x_2} g(x)\, dx.$$

Kesten and Stigum [4] proved the above result for the case when $G(x)$ is the step function

$$(5) \qquad \begin{aligned} G(x) &= 0 \qquad \text{if} \quad x \leqq 1 \\ &= 1 \qquad\qquad x > 1. \end{aligned}$$

This is the Galton-Watson process in discrete time. They considered the multidimensional case. Athreya and Karlin [1] considered the case (here $0 < \lambda < \infty$)

$$(6) \qquad \begin{aligned} G(x) &= 1 - e^{-\lambda x} \qquad \text{for} \quad x > 0 \\ &= 0 \qquad\qquad\qquad x \leqq 0. \end{aligned}$$

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This is the continuous time Markov branching process. Their approach was via split times.

Levinson [6] established the law convergence of $Z(t)/EZ(t)$ under conditions slightly stronger than ours. Harris [3] claimed mean square convergence of $Z(t)/EZ(t)$ when $h''(1) < \infty$ and the absolute continuity of W when in addition to $h''(1) < \infty$, $1 - G(t) = O(e^{-c\tau})$ for some $c > 0$.

Our result is the sharpest known in this direction in as much as (i) we establish the convergence of $Z(t)/EZ(t)$ without any conditions, (ii) we give a necessary and sufficient condition for the nondegeneracy of the limit random variable W and (iii) when W is nondegenerate we establish the absolute continuity without any extra assumptions.

The methods employed in this paper are all extremely simple. Among them are a simplified and sharpened form of Levinson's [6] arguments and a simplification of Stigum's [7] idea to prove absolute continuity of W. One of the important ideas used here is the exploitation of the underlying Galton-Watson process constituted by the size $\{\zeta_n\}$ of the different generations. The key to the understanding of the moment condition $\sum_j j \log jp_j < \infty$ is the simple Lemma 1.

Here is an outline of the rest of the paper. In Section 2 we describe the setting and introduce the necessary terminology and notation. The functional equation (3) is studied in detail in Section 3 where it is shown that a necessary and sufficient condition for (3) to have a nontrivial solution is the finiteness of $\sum j \log jp_j$. The next section explores the connection between the process $\{Z(t); t \geqq 0\}$ and the underlying Galton-Watson process $\{\zeta_n; n = 0, 1, 2, \cdots\}$ and shows that if $\sum j \log jp_j = \infty$ then $Z(t)/EZ(t) \to_p 0$. Assuming $\sum j \log jp_j < \infty$ the convergence in distribution of $Z(t)/EZ(t)$ to a nondegenerate random variable W is shown in Section 5 while Section 6 takes up the proof of absolute continuity. The last section lists some open problems.

2. The basic setup and a lemma. Let $\{Z(t, \omega); t \geqq 0\}$ be an age dependent branching process á lá Harris [3] corresponding to the offspring pgf $h(s) = \sum_{j=0}^{\infty} p_j s^j$ and lifetime distribution function $G(t)$ and defined on a probability space $(\Omega, \mathfrak{F}, P)$ where Ω is the space of all "family histories" and $\mathfrak{F}$ is a "big enough" σ-algebra of subsets of Ω and P is a probability measure as $(\Omega, \mathfrak{F})$ such that the following interpretation of $\{Z(t); t \geqq 0\}$ is valid. We start the process with a certain number $Z(0)$ of particles. Each particle lives a random length of time whose distribution function is $G(t)$ and on death creates a random number of progenies whose pgf is $h(s)$. All the offspring evolve independently of each other and of the parent and in the same manner as the parent. Then $Z(t)$ can be regarded as the number of particles in the system at time t. We now make

ASSUMPTION 1. $P(Z(0, \omega) = 1) = 1.$

ASSUMPTION 2. $m \equiv h'(1-) < \infty.$

ASSUMPTION 3. $G(0+) = 0.$

Under these assumptions it has been shown in [3] that

$$F(s, t) \equiv E(s^{Z(t,\omega)}) = \int_\Omega s^{Z(t,\omega)}\, dP(\omega) \tag{7}$$

where $$0 \leqq s \leqq 1$$

is the unique solution to the integral equation

$$(8) \quad F(s, t) = s(1 - G(t)) + \int_0^t h(F(s, t-u))\, dG(u) \quad \text{for} \quad |s| \leqq 1, \qquad t \geqq 0$$

among those satisfying $|F(s, t)| \leqq 1$.

Further $F(1, t) = 1$ for all $t \geqq 0$ and thus assuring us that $P(Z(t) < \infty) = 1$.

Also under the same assumptions one has

$$(9) \qquad m(t) \equiv EZ(t, \omega)$$

is finite for each finite t and is the unique solution of

$$(10) \qquad m(t) = 1 - G(t) + m \int_0^t m(t-u)\, dG(u)$$

such that it is bounded on each finite t-interval.

It is also known that

$$(11) \qquad P(A \cup B) = 1$$

where $$A = \{\omega : Z(t, \omega) \to \infty\}$$

and $$B = \{\omega : Z(t, \omega) \to 0\}$$

and $P(B)$, called the extinction probability, coincides with q, the smallest non-negative root of the equation $h(x) = x$.

Thus if $m \leqq 1$ then $q = 1$ and so $P(B) = 1$ and hence $Z(t, \omega) \to 0$ almost surely (a.s.) as $t \to \infty$.

In order to make the problem nontrivial we now impose

ASSUMPTION 4. $\quad 1 < m < \infty$.

We call the process *supercritical* in this case.

The question that we seek to answer is what can we say about the limiting behavior of $Z(t)/m(t)$ as $t \to \infty$. Theorem 0 of the previous section is the answer. We break up the proof of Theorem 0 into several bits.

We need one more assumption.

ASSUMPTION 5. $G(t)$ is not lattice. (See [3] for a definition).

Since $m > 1$, and $G(0+) = 0$ there exist a unique positive number α such that

$$(12) \qquad m \int_0^\infty e^{-\alpha t}\, dG(t) = 1.$$

It now follows (see [3]) that

$$(13) \qquad \lim_{t \to \infty} (m(t)/ce^{\alpha t}) = 1$$

where

$$c = (m-1)/(\alpha m^2 \int_0^\infty te^{-\alpha t}\, dG(t)).$$

Thus it suffices to consider the limiting behavior of $Z(t)/ce^{\alpha t}$.

We finish this section with a lemma which explains the moment condition $\sum j \log jp_j < \infty$.

LEMMA 1. *Let X be a nonnegative random variable with $0 < m = EX < \infty$ (of course, E is the expectation operator). Then for any $0 < a < \infty$*

(14) $\int_0^a u^{-2}[E(e^{-uX/m}) - e^{-u}]\, du < \infty$

iff

(15) $$EX\,|\log X| < \infty.$$

PROOF. Since $EX = m$ we have the identity

(16) $$E(e^{-uX/m}) - e^{-u} = E(e^{-uX/m} - 1 + (uX/m)) + 1 - u - e^{-u} \quad \text{for all} \quad u \geqq 0.$$

Noting that for $u \geqq 0$ we have $0 \leqq e^{-u} - 1 + u \leqq u^2/2$ it suffices to prove that

(17) $$\int_0^a u^{-2}E\{e^{-uX/m} - 1 + (uX/m)\}\, du < \infty$$

iff (15) holds. Since $e^{-u} - 1 + u \geqq 0$ for $u \geqq 0$ exchanging orders of integration the left side of (17) becomes

$E\{\int_0^a u^{-2}[e^{-uX/m} - 1 + (uX/m)]\, du\}$ which equals

$$E\{(X/m)\int_0^{aX/m} v^{-2}(e^{-v} - 1 + v)\, dv\}.$$

But $\lim_{T\to\infty} (\int_0^T v^{-2}(e^{-v} - 1 + v)\, dv)(\log T)^{-1} = 1$ and hence (17) holds iff

$$E(X/m)|\log (aX/m)| < \infty$$

which is clearly equivalent to (15). Q.E.D.

We shall find the following consequences of this lemma very useful.

COROLLARY 1. *Let X and Y be independent and nonnegative random variables such that for some $\alpha > 0$, $EXe^{-\alpha Y} = 1$. Then for any $0 < a < \infty$*

(18) $$\int_0^a u^{-2}\{E(e^{-uXe^{-\alpha Y}}) - e^{-u}\}\, du < \infty$$

iff

(19) $$EX\,|\log X| < \infty.$$

PROOF. If $P(Y = 0) = 1$ the assertion is the same as Lemma 1. If for some $\delta > 0, P(Y > \delta) > 0$ then the corollary follows from Lemma 1 by noting that, in view of independence,

$$EXe^{-\alpha Y}\,|\log Xe^{-\alpha Y}| < \infty,$$

iff

$$EX\,|\log X| < \infty. \qquad \text{Q.E.D.}$$

COROLLARY 2. *Let $h(s) \equiv \sum_{j=0}^{\infty} p_j s^j$ and $G(t)$ be a* pgf *and distribution function respectively. Let $G(0) = G(0+) = 0$. Let $1 < m = EX < \infty$ and for $u > 0$*

(20) $$\psi(u) = \int_0^{\infty} u^{-1}\{h(ue^{-\alpha y}) - e^{-u}\}\, dG(y),$$

where α uniquely chosen to satisfy $m \int_0^\infty e^{-\alpha t}\, dG(t) = 1$. Then for any $0 < a < \infty$, $0 < c, r < 1$,

$$\int_0^a u^{-1}\psi(u)\, du < \infty \quad \text{and} \quad \sum_{n=1}^\infty \psi(cr^n) < \infty \tag{21}$$

iff

$$\sum_{j=2}^\infty j \log jp_j < \infty. \tag{22}$$

PROOF. In Corollary 1 take X to be a nonnegative integer valued random variable with pgf $h(s)$ and Y be a nonnegative random variable with distribution function $P(Y < t) = G(t)$ for all $t \geqq 0$. By monotonicity of $\psi(u)$ for u small the two quantities in (21) are finite or infinite at the same time.

COROLLARY 3. *Let* $h(s) \equiv \sum_0^\infty p_j s^j$ *be a* pgf *with* $m = \sum_{j=1}^\infty jp_j < \infty$. *Define* $A(u)$ *on* $[0, 1]$ *by*

$$A(u) = m - u^{-1}(1 - h(1 - u)). \tag{23}$$

Then $A(u)$ *is nonnegative and nondecreasing. Further, for any* r *and* c *in* $(0, 1)$

$$\sum_{n=0}^\infty A(cr^n) < \infty \quad \text{and} \quad \lim_{c \downarrow 0} \sum_0^\infty A(cr^n) = 0 \tag{24}$$

iff

$$\sum_{j=2}^\infty j \log jp_j < \infty. \tag{25}$$

PROOF. For $0 \leqq u \leqq 1$

$$A(1 - u) = m - (1 - u)^{-1}(1 - h(u)).$$

But by mean value theorem the function $(1 - u)^{-1}(1 - h(u))$ is nonnegative and nondecreasing and $\uparrow m$ as $u \uparrow 1$ ($\uparrow$ stands for increasing). This shows $A(u) \geqq 0$ and $\uparrow$ in $[0, 1]$. By monotonicity $\sum_0^\infty A(cr^n) < \infty$ iff

$$\int_0^\infty A(cr^t)\, dt < \infty$$

which is equal to

$$(\log_e r)^{-1} \int_0^c v^{-1} A(v)\, dv. \tag{26}$$

But

$$v^{-1}A(v) = \{h(e^{-u/m}) - e^{-u} + m[1 - e^{-u/m} - (u/m)] + (e^{-u} - 1 + u)\}u^{-2}(uv^{-1})^2$$

where
$$1 - v = e^{-u/m}.$$

Since $x(1 - e^{-x})^{-1}$ is $\geqq 1$ for $x \geqq 0$ and bounded in any finite interval and $0 \leqq e^{-x} - 1 + x \leqq x^2/2$ for $x \geqq 0$ we can conclude in view of (26) that (24) holds iff

$$\int_0^{c'} u^{-2}\{h(e^{-u/m}) - e^{-u}\}\, du < \infty \tag{27}$$

where c' is a constant, $0 < c' < \infty$.

The corollary now follows from Lemma 1 if we take X to be a nonnegative integer valued random variable with $h(s)$ as its pgf. Q.E.D.

3. The functional equation. $\varphi(u) = \int_0^\infty h(\varphi(ue^{-\alpha y}))\, dG(y)$. If $Z(t)/ce^{\alpha t}$ converges in law to a limit random variable W then from the integral equation (8) and bounded convergence theorem we can readily conclude that $\varphi(u) \equiv E(e^{-uW})$, $u \geqq 0$, satisfies the functional equation (3) which we recall here for ease of reference. It is

$$(28) \qquad \varphi(u) = \int_0^\infty h(\varphi(ue^{-\alpha y}))\, dG(y) \qquad \text{for} \quad u \geqq 0.$$

In this section we shall obtain necessary and sufficient conditions for the existence and uniqueness of nontrivial solutions to the above functional equation. For this purpose we define the following classes

$$(29) \quad C = \{\varphi : \varphi \text{ maps } [0, \infty) \text{ onto } (0, 1], \varphi(0) = 1, \lim_{u \downarrow 0} u^{-1}(1 - \varphi(u)) > 0\}$$

$$(30) \quad C_\theta = \{\varphi : \varphi \in C, \lim_{u \downarrow 0} u^{-1}(1 - \varphi(u)) = \theta\} \quad \text{for} \quad 0 < \theta < \infty.$$

Throughout this section we drop our original assumptions and we need to assume only that $m = h'(1) > 1$, $G(0+) < m^{-1}$ and both the following situations do not prevail simultaneously

(a) m is an integer and $h(s) \equiv s^m$.

(b) There exists a $d > 0$ such that $G(d+) - G(d) = 1$.

To start with we have the following result on uniqueness.

THEOREM 1. *Let* $0 < \theta < \infty$. *If* φ_1 *and* φ_2 *are both in* C_θ *and satisfy* (28) *then* $\varphi_1 \equiv \varphi_2$.

PROOF. Let

$$(31) \qquad \psi(u) = |\varphi_1(u) - \varphi_2(u)|/u \qquad \text{for} \quad u > 0.$$

Then from (28),

$$\psi(u) \leqq \int_0^\infty [|h(\varphi_1(ue^{-\alpha y})) - h(\varphi_2(ue^{-\alpha y}))|/u]\, dG(y)$$
$$\leqq m \int_0^\infty (|\varphi_1(ue^{-\alpha y}) - \varphi_2(ue^{-\alpha y})|/u)\, dG(y)$$

since by mean value theorem $|h(x_1) - h(x_2)| \leqq m\,|x_1 - x_2|$ for $0 \leqq x_1, x_2 \leqq 1$. Thus

$$(32) \qquad \psi(u) \leqq m \int_0^\infty \psi(ue^{-\alpha y})e^{-\alpha y}\, dG(y)$$

$$\text{or} \qquad \psi(u) \leqq E\psi(ue^{-\alpha X})$$

where X is a nonnegative random variable with $P(X < x) = m \int_0^x e^{-\alpha y} dG(y)$. Iterating (32) we get

$$(33) \qquad \psi(u) \leqq E\psi(ue^{-\alpha S_n})$$

where $S_n = \sum_1^n X_i$ and X_i, $i = 1, 2, \cdots, \cdots$ are independent random variables distributed as X in (32).

Since φ_1 abd φ_2 are in C_θ and for any $u > 0$

$$|\psi(u)| \leqq |(\varphi_1(u) - 1)u^{-1} - \theta| + |\theta - (1 - \varphi_2(u))u^{-1}|$$

we see that

$$(34) \qquad \lim_{u \downarrow 0} |\psi(u)| = 0$$

and away from 0, $\psi(u)$ is bounded by $2u^{-1}$. Further since $G(0+)$ is less than m^{-1}, we must have $\alpha > 0$ and $EX > 0$. And hence by strong law of large numbers and (34)

$$\psi(ue^{-\alpha S_n}) \to 0 \qquad \text{a.s.}$$

Applying bounded convergence theorem we now get from (33)

$$\psi(u) = 0 \qquad \text{for all} \quad u > 0.$$

Also $\varphi_1(0) = 1 = \varphi_2(0)$ and hence $\varphi_1 \equiv \varphi_2$. Q.E.D.

A necessary condition for the existence of a solution of (3) in C is given by the following

THEOREM 2. *There exists a φ in C satisfying* (3) *only if*

$$(35) \qquad \sum_{j=2}^{\infty} j(\log j)p_j < \infty.$$

PROOF. Suppose (35) does not hold and there exists a φ in C satisfying (3). Clearly

$$\begin{aligned} 0 \leqq g(u) &\equiv (1 - \varphi(u))u^{-1} \\ (36) \qquad &= \int_0^\infty \{1 - h(\varphi(ue^{-\alpha y}))\}u^{-1}\, dG(y) \\ &= m \int_0^\infty [1 - \varphi(ue^{-\alpha y})][1 - m^{-1}A(1 - \varphi(ue^{-\alpha y}))]\, dG(y) \end{aligned}$$

where $A(u)$ is defined by (23). Thus

$$\begin{aligned} (37) \quad 0 \leqq g(u) &= m \int_0^\infty g(ue^{-\alpha y})[1 - m^{-1}A(g(ue^{-\alpha y})ue^{-\alpha y})]e^{-\alpha y}\, dG(y) \\ &= E\{g(ue^{-\alpha X})[1 - m^{-1}A(g(ue^{-\alpha X})ue^{-\alpha X})]\} \end{aligned}$$

where X is as defined by (32) in Theorem 1. Since $\lim_{u \downarrow 0} g(u) > 0$ there exist $c > 0$, $0 < \beta$, $u_0 < 1$ such that

$$(38) \qquad u \leqq u_0 \Rightarrow c \geqq g(u) \geqq \beta > 0.$$

Now by Corollary 3, $A(u)$ is nondecreasing and nonnegative and so if $u \leqq u_0$ (37) yields

$$(39) \qquad 0 < \beta \leqq g(u) \leqq E\{g(ue^{-\alpha X}) \exp(-m^{-1}A(\beta ue^{-\alpha X}))\}.$$

On iterating (39) we get for $u \leqq u_0$

$$(40) \qquad 0 < \beta \leqq g(u) \leqq E\{g(ue^{-\alpha S_n}) \exp(-m^{-1} \textstyle\sum_{m=1}^{n} A(\beta ue^{-\alpha S_m}))\}$$

where S_n is defined in (33). From (40) noting that $g(u)$ is bounded by c for $u \leqq u_0$ we have for $u \leqq u_0$

$$(41) \qquad 0 < \beta \leqq g(u) \leqq cE\{\exp(-m^{-1} \textstyle\sum_{j=1}^{n} A(\beta ue^{-\alpha S_j}))\}.$$

We now claim that $\sum_{j=1}^{\infty} A(\beta ue^{-\alpha S_j}) = \infty$ a.s. if (35) does not hold.

The quickest way to see this is to note that

$$\theta = EX = m \int_0^\infty x e^{-\alpha x} \, dG(x)$$

is finite and hence for some $\epsilon > 0$, $S_j < j(\theta + \epsilon)$ for all large j, a.s. But $A(x)$ is nondecreasing and hence for j large $A(\beta u e^{-\alpha S_j}) \geqq A(\beta u r^j)$ where $0 < r = e^{-\alpha(\theta+\epsilon)} < 1$. However, from Corollary (3) we know that if (35) does not hold then

$$\sum_{j=0}^\infty A(\beta u r^j) = \infty,$$

and this establishes our claim.

From (41) we see that for $u \leqq u_0$

$$0 < \beta \leqq g(u) \equiv 0$$

which is absurd. Q.E.D.

We shall now show that (35) is a sufficient condition for the existence of a solution to (3) in the class C_1 (i.e. in C_θ with $\theta = 1$).

THEOREM 3. *Let* $\varphi_0(u) \equiv e^{-u}$ *and* $\varphi_{n+1}(u) \equiv (T\varphi_n)(u)$ *for* $n = 0, 1, 2, \cdots$ *where for any* φ^* *in* C

$$(T\varphi^*)(u) = \int_0^\infty h(\varphi^*(ue^{-\alpha y})) \, dG(y). \tag{42}$$

If (35) holds then $\varphi(u) \equiv \lim_{n\to\infty} \varphi_n(u)$ exists for each $u \geqq 0$, $\varphi(u)$ belongs to C_1 and $\varphi(u)$ satisfies (3) i.e. $\varphi = T\varphi$.

PROOF. For any $n \geqq 1$ let for $u > 0$

$$\psi_n(u) = |\varphi_n(u) - \varphi_{n-1}(u)|/u. \tag{43}$$

Then as in Theorems 1 and 2 using mean value theorem we get the recurrence inequality

$$0 \leqq \psi_{n+1}(u) \leqq m \int_0^\infty \psi_n(ue^{-\alpha y}) e^{-\alpha y} \, dG(y)$$
$$= E\psi_n(ue^{-\alpha X})$$

where X is defined by (32) in Theorem 1. On iterating the above we get

$$0 \leqq \psi_{n+1}(u) \leqq E\psi_1(ue^{-\alpha S_n}) \tag{44}$$

where S_n has the same meaning as in (33).

If we define $\psi(u) \equiv u^{-1}[\varphi_1(u) - \varphi_0(u)]$ we can easily check that

$$\psi(u) = u^{-1}[E(e^{-uNe^{-\alpha Y}}) - e^{-u}],$$

$$\lim_{u \downarrow 0} \psi(u) = 0 \qquad \text{(use L'Hôpital's rule)}$$

$$\lim_{u \downarrow 0} \psi'(u) = E(Ne^{-\alpha Y})^2 - 1 \qquad \text{(use L'Hôpital's rule)}$$

where N and Y are two independent random variables and N is nonnegative integer valued with $h(s)$ as its pgf while Y is nonnegative valued with distribution $G(y)$. From the choice of α in (12) it follows $E(Ne^{-\alpha Y}) = 1$ and so

$$\lim_{u \downarrow 0} \psi'(u) = \text{Variance of } Ne^{-\alpha Y}$$

which is strictly positive unless both N and Y are degenerate random variables, a case excluded by us.

Thus there exists a $u_0 > 0$ such that $\psi(u)$ is nonnegative and nondecreasing for $0 < u \leqq u_0$ and hence

$$\psi(u) = \psi_1(u) \qquad \text{for} \quad u \leqq u_0. \tag{45}$$

We shall now show that for any $u > 0$

$$0 \leqq \sum_1^\infty \psi_n(u) \equiv \Psi(u) < \infty, \qquad \text{and} \tag{46}$$

$$\lim_{u\downarrow 0} \Psi(u) = 0. \tag{47}$$

From (44)

$$0 \leqq \Psi(u) \leqq \sum_1^\infty E(\psi_1(ue^{-\alpha S_n})).$$

Let $G^*(\lambda) = \int_0^\infty e^{-\lambda y}\, dG(y)$ for $\lambda > 0$. Choose $\mu > 0$ such that

$$e^\mu mG^*(\alpha + 1) < 1. \tag{48}$$

Such a $\mu > 0$ exists since $mG^*(\alpha) = 1$ and hence $mG^*(\alpha + 1) < 1$. For such a μ

$$\begin{aligned}\sum_1^\infty P(S_n \leqq n\mu) &= \sum_1^\infty P(e^{-S_n} \geqq e^{-n\mu}) \\ &\leqq \sum_1^\infty e^{n\mu} E(e^{-S_n}) \\ &= \sum_1^\infty (e^\mu mG^*(\alpha+1))^n < \infty\end{aligned}$$

by (48). For any $u > 0$ $\sup_{0<y<u} \psi_1(y) = \psi_1^*(u)$ is finite since $\lim_{y\downarrow 0}\psi(y) = 0$ and $\psi(y)$ is continuous on $(0, \infty)$. Let for any $u > 0$, $n_0(u)$ be an integer such that $n_0(u)\mu \geqq \alpha^{-1}\log\,(u/u_0)$. Thus for any $u > 0$ using (45)

$$\begin{aligned}0 \leqq \Psi(u) \leqq \psi_1^*(u)\left(\sum_1^{n_0(u)} P(S_n > n\mu)\right) &+ \sum_{k=1}^\infty \psi(u_0(e^{-\alpha\mu})^k \\ &+ \psi_1^*(u) \sum_1^\infty P(S_n \leqq n\mu).\end{aligned}$$

From Corollary 2 we see that $\sum_1^\infty \psi(u_0 e^{-n\alpha\mu}) < \infty$ under (35). This establishes (46). To check (47) choose $u \leqq u_0$ so that $\psi_1(u) = \psi(u)$. Then

$$\begin{aligned}0 \leqq \Psi(u) &\leqq \sum_1^\infty E(\psi(ue^{-\alpha S_n})) \\ &\leqq \psi(u)\sum_1^\infty P(S_n \leqq n\mu) + \sum_1^\infty \psi(u(e^{-\alpha\mu})^n).\end{aligned}$$

But for $0 < r < 1$ and $0 < u \leqq u_0$,

$$\sum_1^\infty \psi(ur^n) \leqq \int_1^\infty \psi(ur^t)\, dt = \left(\int_0^{ur} v^{-1}\psi(v)\, dv\right)(\log_e r)^{-1}$$

by the change of variable $v = ur^t$. Hence by Corollary 2

$$\lim_{u\downarrow 0} \sum_1^\infty \psi(ur^n) = 0.$$

We know already that $\lim_{u\downarrow 0}\psi(u) = 0$. Thus (47) is established also. Clearly (46) and (47) imply that $\lim_{n\to\infty}\varphi_n(u) = \varphi(u)$ exists for all $u > 0$ and $\varphi(u) \varepsilon C_1$. To see this last point note that for $u > 0$

$$0 \leqq u^{-1}|\varphi(u) - \varphi_0(u)| \leqq \Psi(u)$$

and hence

$$\lim_{u\downarrow 0}|u^{-1}(1-\varphi(u))-1| = \lim_{u\downarrow 0}[u^{-1}(1-\varphi_0(u))-1-u^{-1}(\varphi(u)-\varphi_0(u))]$$
$$\leqq \lim_{u\downarrow 0}|u^{-1}(1-\varphi_0(u))-1| + \lim_{u\downarrow 0}\Psi(u)$$
$$= 0.$$

That $\varphi(u)$ satisfies (3) is obvious from bounded convergence theorem and the relation

$$\varphi_{n+1}(u) = T\varphi_n(u). \qquad \text{Q.E.D}$$

COROLLARY 4. *For any* $0 < \varphi < \infty$ *let* $\varphi_\theta(u) = \varphi(\theta u)$ *where* $\varphi(u)$ *is defined in Theorem* 3. *Then* $\varphi_\theta(u)$ *satisfies the functional equation* $T\varphi_\theta \equiv \varphi_\theta$ *and is the unique* solution in C_θ.

PROOF. Trivial.

4. The imbedded Galton-Watson process. It is well known [3] that if $\zeta_n(\omega)$ denotes the number of particles belonging to the kth generation for the family history corresponding to ω then the sequence of random variables $\{\zeta_n(\omega); n = 0, 1, 2, \cdots\}$ (note that we can always assume that the σ-algebra on $\mathfrak{F}$ is big enough to make the $\zeta_n(\omega)$'s random variables) forms a Galton-Watson process in discrete time with $h(s)$ as its associated pgf. If $G(0+) = 0$ and $m = h'(1-) < \infty$ then one can construct the process $\{Z(t, \omega); t \geqq 0\}$ in two stages. First one gets the sequence $\{\zeta_0(\omega) \equiv 1, \zeta_1(\omega), \zeta_2(\omega), \cdots\}$ and then independently generates lifetimes corresponding to the particles that have been created and construct $Z(t, \omega)$ as those particles that are "born before or at t" and "alive" at t. These terms can be precisely defined but as this is all done in [3] we omit details. For any $t \geqq 0$ and ω in Ω we can write

$$Z(t, \omega) \equiv \sum_{k=0}^{\infty} Y_k(t, \omega), \tag{49}$$

where for $k = 0, 1, 2, \cdots, \cdots$

$$Y_k(t, \omega) \equiv \sum_{j=1}^{Y_k(\omega)} \delta_{kj}(t, \omega) \qquad \text{if} \quad \zeta_k(\omega) \neq 0$$
$$\equiv 0 \qquad \text{if} \quad \zeta_k(\omega) = 0$$

and for $j = 1, 2, \cdots, \zeta_k(\omega)$

$$\delta_{kj}(t, \omega) = 1$$

if the jth particle belonging to the kth generation is born before or at t and is alive at t and 0 otherwise.

That is for any k and t, $Y_k(t)$ represents the number of particles belonging to the kth generation present at time t.

Let $\mathfrak{F}_1 \equiv \sigma(\zeta_0(\omega), \zeta_1(\omega), \cdots, \zeta_n(\omega), \cdots)$ be the sub σ-algebra of $\mathfrak{F}$ generated by the random variables $\zeta_0(\omega), \zeta_1(\omega), \zeta_2(\omega), \cdots, \cdots$

One of the immediate consequences of (49) is to note that the mean function $m(t) \equiv EZ(t, \omega)$ can be written as

$$m(t) = \sum_{k=0}^{\infty} m^k (G_k(t) - G_{k+1}(t)) \tag{50}$$

where $G_n(\cdot)$ is the n-fold convolution of $G(\cdot)$ with itself. Of course, (50) is the well known solution of the socalled renewal equation (10) that $m(t)$ satisfies.

We arrive at (50) via the following simple

LEMMA 2. *For any k and t one has*

$$E(Y_k(t) \mid \mathfrak{F}_1) = \zeta_k p_k(t) \qquad \text{a.s.} \tag{51}$$

where $p_k(t) = G_k(t) - G_{k+1}(t)$.

PROOF. If $\zeta_k(\omega) = 0$ then (51) is evident. If $\zeta_k(\omega) \neq 0$ then by symmetry

$$\begin{aligned} E(Y_k(t) \mid \mathfrak{F}_1) &= \zeta_k E(\delta_{k1}(t, \omega) \mid \mathfrak{F}_1) \\ &= \zeta_k P(\delta_{k1}(t, \omega) = 1 \mid \mathfrak{F}_1). \end{aligned}$$

A particle belonging to the kth generation is "born" at or before t" and "alive at t" if and only if the sum of the life times of its parent, grandparent, great grand parent, etc. is less than or equal to t while if you add the life time of this particle to this sum it exceeds t. Thus by the independence of life times and the ζ_i's

$$P(\delta_{k1}(t, \omega) = 1 \mid \mathfrak{F}_1) = \text{Prob } (S_k \leqq t < S_{k+1})$$

where $S_k = Y_1 + Y_2 \cdots + Y_k$ for $k = 0, 1, 2, \cdots$ and Y_i's are iidrv with distribution function $G(t)$.

This proves the lemma since $p_k(t) = \text{Prob } (S_k \leqq t < S_{k+1})$. Q.E.D.

Clearly $E(\zeta_k(\omega)) = m^k$ if we assume $P(\zeta_0(\omega) = 1) = 1$ and in this case Lemma 2 yields (50).

Now we are ready to prove (1) namely that unless (35) holds $Z(t)/EZ(t)$ must go to zero in law.

THEOREM 4. *If* (35) *does not hold then*

$$Z(t, \omega)/m(t) \rightarrow_p 0.$$

PROOF. We know from Kesten and Stigum's result [4] for the Galton-Watson process that if (35) does not hold then

$$\lim_{n\to\infty} \zeta_n(\omega)/m^n = 0 \qquad \text{a.s.} \tag{51}$$

Let η_1, η_2 and ϵ be three arbitrary positive numbers in (0, 1). By Egoroff's theorem (see [5]) there exists a set $A \varepsilon \mathfrak{F}_1$ such that $P(A) > 1 - \eta_1$ and on A the convergence in (51) is uniform. Thus for any $\eta_2 > 0$ there exists an N such that $\omega \varepsilon A$, $n \geqq N$ implies $\zeta_n(\omega)/m^n < \eta_2$. Thus

$$P(Z(t, \omega) > \epsilon m(t)) \leqq \eta_1 + P(\omega{:}Z(t, \omega) > \epsilon m(t), \omega \varepsilon A). \tag{52}$$

But

$$\begin{aligned} P(\omega{:}Z(t, \omega) > \epsilon m(t), \omega \varepsilon A) &\leqq P(\omega{:} \textstyle\sum_{k \leqq N} Y_k(t, \omega) > \epsilon 2^{-1} m(t), \omega \varepsilon A) \\ &\quad + P(\omega{:} \textstyle\sum_{k > N} Y_k(t, \omega) > \epsilon 2^{-1} m(t), \omega \varepsilon A) \\ &= B_1(t) + B_2(t) \text{ (say).} \end{aligned}$$

Now

$$\begin{aligned} B_1(t) &\leqq P(\omega: \textstyle\sum_{k\leqq N} Y_k(t,\omega) > \epsilon 2^{-1} m(t)) \\ &\leqq 2\epsilon^{-1} E[\textstyle\sum_{k\leqq N} Y_k(t,\omega))/m(t)] \\ &= 2\epsilon^{-1}[\textstyle\sum_{k\leqq N} m^k p_k(t))/m(t)]. \end{aligned}$$

Since $p_k(t) \downarrow 0$ and $m(t) \uparrow \infty$ as $t \uparrow \infty$

(53) $$\limsup_{t\to\infty} B_1(t) = 0 \qquad \text{for every } \epsilon > 0.$$

Also if D is the set $\{\omega: \sum_{k\geqq N} Y_k(t,\omega) > \epsilon 2^{-1} m(t)\}$ then,

$$B_2(t) \leqq E[\chi_A(\omega)\chi_D(\omega)]$$

where $\chi_A(\omega)$ as usual is the indicator function of the set A. Since $A \varepsilon \mathfrak{F}_1$ by conditioning an $\mathfrak{F}_1$ we get

$$\begin{aligned} B_2(t) &\leqq E[\chi_A(\omega) E[\chi_D(\omega) \mid \mathfrak{F}_1)] \\ &\leqq 2(\epsilon m(t))^{-1} E(\chi_A(\omega) \textstyle\sum_{k\geqq N} \zeta_k p_k(t)) \\ &\leqq 2(\epsilon m(t))^{-1} \eta_2 \textstyle\sum_{k\geqq N} m^k p_k(t)) \\ &\leqq 2\epsilon^{-1}\eta_2 . \end{aligned}$$

This with (52) and (53) implies for any $\epsilon > 0$

$$\limsup_{t\to\infty} P(\omega: Z(t,\omega) > \epsilon m(t)) \leqq \eta_1 + 2\epsilon^{-1}\eta_2 .$$

But η_1 and η_2 being arbitrary the lemma is now proved. Q.E.D.

REMARKS.

1. Note that all we need for Theorem 4 is that $G(t)$ be such that the representation (49) and Lemma 2 are valid. This does not need, for eg, that $G(0+) = 0$.

2. Although Theorem 4 asserts only convergence to zero in probability one could with a little more work establish the convergence with probability one. We omit this.

5. Convergence of $Z(t)/EZ(t)$ to a nondegenerate distribution. In Section 4 we proved (1) which ways that (35) is a necessary condition for $Z(t)/m(t)$ to converge to a nondegenerate limit distribution. This section will establish the sufficiency of (35). We shall, in fact, show that $Z(t)/m(t)$ converges in law to a distribution on the nonnegative reals whose Laplace transform or the moment generating function is given by $\varphi(u)$ of Theorem 3. We follow closely Levinson's [6] route. The first step is

THEOREM 5. *With the notations and assumptions of Section* 2 *we assert that if* (35) *holds then*

(54) $$\lim_{u\downarrow 0} \sup_{t\geqq 0} |H(u,t)| = 0$$

where $H(u,t) \equiv (m(t)/ce^{\alpha t}) - ([1 - F_1(u,t)]/u)$ *and* $F_1(u,t) = F(e^{-u(ce^{\alpha t})^{-1}}, t)$ *with* F *as in* (7).

PROOF. Since $H(u, t) = Eu^{-1}[uX - 1 + e^{-uX}]$ where X is the random variable $Z(t)/ce^{\alpha t}$ and since $x^{-1}(e^{-x} - 1 + x) \geqq 0$ for $x > 0$ and nondecreasing we have $H(u, t) \geqq 0$ and nondecreasing in u for all $t > 0$ and $u > 0$ and thus $|H(u, t)| = H(u, t)$. From (8) and (9) we get

$$\begin{aligned} 0 \leqq H(u, t) &= \int_0^t u^{-1}\{mue^{-\alpha y}m(t-y) - 1 + h(F_1(ue^{-\alpha y}, t-y))\}\, dG(y) \\ &\quad + [(u/c)e^{-\alpha t} - 1 + \exp(-(u/c)e^{-\alpha t})]u^{-1}(1 - G(t)) \\ &\leqq \int_0^t u^{-1}\{ue^{-\alpha y}mH(ue^{-\alpha y}, t-y) \\ &\quad + m[1 - F_1(ue^{-\alpha y}, t-y)]A(1 - F_1(ue^{-\alpha y}, t-y))\}\, dG(y) \\ &\quad + (u/2c^2) \end{aligned} \tag{55}$$

using the fact $0 \leqq (e^{-x} - 1 + x) \leqq (x^2/2)$ for $x \geqq 0$ and both $1 - G(t)$ and $e^{-\alpha t}$ are less than one for $t > 0$. Here $A(u)$ is the function defined by (23). For any $T > 0$, $u > 0$ let

$$H_T(u) = \sup_{t \leqq T} H(u, t). \tag{56}$$

Since $H(u, t)$ is a continuous function of u and t in $[0, \infty)$ there exists a t_0 in $[0, T]$ such that

$$H_T(u) = H(u, t_0). \tag{57}$$

Since $H(u, t)$ converges to zero uniformly for t in finite intervals we need to consider only the case when $t_0 \to \infty$ as $T \to \infty$.

Also from (13) and the nonnegativity of H we get

$$0 \leqq (ue^{-\alpha y})^{-1}(1 - F_1(ue^{-\alpha y}, t-y)) \leqq (ce^{\alpha(t-y)})^{-1}m(t-y) \leqq c_1 \tag{58}$$

where c_1 is some constant independent of u, y, t all $\geqq 0$. Thus (57) yields

$$H_T(u) \leqq \int_0^{t_0} H(ue^{-\alpha y}, t_0 - y)\, d\tilde{G}(y) + c_1 A(c_1 u) + (u/2c^2), \tag{59}$$

where function $\tilde{G}(v) \equiv m \int_0^v e^{-\alpha y}\, dG(y)$ in v is continuous, nondecreasing, is zero at zero and one at ∞. Clearly, there exists a v such that $\tilde{G}(v) = \frac{1}{2}$. For $t_0 > v$, observe that since $H(u, t)$ is nondecreasing in u we have

$$H(ue^{-\alpha y}, t_0 - y) \leqq H_T(u) \qquad \text{for } y \text{ in } [0, v]$$

and

$$H(ue^{-\alpha y}, t_0 - y) \leqq H_T(ue^{-\alpha v}) \qquad \text{for } y \text{ in } [v, t_0].$$

Breaking up the integral appropriately we get from (59)

$$H_T(u) \leqq (\tfrac{1}{2})H_T(u) + (\tfrac{1}{2})H_T(ue^{-\alpha v}) + c_1 A(c_1 u) + (u/2c^2)$$

or

$$H_T(u) \leqq H_T(ue^{-\alpha v}) + 2c_1 A(c_1 u) + (u/2c^2)$$

which on iterating yields

(60) $H_T(u) \leqq (\frac{1}{2}c^2)u(1 - e^{-\alpha v})^{-1} + 2c_1 \sum_{r=0}^{n-1} A(c_1 ue^{-r\alpha v}) + H_T(ue^{-\alpha nv})$.

Noting again that for $0 < T < \infty$ $\lim_{u\downarrow 0} H_T(u) = 0$ we have from (60)

(61) $H_T(u) \leqq (\frac{1}{2}c^2)u(1 - e^{-\alpha v})^{-1} + \sum_{r=0}^{\infty} A(c_1 ue^{-r\alpha v}) = H(u)$ (say)

and the right side being independent of T we get on letting $T \uparrow \infty$

$$\sup_{t\geqq 0} H(u, t) \leqq H(u).$$

It remains to show that $\lim_{u\downarrow 0} H(u) = 0$. But this is immediate from (35) and Corollary 3. Q.E.D.

The converse to the assertion in Theorem 4 is an easy consequence of the following result which makes crucial use of the above theorem.

THEOREM 6. *With the notations and assumptions of Section* 2 *we assert that if* (35) *holds then for each* $0 < u < \infty$

$$\lim_{t\to\infty} |u^{-1}(F_1(u, t) - \varphi(u))| = 0 \tag{62}$$

where $\varphi(u)$ *is defined in Theorem* 3 *and* F_1 *is defined in Theorem* 5.

PROOF. Let

$$K(u, t) = u^{-1}\{F_1(u, t) - \varphi(u)\}. \tag{63}$$

First observe that

$$\limsup_{u\downarrow 0} \limsup_{t\uparrow\infty} |K(u, t)| = 0. \tag{64}$$

This follows easily by majorizing $K(u, t)$ by

$$|K(u, t)| \leqq |u^{-1}(F_1(u, t) - 1) - (m(t)/ce^{\alpha t})| + |(m(t)/ce^{\alpha t}) - 1| + |1 + u^{-1}(1 - \varphi(u))|,$$

and then using Theorem 5, (13) and Theorem 3. We are aiming at proving (62) or equivalently

$$\limsup_{t\uparrow\infty} |K(u, t)| \equiv \lim_{T\to\infty} (\sup_{t\geqq T} |K(u, t)|) \equiv \lim_{T\uparrow\infty} K_T(u) \quad \text{(say)} = 0.$$

By usual arguments

$$\begin{aligned} K(u, t) = & \int_0^t (ue^{-\alpha y})^{-1}\{h(F_1(ue^{-\alpha y}, t - y)) - h(\varphi(ue^{-\alpha y}))\}e^{-\alpha y}\, dG(y) \\ & + \int_t^\infty (ue^{-\alpha y})^{-1}\{e^{-u/ce^{\alpha t}} - h(\varphi(ue^{-\alpha y}))\}e^{-\alpha y}\, dG(y) \\ \leqq & I_1 + I_2 \quad \text{(say)}. \end{aligned}$$

Now

$$\begin{aligned} |I_2| \leqq & \int_t^\infty u^{-1}|e^{-(u/ce^{\alpha t})} - 1|\, dG(y) + \int_t^\infty (|1 - h(\varphi(ue^{-\alpha y}))|/ue^{-\alpha y})e^{-\alpha y}\, dG(y) \\ \leqq & (ce^{\alpha t})^{-1} + m\int_t^\infty (ue^{-\alpha y})^{-1}(1 - \varphi(ue^{-\alpha y}))e^{-\alpha y}\, dG(y) \\ & \qquad \text{(by mean value Theorem on } h) \\ \leqq & (ce^{\alpha t})^{-1} + (1 - \tilde{G}(t)) \qquad \text{(see (59) for a definition of } \tilde{G}) \end{aligned}$$

since $0 \leqq u^{-1}(1 - \varphi(u)) \leqq 1$. For $t > T$

$$|I_1| \leqq \int_0^{t-T} + \int_{t-T}^{t} = I_{11} + I_{12} \quad \text{(say)}.$$

Let $t > 2T$. Then, in view of (64), and mean value theorem on h

$$|I_{12}| \leqq c_2(1 - \tilde{G}(T)) \qquad \text{where } c_2 \text{ is some constant.}$$

As for I_{11} for any $t > T$ again by mean value theorem on h

$$|I_{11}| \leqq m \int_0^{t-T} K(ue^{-\alpha y}, t - y)e^{-\alpha y}\, dG(y) \leqq \int_0^{t-T} K_T(ue^{-\alpha y})\, d\tilde{G}(y) \leqq EK_T(ue^{-\alpha X})$$

where X is a random variable with $\tilde{G}$ as its distribution function.

Combining the above arguments we get for $t > 2T$

$$|K(u, t)| \leqq EK_T(ue^{-\alpha X}) + c_2(1 - \tilde{G}(T)) + (ce^{\alpha t})^{-1} + (1 - \tilde{G}(T))$$

or equivalently

$$K_{2T}(u) \leqq E(K_T(ue^{-\alpha X})) + (ce^{\alpha T})^{-1} + (c_2 + 1)(1 - \tilde{G}(T)).$$

On letting $T \to \infty$ this yields by bounded convergence theorem

$$K(u) \leqq EK(ue^{-\alpha X})$$

which on iteration yields

$$K(u) \leqq EK(ue^{-\alpha S_n}) \tag{65}$$

where $S_n = X_1 + \cdots + X_n$, X_i are iidrv with the same distribution as X.

By strong law of large numbers and bounded convergence theorem (65) implies for any $u > 0$

$$0 \leqq K(u) \leqq K(0+).$$

But $K(0+) = 0$ by (64) and the theorem is proved. Q.E.D.

It only remains to prove (c) and (d) of Theorem 0. We turn to this now.

6. Absolute continuity of W. Just from the facts that $E(e^{-uW})$ coincides with $\varphi(u)$ and hence satisfies (3) and $EW = 1$ we can deduce a lot of things about W the most important of all being the absolute continuity of W. If the higher moments of the offspring distribution exist, one can, by differentiating both sides of (3) determine the corresponding moments of W (see [3]). Throughout this section we assume (35) holds so that $P(W = 0) < 1$.

Our proof of (4) depends on the following lemma on characteristic functions. We state the result in a slightly more general fashion than we need.

LEMMA 3. *Let $F(x)$ be a cumulative distribution function* (cdf) *on* $(-\infty, \infty)$. *That is $F(x)$ is nonnegative, nondecreasing and left continuous, $F(-\infty) = 0$, $F(\infty) = 1$. Let $F(0+) - F(0) = p \geqq 0$. Let $\psi(t) = \int_0^\infty e^{itx}\, dF(x)$ be the characteristic function of $F(x)$. Suppose*

$$\lim_{t\to\infty} |\psi(t) - p| = 0, \quad \int_{-\infty}^{+\infty} |x|\, dF(x) < \infty \quad (\text{so that } \psi'(t) \text{ exists for all } t) \quad \text{and}$$

$$\int_{-\infty}^{+\infty} |\psi'(t)|\, dt < \infty.$$

Then there exists a nonnegative continuous function $g(u)$ defined for all real u except at 0 *such that*

$$(67) \qquad F(x) - F(0+) = \int_{0+}^{x} g(u)\,du \qquad \text{for} \quad x > 0 \quad \text{and}$$
$$F(0) - F(x) = \int_{x}^{0} g(u)\,du \qquad \text{for} \quad x \leqq 0.$$

PROOF. If $p \neq 0$ set

$$F^*(x) = (1-p)^{-1}F(x) \qquad \text{if} \quad x \leqq 0$$
$$= (1-p)^{-1}[F(x) - p] \qquad \text{if} \quad x > 0.$$

Then $F^*(x)$ is a cdf with 0 as a continuity point. Also

$$\psi^*(t) \equiv \int_{-\infty}^{+\infty} e^{itx}\,dF^*(x) = (1-p)^{-1}(\psi(t) - p).$$

Thus our assumptions imply

$\lim_{|t|\to\infty} \psi^*(t) = 0$, $\int_{-\infty}^{+\infty} |x|\,dF^*(x) < \infty$ (so that $\psi^{*\prime}(t)$ exists for all t) and

$$\int_{-\infty}^{+\infty} |\psi^{*\prime}(t)|\,dt < \infty.$$

Therefore, if the theorem is true in the case $p = 0$ then there exists a continuous nonnegative function $g^*(u)$ defined for $u \neq 0$ such that

$$F^*(x) - F^*(0) = \int_{0+}^{x} g^*(u)\,du \qquad \text{if} \quad x > 0,$$
$$F^*(x) - F^*(x) = \int_{x}^{0} g^*(u)\,du \qquad \text{if} \quad x \leqq 0.$$

This and the definition of F^* imply (67) with $g(u) = (1-p)g^*(u)$. Hence we need to consider only the case $p = 0$. Further we need to establish (67) only for $x > 0$. The argument for $x < 0$ is entirely analogous. By a classical inversion formula [5] for any two continuity points x_1 and x_2 of $F(x)$ with $0 < x_1 < x_2 < \infty$

$$(68) \quad F(x_2) - F(x_1) = \lim_{T\to\infty} \int_{-T}^{T} \{(e^{-itx_1} - e^{-itx_2})(2\pi i t)^{-1}\}\psi(t)\,dt.$$

But

$$\int_{-T}^{T} (2\pi i t)^{-1}(e^{-itx_1} - e^{-itx_2})\psi(t)\,dt = \int_{-T}^{T} (2\pi)^{-1}\left(\int_{x_1}^{x_2} e^{-itu}\,du\right)\psi(t)\,dt.$$

Since $e^{itu}\psi(t)$ is a bounded function of t and u in the finite set $[x_1, x_2] \times [-T, +T]$ on interchanging orders of integration we get

$$F(x_2) - F(x_1) = \lim_{T\to\infty} \int_{x_1}^{x_2} g_T(u)\,du$$

where

$$(69) \qquad g_T(u) = (2\pi)^{-1} \int_{-T}^{T} e^{-itu}\psi(t)\,dt.$$

On integrating by parts, since $0 < x_1 \leqq u \leqq x_2 < \infty$,

$$g_T(u) = (2\pi u)^{-1}[e^{iTu}\psi(-T) - e^{-iTu}\psi(T)] + (2\pi u)^{-1} \int_{-T}^{T} e^{-itu}\psi'(t)\,dt$$

by hypothesis $\lim_{T\to\infty} |\psi(\pm T)| = 0$ and

$$\int_{-T}^{T} |e^{-itu}\psi'(t)|\,dt \leqq \int_{-\infty}^{+\infty} |\psi'(t)|\,dt < \infty.$$

Hence

$$\sup_{x_1 \leqq u \leqq x_2, T \geqq 0} |g_T(u)| < \infty \qquad \text{and}$$

$$\lim_{T\to\infty} g_T(u) \equiv g(u) \quad \text{exists and} \quad = (2\pi u)^{-1} \int_{-\infty}^{+\infty} e^{-itu} \psi'(t)\, dt.$$

Clearly since $\int_{-\infty}^{+\infty} |\psi'(t)|\, dt < \infty$, $g(u)$ is a continuous function of u for $u > 0$. (In fact, $ug(u)$ is uniformly continuous on $u > 0$.)

From (69) we now get by bounded convergence theorem

$$F(x_2) - F(x_1) = \int_{x_1}^{x_2} g(u)\, du$$

which on letting $x_1 \downarrow 0$ through continuity points of $F(x)$ becomes (67) for any continuity point x_2. Since F is left continuous and the continuity points are dense on the real line (67) holds for all $x > 0$. Q.E.D.

We now return to the proof of absolute continuity of W. All we need to do is to check that the hypotheses of Lemma 3 hold for $F(x) \equiv P(W \leqq x)$. This we do in the following lemmas.

Lemma 4. *The distribution of W is not concentrated at one point.*

Proof. Since $EW = 1$ and $P(W = 0) = q$ if $q > 0$ there is nothing to prove. If $q = 0$ then since $EW = 1$, W is concentrated at one point means $P(W = 1) = 1$. Thus $\varphi(u) \equiv E(e^{-uW}) \equiv e^{-u}$ and since

$$\varphi(u) = \int_0^\infty h(\varphi(ue^{-\alpha y}))\, dG(y) \qquad \text{we get} \qquad e^{-u} = \int_0^\infty h(e^{-ue^{-\alpha y}})\, dG(y).$$

But $G(0+) = 0$ and $h(s_1) < h(s_2)$ for $0 \leqq s_1 < s_2 \leqq 1$ implying

$$e^{-u} = \int_0^\infty h(e^{-ue^{-\alpha y}})\, dG(y) < e^{-u}$$

which is absurd.

Lemma 5. *If we denote $E(e^{itW})$ by $\varphi(it)$ then $|\varphi(it)| < 1$ for $t \neq 0$.*

Proof. From Lemma 4 we infer that (see [2], pp. 475) there exists a $\delta > 0$ such that

$$|\varphi(it)| < 1 \qquad \text{for all} \quad |t| < \delta, \qquad t \neq 0.$$

Clearly $\varphi(it)$ satisfies

$$\varphi(it) = \int_0^\infty h(\varphi(ite^{-\alpha y}))\, dG(y)$$

or

$$\varphi(it) = Eh(\varphi(itX))$$

where $X = e^{-\alpha Y}$ and Y is a random variable with

$$P(Y < t) = G(t) \qquad \text{for} \quad t \geqq 0.$$

Since $G(0+) < 1$, there exists a $\eta > 0$ and < 1 such that $P(X \leqq 1 - \eta) > 0$. Choose t such that $|t| < \delta(1 - \eta)^{-1}$. Then

$$|\varphi(it)| \leqq E[h(|\varphi(itX)|); \qquad X < 1 - \eta] + P(X > 1 - \eta).$$

On the set $\{X \leqq 1 - \eta)\}$ we must have $0 < |itX| < \delta$ since $P(Y < \infty) = 1$.

Hence $h(|\varphi(itX)|)$ is strictly less than one on $\{X \leqq 1 - \eta\}$ if $0 < |t| < \delta(1-\eta)^{-1}$. Thus

$$\begin{aligned} &|\varphi(it)| < 1 \quad \text{for} \quad 0 < |t| < \delta \\ \Rightarrow &|\varphi(it)| < 1 \quad \text{for} \quad 0 < |t| < \delta(1-\eta)^{-1} \\ \Rightarrow &|\varphi(it)| < 1 \quad \text{for} \quad 0 < |t| < \delta(1-\eta)^{-r} \end{aligned}$$

for any nonnegative integer r. Q.E.D.

LEMMA 6. $\limsup_{|t|\to\infty} |\varphi(it)| < 1$.

PROOF. Suppose not. Let $\limsup_{t\to\infty} |\varphi(it)| = 1$. Let $0 < t_0 < \infty$. Then by Lemma 5 $|\varphi(it_0)| < 1$. Let $0 < \epsilon < 1 - |\varphi(it_0)|$.

Since $|\varphi(it)|$ is continuous in t, goes to 1 as $t \to 0$ and by assumption $\limsup_{t\to\infty} |\varphi(it)| = 1$ there exists $t_1 < t_0$ and $t_2 > t_0$ such that

$$|\varphi(it)| < 1 - \epsilon \qquad \text{for} \quad t_1 < t < t_2$$

$$|\varphi(it_2)| = |\varphi(it_1)| = 1 - \epsilon.$$

But as in Lemma 5

$$(1-\epsilon) = |\varphi(it_2)| = |Eh(\varphi(it_2X))| \leqq E[h(|\varphi(it_2X)|; X > t_1t_2^{-1}] + P(X \leqq t_1t_2^{-1}).$$

That is

$$\begin{aligned} &1 - \epsilon \leqq h(1-\epsilon)P(X > t_1t_2^{-1}) + P(X \leqq t_1t_2^{-1}) \\ \text{(70)} \qquad &\text{or} \qquad P(X > t_1t_2^{-1})[1 - h(1-\epsilon)] < \epsilon \\ &\text{or} \qquad P(X > t_1t_2^{-1})[1 - h(1-\epsilon)]\epsilon^{-1} < 1. \end{aligned}$$

Let $\epsilon \downarrow 0$. Then $t_1 \downarrow 0$ and since $t_2 \geqq t_0$

$$P(X > t_1t_2^{-1}) \geqq P(X > t_1t_0^{-1}) \uparrow P(X > 0) = 1.$$

Also $$(1 - h(1-\epsilon))\epsilon^{-1} \uparrow m.$$

Thus (70) implies $m < 1$ and this contradicts our assumption that $m > 1$.

LEMMA 7. $\limsup_{|t|\to\infty} |\varphi(it)| \leqq q$.

PROOF. Let $$\beta_T = \sup_{|t|>T} |\varphi(it)|.$$

Let $\epsilon > 0$ be arbitrary. For any ϵ

$$|\varphi(it)| \leqq E[h(|\varphi(itX)|); X \geqq \epsilon] + P(X \leqq \epsilon).$$

Hence choosing $|t| > T$

$$\beta_T \leqq h(\beta_{T\epsilon}) + P(X \leqq \epsilon). \tag{71}$$

Let $T \uparrow \infty$. Then $\beta_T \downarrow \beta \equiv \limsup_{|t|\to\infty} |\varphi(it)|$. Now (71) implies

$$\beta \leqq h(\beta) + P(X \leqq \epsilon)$$

and ϵ being arbitrary

$$\beta \leqq h(\beta).$$

By Lemma 6, $\beta < 1$. Also $h(x)$ is convex in [0, 1]. Thus $\beta \leqq q$. Q.E.D.

LEMMA 8. $\limsup_{|t|\to\infty} |\varphi(it) - q| = 0.$

PROOF. As in Lemma 5

$$\varphi(it) - q = E[h(\varphi(itX) - q)]$$
$$= E[R(\varphi(itX))(\varphi(itX) - q)],$$

where
$$R(x) = (x - q)^{-1}(h(x) - q), \qquad x \neq q,$$
$$= h'(q), \qquad x = q.$$

Let
$$l_T = \sup_{|t|>T} |\varphi(it) - q|.$$

For any $t > 0$ proceeding as in Lemma 6

(72) $$l_T \leqq R(\beta_{T\epsilon})l_{T\epsilon} + cP(X < \epsilon)$$

where $c = 2\sup_{0\leqq x\leqq 1} R(x) < \infty$, β_T is as defined in Lemma 7. On letting $T \uparrow \infty$ (72) yields (since $\beta_T \downarrow \beta \leqq q$ and $\lim_{x\downarrow\beta} R(x) \leqq \lim_{x\to q} R(x) = h'(q) = \gamma$ (say))

(73) $$l \leqq \gamma l + cP(X < \epsilon)$$

where

$$l = \lim_{T\uparrow\infty} l_T = \limsup_{|t|\to\infty} |\varphi(it) - q|.$$

Since $\epsilon > 0$ is arbitrary we get from (73)

$$l \leqq \gamma l.$$

But $\gamma = h'(q) < 1$ and l is finite and so $l = 0$. Q.E.D.

LEMMA 9. $\int_{-\infty}^{+\infty} |\varphi'(it)|\, dt < \infty.$

PROOF. Since $\varphi(it) = Eh(\varphi(itX))$ where X is as in Lemma 5, we get

(74) $$\varphi'(it) = E[h'(\varphi(itX))\varphi'(itX)X].$$

By Lemma 7, there exists a $\tau > 0$ and θ in (0, 1) such that

(75) $$|h'(\varphi(iy))| < \theta < 1 \qquad \text{for} \quad |y| > \tau.$$

Let

$$M_T = \int_{\tau<|t|<T} |\varphi'(it)|\, dt \qquad \text{if} \quad T > \tau$$
$$= 0 \qquad \text{if} \quad T \leqq t.$$

Then

$$M_T \leqq \int_{\tau<|t|<T} E[h'(|\varphi(itX)|)\, |\varphi'(itX)|\, X]\, dt$$
$$= E(\textstyle\int_{\tau X<|y|<TX} h'(|\varphi(iy)|)|\varphi'(iy)|\, dy)$$
$$= E(\textstyle\int_{\tau<|y|<TX} + \int_{\tau X<|y|<\tau})$$
$$\leqq \theta EM_{TX} + \textstyle\int_{|y|<\tau} h'(|\varphi(iy)|)\, |\varphi'(iy)|\, dy$$
$$= \theta EM_{TX} + c \quad \text{(say)}$$

where c is a constant independent of T. Iterating

$$M_T \leqq \theta E(M_{TX_1X_2\cdots X_n}) + (\theta^{n-1} + \theta^{n-2} + \cdots + \theta + 1)c \tag{76}$$

for any integer n. Applying strong law of large numbers to the variables $-\log_e X_i$ we conclude that $P_n \equiv X_1X_2 \cdots X_n \to 0$ a.s. Since $M_{TX_1\cdots X_n} \leqq M_T$ for all n by bounded convergence theorem (76) yields now

$$M_T \leqq c(1-\theta)^{-1} < \infty. \tag{77}$$

The right side of (77) being independent of T we get on letting $T \uparrow \infty$.

$$\int_{|t|>\tau} |\varphi'(it)|\, dt < \infty.$$

But $\varphi'(it)$ being continuous is bounded in $[-\tau, \tau]$ and

$$\int_{|t|\leqq\tau} |\varphi'(it)|\, dt < \infty. \qquad \text{Q.E.D.}$$

The assertion (4) now follows quite easily from Lemmas 3, 8 and 9. This with Theorems 4 and 6 completes the proof of Theorem 0, our main result.

7. Concluding remarks. A strengthening of Theorem 0 would be to prove that $Z(t, \omega)/m(t)$ converges with probability one to a random variable $W(\omega)$. For the cases when G satisfies (5) or (6) martingale arguments yield this very quickly (see [3], [1]). For a general G too we conjecture that a martingale argument would work. The support for this comes from the following (see [3] for details).

Let:

$Z(x, y, t)$ = number of particles living at time t and being of age less than or equal to y given that the branching process started with one particle of age x at time $t = 0$,

$$\begin{aligned} M(x, y, t) &= EZ(x, y, t), \\ V(x) &= e^{\alpha x}(1 - G(x))^{-1} \textstyle\int_x^\infty e^{-\alpha u}\, dG(u), \\ A(x) &= (\textstyle\int_0^x e^{-\alpha t}(1 - G(t))\, dt)(\textstyle\int_0^\infty e^{-\alpha t}(1 - G(t))\, dt)^{-1}, \\ V_t &= \textstyle\sum_{i=1}^{Z(t)} V(x_i) \qquad \text{where} \quad x_1, x_2, \cdots, x_{Z(t)} \end{aligned} \tag{78}$$

are the ages of the $Z(t)$ particles in the system at time t.

One can now verify that $M(x, y, t)$ satisfies an integral equation similar to (10) and that $V(x)$ is a right eigenfunction for $M(x, y, t)$ with eigenvalue $e^{\alpha t}$. That is,

$$\int_0^\infty V(y)\, d_yM(x, y, t) = e^{\alpha t}V(x). \tag{79}$$

This implies that the process $\{V_t(\omega)e^{-\alpha t}; t \geqq 0\}$ is a martingale and this being nonnegative we get

$$\lim_{t\to\infty} V_t(\omega)e^{-\alpha t} \quad \text{exists and} \quad = W'(\omega) \quad (\text{say}). \tag{80}$$

Now look at $V_t(Z(t))^{-1}$, of course only on the set of nonextinction. If we could conclude that this converges to a nonzero quantity with probability one then we get from (80) that

$$\lim_{t\to\infty} Z(t)e^{-\alpha t} \quad \text{exists with probability one.} \tag{81}$$

The measure $A(x, t) \equiv (Z(y, x, t))(Z(t))^{-1}$ is the age distribution at time t. Also $V(x)$ is a continuous function with values in [0, 1]. So if the age distribution converges with probability one then we are done since

$$V_t(Z(t))^{-1} = \int V(x)\, d_x A(x, t). \tag{82}$$

At this moment the convergence of age distribution $A(x, t)$ remains a conjecture. One can show that $A(x, t)$, if it converges to something, that limit must be $A(x)$. For this reason $A(x)$ will be called *limiting age distribution.* The quantity $V(x)$ is called *reproductive value* by Harris. (See [3] where the reader may find more material on this subject.) If one proves (81) then (1) will follow from Theorem 2 and we don't need the ζ_n argument given in Theorem 4.

Another open problem is to find the relation, if any, between the limit distributions of $(m(t))^{-1}Z(t, \omega)$ and $\zeta_n(\omega)m^{-n}$ assuming $m > 1$ and $\sum_j j \log jp_j < \infty$ so that they are nontrivial.

It will be useful in studying the order of magnitude of $Z(t) - m(t)W$ (providing $Z(t)(m(t))^{-1}$ converges to W with probability one which is the case when G satisfies (5) or (6)) to know the relation between the tails of the distribution $\{p_j\}$ and the distribution of W.

The treatment here can easily be adapted to the simpler case when G satisfies (5) thus yielding an easier proof of one type case of the theorem of Kesten and Stigum [4].

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